Answers to LTCC Graph Theory Examination 2011

1 Note: a function $f : V(G) \to V(H)$ such that $xy \in E(G) \Rightarrow f(x)f(y) \in E(H)$ is called a *homomorphism* from G to H. It is convenient to use this term throughout the solutions.

(a) We claim that $G \leq_E K_t$ if and only if G is t-colourable. Indeed, a function $f : V(G) \to [t] = V(K_t)$ is a t-colouring if and only if, for every pair of adjacent vertices x and y in G, $f(x)f(y)$ is an edge of K_t, i.e., $f(x) \neq f(y)$.

We claim that $K_t \leq_E G$ if and only if G contains a clique of size t. Indeed, if there is a t-clique in G, then a map taking the vertices of K_t to different vertices of this clique is a homomorphism from K_t to G. Conversely, if f is a homomorphism from K_t to G, then the vertices $f(1), \ldots, f(t)$ form a t-clique.

Therefore we have both $G \leq_E K_t$ and $K_t \leq_E G$ if and only if G is a graph with clique number and chromatic number equal to t.

(b) We have to show that $G \leq_E G$ for all graphs G: this is obvious as the identity function f is a homomorphism.

Now suppose that $G \leq_E H$ and $H \leq_E J$: we have to show that $G \leq_E J$. For this, take homomorphisms $f : V(G) \to V(H)$ and $g : V(H) \to V(J)$. The composition gf is then a map from $V(G)$ to $V(J)$ such that $xy \in E(G) \Rightarrow f(x)f(y) \in E(H) \Rightarrow gf(x)gf(y) \in E(J)$, as required.

(c) Suppose $\chi(H) > \chi(G) = s$ and $H \leq_E G$. As also $G \leq K_s$ (from (a)), we have $H \leq K_s$ (from (b)), and therefore $\chi(H) \leq s$ (from (a) again), a contradiction.

(d) If H has at least one cycle, then it certainly has at least one edge (i.e., a copy of K_2). Now, if G is bipartite, then $G \leq_E H$ by (a).

Conversely, suppose G is not bipartite, let f be a homomorphism from G to H, and consider the set of vertices $f(G) = \{f(x) : x \in V(G)\}$. Note that, if $x = x_0x_1 \cdots x_s = y$ is a path in G, then $f(x) = f(x_0)f(x_1) \cdots f(x_s) = f(y)$ is a path in H, so $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in V(G)$.

Let $\mathrm{diam}(G) = d$, so that $g(H) \geq 2d + 2$. Suppose that $f(G)$ contains a cycle C of H, and let $C = u_0u_1 \cdots u_r = u_0$ be a shortest such cycle, so $r \geq g(H) \geq 2d + 2$. Now consider vertices $x, y \in V(G)$ such that $u_0 = f(x)$ and $u_{d+1} = f(y)$. Since $d(x, y) \leq d$ in G, we also have $d(u_0, u_{d+1}) \leq d$ in H. So there is a path $u_0 = v_0v_1 \cdots v_t = u_{d+1}$ in H, and this enables us to find a cycle shorter than C in H, a contradiction.

Therefore $f(G)$ contains no cycle. Therefore it is a forest, and thus bipartite, so there is a homomorphism g from the induced subgraph $H[f(G)]$ to K_2. Then the composition gf is a homomorphism from G to K_2, and so G is bipartite.

(e) The relevant result from the course is that, for all g and s, there is a (connected) graph with girth at least G and chromatic number at least s.

We shall construct an infinite antichain $G_1, G_2, \ldots$. We start with $G_1 = K_3$ (any non-bipartite graph will do). Given $G_1, \ldots, G_k$, we let d be the maximum diameter of graphs $G_1, \ldots, G_k$, and s the maximum of the chromatic numbers. Now let G_{k+1} be a connected graph with girth at least $2d + 2$ and chromatic number at least $s + 1$. Then we don't have $G_j \leq_E G_{k+1}$ for any $j \leq k$ (by (d)), and we don't have $G_{k+1} \leq_E G_j$ for any $j \leq k$ (by (c)).

(f) If f is a homomorphism from G to H, then restricting f to the induced subgraph $G[X]$ of G also yields a homomorphism from $G[X]$ to H.

Suppose that, for every finite induced subgraph $G[X]$ of G, there is a homomorphism from $G[X]$ to H. Now enumerate the vertices of G as $V(G) = \{v_1, v_2, \dots\}$; for each n, set $X_n = \{v_1, \dots, v_n\}$, and choose a homomorphism f_n from $G[X_n]$ to H.

Now we use a standard argument. We start with $N_1 = \mathbb{N}$. Choose h_1 in $V(H)$ such that $f_n(v_1) = h_1$ for infinitely many $n \in N_1$, and let $N_2 = \{n \in N_1 : f_n(v_1) = h_1\}$, and continue in this way: given an infinite set N_k, choose $h_k \in V(H)$ such that $f_n(v_k) = h_k$ for infinitely many $n \in N_k$, and set $N_{k+1} = \{n \in N_k : f_n(v_k) = h_k\}$.

In this way, we define a function $f : V(G) \to V(H)$. Suppose f is not a homomorphism; then there are adjacent $x, y \in V(G)$ such that $f(x)f(y) \notin E(H)$. But there is some m such that $x, y \in X_m$, and f agrees with infinitely many f_n on X_m; in particular, f agrees with f_n for some $n \geq m$, so that f_n is not a homomorphism on $G[X_m]$, a contradiction.

For the other part, one easy example is to take G a countable clique, and H a disjoint union of finite cliques $C_1, C_2, \dots$, with $|C_i| = i$ for each i.

(g) Given a finite graph G, a certificate for G being a positive instance of Maps-to-H is a homomorphism f from G to H. Given G and f, it is easy to check that $f(x)f(y) \in E(H)$ for all $xy \in E(G)$, and indeed this can be done in time (roughly) quadratic in the number of vertices of G.

Maps-to-H is NP-complete for $H = K_3$, since this is equivalent to 3-colourability.

Maps-to-H is in P for $H = K_2$, since this is equivalent to 2-colourability.

(h) Maps-from-H is in P for every finite H. To see this, consider the algorithm that runs through all possible functions from $V(H)$ to $V(G)$, and checks to see whether the function is a homomorphism. Each check takes time polynomial in G (at worst), and there are only $|V(G)|^{|V(H)|}$ such functions. For any fixed H, this is a polynomial in $|V(G)|$.