London Taught Course Centre

2009 mock examination (based on the 2008 examination)

Graph Theory

Instructions to candidates

This part of the exam has **three questions**. You are required to **answer all three questions**.
All questions carry equal numbers of marks.

In your answers, you are allowed to quote results from any textbooks mentioned in the notes for the course. You are not expected to use any other sources.

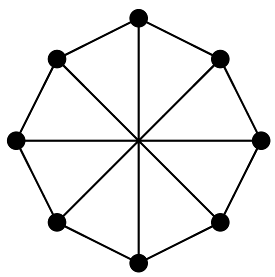
All your answers should be justified in detail. Illustrative diagrams are encouraged, but should not constitute the entirety of your answer to any part of the question.

1 Let G be a graph with a countably infinite vertex set $V = \{x_1, x_2, \dots\}$. A *matching* in G is a collection M of edges of G so that every vertex is an endpoint of at most one edge in M. A matching M is said to *cover* a subset U of V if every vertex of U is an endpoint of an edge in M.

(a) Suppose that every vertex of G has finite degree. Suppose also that, for every finite subset U of V, there is a matching $M(U)$ covering U. Show, in detail, that there is a matching covering V.

(b) Give an example of a graph G with vertex set $V = \{x_1, x_2, \dots\}$ such that, (i) for every finite subset U of V, there is a matching covering U, (ii) there is no matching covering V.

2 For $k \geq 2$, let H_k be the graph formed from the cycle C_{2k} on $2k$ vertices by putting an edge between each pair of vertices that are opposite on the cycle. The graph H_4 is shown below.



(a) For which values of k is H_k planar?

(b) What is the smallest k such that H_k contains K_5 as a minor?

(c) For which values of k can the graph H_k be embedded on the torus?

Justify your answers carefully.

3 (a) Show that the set of all trees, with subgraph ordering $\leq_S$ as the ordering relation, is not a well-quasi-ordering.

Let G, H be graphs so that H is obtained from G by a sequence of suppressions of vertices of degree two.

(b) Show that for all $k \geq 3$, if H is k-choosable, then G is k-choosable.

(c) Give an example showing that we cannot take $k = 2$ in part (b).

Answers

1 (a) Consider the finite sets $U_k = \{x_1, \dots, x_k\}$, for $k \in \mathbb{N}$. For each of these sets, there is a matching M_k covering U_k.

The vertex x_1 is covered by each matching M_k, ≥ 1. As x_1 has finite degree, there is some neighbour y_1 of x_1 such that the edge x_1y_1 is present in infinitely many of the M_k. Let $M_1^1, M_2^1, M_3^1, \dots$ be an infinite subsequence of the M_k such that x_1 is matched with y_1 in each M_k^1.

Now we continue inductively. We claim that, for each $j \in \mathbb{N}$, there is a sequence $y_1, \dots, y_j$ of vertices of V, and an infinite subsequence $M_1^j, M_2^j, \dots$ of the sequence M_k of matchings, such that each M_1^j includes all the edges x_1y_1, x_2y_2, …, x_jy_j. The case $j = 1$ is proved already.

Given the claim for j, we consider x_{j+1}. If x_{j+1} is one of the y_i, we set $y_{j+1} = x_i$, and $M_i^{j+1} = M_i^j$ for each i. If not, we use the fact that x_{j+1} has finite degree in G. So there is one neighbour y_{j+1} of x_{j+1} such that the edge $x_{j+1}y_{j+1}$ occurs in an infinite subsequence $M_1^{j+1}, M_2^{j+1}, \dots$ of the sequence $M_1^j, M_2^j, \dots$ of matchings. This establishes the claim for $j + 1$, and the whole claim follows by induction on j.

Now we claim that the set M of edges x_iy_i forms a matching covering all of V (every edge is listed twice). Clearly each vertex x_j in V does appear in M. Moreover, if any vertex is in more than one edge of M, then it is also in more than one edge of some matching M_k.

(b) The simplest example has edges $x_2x_3, x_4x_5, x_6x_7, \dots, x_1x_2, x_1x_4, x_1x_6, \dots$. In any matching covering the whole of V, all the edges $x_{2k}x_{2k+1}$ must appear, but then x_1 cannot be covered. However, for any finite subset U of V, all vertices of $U \setminus \{x_1\}$ can be covered by edges $x_{2k}x_{2k+1}$, and then x_1 (if in U) can be matched to any other vertex x_{2m} not so far matched.

2 (a) $H_2 = K_4$ is planar. For $k \geq 3$, H_k is non-planar, since it contains $K_{3,3}$ as a topological minor, as shown below. Specifically, if the cycle is $x_1x_2 \cdots x_{2k}x_1$, let one vertex class be $\{x_1, x_3, x_{k+2}\}$ and the other be $\{x_2, x_{k+1}, x_{k+3}\}$. Seven of the nine edges of the $K_{3,3}$ are present in the graph: the other two appear subdivided as the paths $x_{k+3} \cdots x_1$ and $x_3 \cdots x_{k+1}$.

(b) Suppose that H_k contains a K_5 minor, with vertex sets $C_1, \dots, C_5$. Each of the C_i has at least 4 edges coming out of it – one to each of the other C_j, so none can consist of a single vertex. So $k \geq 5$.

Finally, we note that H_5 does contain a K_5-minor: set $C_i = \{x_{2i-1}, x_{2i}\}$, for $i = 1, \dots, 5$.

So the smallest k such that H_k contains a K_5-minor is $k = 5$.

(c) All the graphs H_k can be embedded on the torus. To construct the embedding, place vertices $x_1, \dots x_k$ in one row, joined in a path, with $x_{k+1}, \dots, x_{2k}$ as a path in a second row beneath them. Vertical edges join x_j to x_{k+j} for each j. The two remaining edges are x_kx_{k+1} and $x_{2k}x_1$: these go "round the torus" in any of several ways.

3 (a) For $k \geq 1$, let T_k be the tree formed by taking a path with $k + 1$ vertices and then adding four new vertices, two of whom are adjacent to each of the end vertices of the path. We claim that for $k \neq \ell$, T_k is not a subgraph of T_ℓ. The easiest way to see this is by observing that every proper connected subgraph of T_ℓ has at most one vertex of degree 3, while T_k has two vertices of degree 3.

(b) Suppose $k \geq 3$ and H is k-choosable. Let L be a list assignment of k colours to each vertex of G. Each vertex of H corresponds to a vertex of G. Let L_H be list assignment to the vertices of H corresponding to the lists given to the vertices of G. Then H can be properly coloured using colours from each of the vertices' lists. This corresponds to a partial colouring of G (proper, since if two vertices from G are present in H and adjacent in G, then they are also adjacent in H). The only vertices not coloured yet in G are the vertices of degree two that were suppressed. But since these have only two neighbours and $k \geq 3$ colours in their lists, they can be coloured without any problems.

(c) All n-cycles with n even are 2-choosable, while those with n odd are not 2-choosable (question in the homework). So if we take H a 4-cycle and G a 5-cycle, then H can be obtained from G by a suppressing one vertex of degree two. But these two graphs fail the implication "H k-choosable $\Rightarrow$ G k-choosable".