(a) We first do the case \( k = 2 \). For \( n = 2 \ell \) even, we can draw \( C_{2\ell}^2 \) in the plane as follows: first draw the cycle \( v_1v_2\ldots v_nv_1 \). Then draw the additional edges that go between odd-indexed vertices \( v_1v_3, v_3v_5, \ldots, v_{2\ell-1}v_1 \) inside the cycle and the ‘even’ additional edges outside the cycle.

For \( n \) odd, the graph \( C_n^2 \) is not planar. The easy way to show this is by using arguments similar to the previous paragraph. First draw the cycle in a plane. Suppose the edge \( v_1v_3 \) is drawn on the inside of the cycle. Then the edge \( v_2v_4 \) must be drawn on the outside of the cycle. And then the edge \( v_3v_5 \) must be on the inside again. Etc., until we reach a problem with the last edge. If \( v_1v_3 \) is on the outside, we reach a similar contradiction.

For \( k \geq 3 \), the graph \( C_k^n \) is never planar. To see that, again we draw the cycle first. Then for the edges \( v_1v_{k+1} \) and \( v_2v_{k+2} \) we have: one must be on the inside and one must be on the outside of the cycle. But then there is no way to add the edge \( v_3v_{k+3} \) without crossing some other edge.

We obtain that \( C_k^n \) is planar if, and only if, \( n \) is even and \( k = 2 \).

(b) The graphs \( C_k^n \) with \( n \geq 2 \ell + 1 \geq 5 \) have maximum degree \( \Delta = 4 \) and are connected. From Brooks’ Theorem we know that the only connected graphs \( G \) that have \( \chi(G) = \Delta + 1 \) are complete graphs and odd cycles. Obviously, \( C_k^n \) is not a cycle; and the only choice for \( n, k \) that makes \( C_k^n \) a complete graph is \( n = 5, k = 2 \). So we have \( \chi(C_5^2) = 5 \); while in all other cases \( \chi(C_k^n) \leq 4 \).

Now take \( n = 6 \ell + 1 \) for some \( \ell \geq 1 \) and \( k = 2 \). We will show that \( \chi(C_{6\ell+1}^2) = 4 \). Note that those graphs are not bipartite (since \( 6 \ell + 1 \) is odd), hence have chromatic number at least three. Suppose we try to colour \( C_{6\ell+1}^2 \) using three colours only, and consider the colouring on the cycle \( v_1v_2\ldots v_{6\ell+1}v_1 \). Since \( 6 \ell + 1 \) is not a multiple of three, we can’t colour the cycle in a regular pattern 1,2,3,1,2,3,… In particular, at some point on the cycle we see the colour pattern \( c_1, c_2, c_1 \) on three consecutive vertices on the cycle. But that means the additional edges between vertices at distance two on the cycle introduces an edge between the two vertices with colour \( c_1 \). So the colouring is no longer proper.

So in order to colour \( C_{6\ell+1}^2 \) we need at least four colours. Together with the conclusion in the first paragraph of this part, we get \( \chi(C_{6\ell+1}^2) = 4 \) for all \( \ell \geq 1 \).

(c) A little bit of playing with small values of \( n \) and \( k \) should lead to the observation that \( C_6^3 \) is isomorphic to the complete bipartite graph \( K_{3,3} \). And \( K_{3,3} \) is the standard example of a graph for which the choice number is not equal to the chromatic number.
(a) Let $C \subset A$ and $D \subset B$ be sets such that $|C| \geq \frac{1}{2} |A|$ and $|D| \geq \frac{1}{2} |B|$. Take $C' \subset C$ and $D' \subset D$ such that $|C'| \geq (2\varepsilon) |C|$ and $|D'| \geq (2\varepsilon) |D|$. We see that $|C'| \geq (2\varepsilon) |A|/2 = \varepsilon |A|$ and $|D'| \geq (2\varepsilon) |B|/2 = \varepsilon |B|$. From $\varepsilon$-regularity, we obtain that $|d(A, B) - d(C', D')| < \varepsilon$.

Furthermore, $|C| \geq \frac{1}{2} |A| > \varepsilon |A|$ and $|D| \geq \frac{1}{2} |B| > \varepsilon |B|$, hence $|d(A, B) - d(C, D)| < \varepsilon$. By the triangle inequality,

$$|d(C, D) - d(C', D')| \leq |d(A, B) - d(C', D')| + |d(A, B) - d(C, D)| < 2\varepsilon.$$  

(b) Notice that $e_G(A, B) = |A| |B| - e_G(A, B)$, hence $d_G(A, B) = 1 - d_G(A, B)$. So,


From this, it follows that $A, B$ is $\varepsilon$-regular in $G$ if, and only if, $A, B$ is $\varepsilon$-regular in $\bar{G}$.

(c) We follow the hint. Let $A'$ be the set of all vertices in $A$ with less than $(1/2 - \varepsilon) |B|$ neighbours in $B$. Then, $e(A', B) < |A'| (1/2 - \varepsilon) |B|$ and $d(A', B) < 1/2 - \varepsilon$.

On the other hand, if $|A'| \geq \varepsilon |A|$, then $d(A', B) > d(A, B) - \varepsilon \geq 1/2 - \varepsilon$. This is a contradiction.

(d) Since $\varepsilon < 1/10$ and $|A|, |B| \geq 100$, we have that $|N(a) \setminus \{b\}| \geq (1/2 - \varepsilon) |B| - 1 \geq \varepsilon |B|$ and $|N(b) \setminus \{a\}| \geq (1/2 - \varepsilon) |A| - 1 \geq \varepsilon |A|$. So, by $\varepsilon$-regularity, the pair $N(b) \setminus \{a\}, N(a) \setminus \{b\}$ has density at least $d(A, B) - \varepsilon \geq 1/2 - \varepsilon > 0$. Hence, there is an edge $xy$ with $x \in N(a) \setminus \{b\}$ and $y \in N(b) \setminus \{a\}$.

Thus, we have the path $a, x, y, b$.

In a similar way, we have that $|N(x) \setminus \{a, y\}| \geq (1/2 - \varepsilon) |A| - 2 \geq \varepsilon |A|$ and $|N(y) \setminus \{b, x\}| \geq (1/2 - \varepsilon) |B| - 2 \geq \varepsilon |B|$. So, by $\varepsilon$-regularity, the pair $N(x) \setminus \{a, y\}, N(y) \setminus \{b, x\}$ has density at least $d(A, B) - \varepsilon \geq 1/2 - \varepsilon > 0$. Hence, there is an edge $wz$ with $w \in N(y) \setminus \{b, x\}$ and $z \in N(x) \setminus \{a, y\}$.

Thus, we have the path $a, x, z, w, y, b$. 