London Taught Course Centre

2009 examination

Graph Theory

Answers

1 Suppose G is coloured with the colours $\{1, 2, \ldots, k\}$. For every edge uv, the colours of u and v are different. Orient this edge from the smaller to the larger colour. If we do this for all edges, then the resulting orientation G^* of G has the property that for every path, the vertices along the path are coloured with increasing order. This shows that the orientation is acyclic, and that a path has at most k vertices, hence at most $k-1$ edges.

For the converse, we use induction on k. The case $k = 1$ is trivial. So suppose G has an acyclic orientation G^* without paths of length k. Since G^* is acyclic, there must be vertices with outdegree zero. Let A_k be those vertices. Let G^- be the oriented graph obtained by removing the vertices in A_k from G^*.

We claim that G^- has no path of length $k-1$. For this, let P be a path of maximum length in G^-, and let v be the last vertex in P. Then v must have outdegree zero. For suppose not. If v has an edge to u and u is not on P, then we could extend P with u, contradicting that P has maximum length. And if u is on P, then the part of P from u to v together with the edge vu would form a cycle.

Since v has outdegree zero in G^-, but not in G^* (otherwise it would be in A_k and removed from G^*), there must be an edge from v to some $v^+ \in A_k$. Adding v^k to P gives a path in G^*. Since that path has length less than k, P has length less than $k-1$.

So now we know that G^- has no path of length $k-1$. By induction, we can colour G^- with $k-1$ colours. Then we can use colour k for the vertices in A_k (who form an independent set, since they have outdegree zero, so there can be no edges between them). That gives a k-colouring of G, as required.

2 [This answer is rather brief: students less familiar with these calculations will probably find themselves writing more. I am also not mentioning an experimental phase where an appropriate function $p = p(n)$ is chosen.]

Consider the random graph $G_{n,p}$, where $p = t \log n / n^{1/2}$, and t is a constant to be chosen later.

The expected number of independent sets of size $r = \lceil n^{1/2} \rceil$ in $G_{n,p}$ is

$$\binom{n}{r}(1-p)^{\binom{r}{2}} \le \Big(\frac{en}{r}(1-p)^{(r-1)/2}\Big)^r \le \big(en^{1/2}e^{-p(r-1)/2}\big)^r \le \big(e^{1+p/2}n^{1/2}e^{-t\log n/2}\big)^r.$$

For $t > 1$, this quantity is less than $1/3$ for sufficiently large n.

For any pair (x, y) of vertices, the number $N(x, y)$ of common neighbours is a Binomial random variable with parameters $(n-2, p^2)$, and therefore has mean $\mu = (n-2)p^2 \le t^2 \log n$. For $c = t^2 + a$, provided $a > 0$, the probability that $N(x, y)$ is as large as $c \log^2 n$ is at most

$$\mathbb{P}(N(x,y) \ge \mu + a\log^2 n) \le \exp\left(-\frac{(a\log^2 n)^2}{2(t^2 + a/3)\log^2 n}\right) = e^{-\varepsilon \log^2 n},$$

by the Chernoff bound, where ε is a positive constant depending on t and a.

Hence the expected number of pairs $\{x, y\}$ with at least $c \log^2 n$ common neighbours is at most $n^2 e^{-\varepsilon \log^2 n}$, which tends to zero as $n \to \infty$. Therefore the probability that there is a pair with at least $c \log^2 n$ common neighbours is at most $1/3$ for sufficiently large n.

Hence the probability that our random graph $G_{n,p}$ has the two properties is at least $1/3$, for sufficiently large n, and in particular there is such a graph for all sufficiently large n.

For the conclusion, we need to take constants $t > 1$, and $c > t^2$. So the result follows with any value of c strictly greater than 1.

3 Let $d = e(V_1, V_2)/|V_1|\,|V_2|$ be the *density* of the pair. If the pair is ε-regular with $\varepsilon = 1/5$, then, since $2 \geq (2\varepsilon)3$, each pair of 2-element subsets has density in $(d - \varepsilon, d + \varepsilon)$. Thus, for all 2-elements subsets A, B, A', B' with $A, A' \subset V_1$, $B, B' \subset V_2$, we have:

$$\left| \frac{e(A,B)}{4} - \frac{e(A',B')}{4} \right| < 2\varepsilon,$$

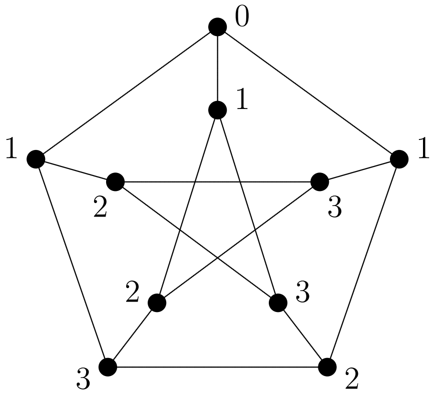
and so $|e(A,B) - e(A',B')| < 8/5$. As this is an integer, it must be at most 1.

For the second part, there are only four such bipartite graphs up to isomorphism, one for each value of d. If $d = 0$ or $d = 3$, the bipartite graph across the pairs is empty or complete bipartite, respectively, and the result is trivial.

If $d = 1$, the graph across the pair (V_1, V_2) is a matching, and the overall density is $1/3$. The condition of $\frac{1}{5}$-regularity amounts to saying that, for every pair of sets A, B of size 2, $A \subset V_1$, $B \subset V_2$, $|e(A,B)/4 - 1/3| < 1/5$, which means that $e(A,B)$ equals 1 or 2. This indeed holds.

The case $d = 2$ follows on taking complements.

Consider the partition of the Petersen graph, with $n = 10$ vertices, as shown. Here vertices in set V_i are labelled i.



We claim that this partition is ε-regular for $\varepsilon = \frac{1}{5}$. The set V_0 has $1 \leq \varepsilon n$ vertices, as required, and the other three sets all have the same size, 3. Each of the three pairs induces either a matching or the complement of a matching, and we have just seen that all these pairs are ε-regular. (We would not be allowed any irregular pairs, as $3\varepsilon < 1$.)