London Taught Course Centre

2011 examination

Graph Theory

Instructions to candidates

This part of the exam has **one question, with several parts**. You are required to **answer all parts of the question**.

Justify all your answers.

1 We define a relation $\leq_E$ on graphs (finite or countably infinite) by setting $G \leq_E H$ if there is a function $f : V(G) \to V(H)$ such that $xy \in E(G) \Rightarrow f(x)f(y) \in E(H)$ – so f maps adjacent vertices of G to (different) adjacent vertices of H.

(a) For $t \geq 2$, let K_t denote the complete graph on t vertices.

For $t \geq 2$, describe the family of graphs G such that both $G \leq_E K_t$ and $K_t \leq_E G$.

(b) Show that $\leq_E$ is a quasi-order on the family of all graphs.

(c) Suppose that G and H are finite graphs with $\chi(H) > \chi(G)$. Show that $H \not\leq_E G$.

(d) For a graph G with at least one cycle, define the *girth* $g(G)$ of G to be the length of a shortest cycle in G. For a finite connected graph G, and two vertices x and y in G, define the *distance* $d(x, y)$ between x and y to be the length of a shortest path from x to y. Define the *diameter* $\operatorname{diam}(G)$ of a finite connected graph G to be the maximum distance between any pair of vertices of G.

Suppose that G and H are finite connected graphs such that H has at least one cycle, and $g(H) \geq 2\operatorname{diam}(G) + 2$. Show that $G \leq_E H$ if and only if G is bipartite.

(e) Deduce from (c), (d), and a result in the course that there is an infinite $\leq_E$-antichain of finite graphs, i.e., an infinite sequence $G_1, G_2, \ldots$ of finite graphs such that $G_i \not\leq_E G_j$ whenever $i \neq j$.

(f) Suppose that H is a finite graph, and G is a countably infinite graph. Show that $G \leq_E H$ if and only if $F \leq_E H$ for all finite subgraphs F of G.

Show also that this fails if H is allowed to be countably infinite: give an example of two graphs G and H with vertex set $\mathbb{N}$ such that $F \leq_H$ for all finite subgraphs F of G, but $G \not\leq_E H$.

(g) For a fixed finite graph H, consider the following decision problem.

MAPS-TO-H
Instance: A finite graph G.
Question: Is it true that $G \leq_E H$.

Explain why Maps-to-H is in NP for all finite graphs H.

Give an example of an H for which Maps-to-H is NP-complete, and an example of an H for which Maps-to-H is in P.

Informal arguments will suffice.

(h) For a fixed finite graph H, consider the following decision problem.

MAPS-FROM-H
Instance: A finite graph G.
Question: Is it true that $H \leq_E G$.

For which finite graphs H is Maps-from-H in P?

Informal arguments will suffice.