

London Taught Course Centre

2010 examination

Graph Theory

Answers

- 1 (a) The idea is to use the greedy algorithm for vertex colouring. Set $r = \max_{t \leq n} 2f(t)/t$. For every subset X of the vertex set, the number of edges in $G[X]$ is at most $f(|X|)$, so the average degree is at most $2f(|X|)/|X|$, and thus there is a vertex of degree at most r in $G[X]$. We now order the vertices, choosing at each stage j a vertex v_j of degree at most r in $G[V \setminus \{v_1, \dots, v_{j-1}\}]$. Then we run the greedy algorithm with the vertices in this order to obtain an $(r + 1)$ -vertex-colouring of G .
- (b) The fact that $f(4) < 6$ is sufficient to show that an f -sparse graph G contains no copy of K_4 (which has 6 edges).

For the converse, we need Turán's Theorem. If G contains no K_4 , then nor does any induced subgraph $G[X]$, and so, for any set X of size m , $G[X]$ contains no more edges than the Turán graph $T_3(m)$. It remains to be shown that this number is at most $m^2/3$. The number $t_3(m)$ of edges in the Turán graph is at most

$$\frac{m(m-1)}{2} - \frac{3(m/3)(m/3-1)}{2} = m^2/3.$$

- (c) Let G be an f -sparse graph. If we remove a vertex or an edge from G , then the resulting graph is obviously f -sparse as well.

Let xy be an edge of G , and suppose G' is the graph obtained by contracting xy . Denote the new vertex representing the edge xy by z . Take a set X' of vertices of G' . If $z \notin X'$, then $G'[X']$ and $G[X']$ have the same number of edges, which is at most $f(|X'|)$. If $z \in X'$, then set $X = (X' \setminus \{z\}) \cup \{x, y\} \subseteq V(G)$ and note that $G'[X']$ has at least one edge fewer than $G[X]$ (the contracted edge xy has gone). So the number of edges of $G'[X']$ is at most $f(|X|) - 1 = |X| + C - 1 = |X'| + C = f(|X'|)$. This completes the proof that G' is also f -sparse.

So each of the 3 minor operations (vertex removal, edge removal and edge contraction) results in another f -sparse graph, proving that f -sparsity is closed under taking minors.

Many examples are possible to show the second part. Here is one.

Take $f(x) = 2$ for $x = 0, 1, 2, 3$ and $f(x) = \binom{x}{2}$ if $x \geq 4$. Then any cycle C_n with $n \geq 4$ is f -sparse. But C_3 , which is a minor of all longer cycles, is not f -sparse: If we take $X = V(C_3)$, then $C_3[X]$ has $3 > f(3)$ edges.

- (d) Let G be a connected planar graph with girth at least g , i.e., G has no cycle of length less than g . Consider a plane embedding of G ; let f_i be the number of faces with i sides in the embedding, and f be the total number of faces. Also let v be the number of vertices and e the number of edges. Then Euler's formula is that $v + f - e = 2$. We also have $2e = \sum_{i \geq g} i f_i \geq g f$. Therefore $2e \geq g(e - v + 2)$, so $(g - 2)e \leq g(v - 2)$, or $e \leq \frac{g}{g-2}(v - 2)$. The same inequality follows for disconnected G as well, on summing over components.

Now if G is a planar graph with girth at least G , then so is $G[X]$ for any $X \subseteq V(G)$. The above inequality then tells us that the number of edges of $G[X]$ is at most $\frac{g}{g-2}(|X| - 2)$ for each $X \subseteq V(G)$, i.e., that G is f -sparse for $f(x) = \frac{g}{g-2}(x - 2)$.

(e) The number of edges in a random graph $G_{m,p}$ is a Binomial random variable Z with mean $\mu = pm(m-1)/2$. We set

$$t = \frac{1}{2}pm + m^{3/2}\sqrt{\frac{p \log n}{2}} + \frac{m \log n}{3},$$

so that $f(m) = \mu + t$. By the Chernoff bound,

$$\Pr(Z \geq \mu + t) \leq \exp\left(-\frac{t^2}{\mu + t/3}\right).$$

In order to show the required bound, we need to show that $t^2/(\mu + t/3) \geq m \log n$, or

$$t^2 - t\frac{m \log n}{3} - \mu m \log n \geq 0 \text{ or } \left(t - \frac{m \log n}{6}\right)^2 \geq \mu m \log n + \frac{m^2 \log^2 n}{36}.$$

The given value of t does satisfy this; indeed

$$\left(t - \frac{m \log n}{6}\right)^2 \geq \frac{1}{2}m^3 p \log n + \left(\frac{m \log n}{6}\right)^2 \geq \mu m \log n + \frac{m^2 \log^2 n}{36},$$

as claimed.

Now we take a random graph $G_{n,p}$, say with vertex set $[n]$. For each subset X of $[n]$ of each size m , the graph induced on X by $G_{n,p}$ is distributed as a random graph $G_{m,p}$, so the probability that $G_{n,p}[X]$ has more than $f(m)$ edges is at most n^{-m} .

Therefore the expected number of subsets X of $[n]$ with more than $f(|X|)$ edges is bounded above by

$$\sum_{m=2}^n \binom{n}{m} n^{-m} \leq \sum_{m=2}^n \frac{1}{m!} \leq e - 2.$$

Hence the probability that there is such a subset X is at most $e - 2$, i.e., the probability that $G_{n,p}$ is f -sparse is at least $3 - e$.