MA 409 Mock Examination

General advice: The mock paper below is intended only as an indication of the style of the forthcoming examination. It therefore uses questions and exercises that have already been set. If you want more exercises of a similar form, you already have some in the final execise set and you may find useful to look at past papers of the MA305 course. But note that MA305 does not include control under uncertainty as a topic. I always offer the following advice for revision. Imagine *you* are the Examiner. What questions would you set as an Examiner? When we meet to revise for the examination, we will consider this advice and your potential answers.

Specific points: The MA409 Examination lasts two hours. The paper has six questions carrying equal numbers of marks. You may attempt as many as you like. Only your best four answers will count.

Question 1. Derive the Euler-Lagrange equation corresponding to the problem of finding an extremal of the functional

$$F(x) = \int_0^1 f(x(t), \dot{x}(t), t)dt.$$

You may make use of any lemma without giving proof but you should quote such a result correctly.

Use the result to solve the catenary problem, i.e. find the differentiable function $x(t)$ which minimises the (energy) functional

$$\int_0^1 x\sqrt{1+\dot{x}^2}dt$$

subject to the isoperimetric constraint

$$\int_0^1 \sqrt{1+\dot{x}^2}dt = 2.$$

Question 2. A simplified model of economic growth calls for the choice of a non-negative function $c(t)$ describing per capita consumption rate which maximises the discounted utility:

$$\int_0^T e^{-\beta t}\sqrt{c(t)}\, dt$$

subject to

$$\begin{aligned} \dot{k} &= \alpha k - c, \\ k(0) &= k_0, \end{aligned}$$

where α, β are positive constants with $\alpha > \beta$, and k is capital stock per capita with k_0 its initial value. Show that if the consumption is unrestricted, the optimal trajectory is exponentially growing. Show also that at any moment of time the Hamiltonian of the problem is a strictly concave function $\mathcal{H}(c)$ in c. Justify the conclusion that if a ceiling of $\bar{c}$ is placed on $c(t)$ consumption rises to this value and is constant thereafter. You may find it helpful to compare $\mathcal{H}(\bar{c})$ with $\mathcal{H}(0)$.

Question 3. State the Pontryagin Principle in a form suited to deriving the minimum time trajectory taking a dynamical system from a given initial state to rest at the origin (i.e. $x = 0, \quad \dot{x} = 0$) when the governing equation of the system is:

$$\ddot{x} = \dot{x} + 2x + u, \qquad |u| \leq 1.$$

Use the Pontryagin Principle to show that the optimal control is of 'bang-bang' type and that at most one switch of control takes place. Find the singular points in the $(x, \dot{x})$ phase plane corresponding to the two constant controls $u = \pm 1$ and the linear trajectories through them. Show that all non-linear trajectories are asymptotic to a linear trajectory; you should give the equation of the asymptote. By considering the eigenvalues of the associated first-order formulation say what shape of trajectories to expect in general. Sketch the trajectories $u = \pm 1$ which pass through the origin of the phase plane. Sketch also the switching curve and indicate how it is used to obtain optimal trajectories from controllable initial states $(x, \dot{x})$ in the phase plane.

Question 4. Illustrate Bellman's Principle by solving the problem of minimizing over continuously differentiable functions $x(t)$ the integral

$$\int_0^\infty (x^{2m} + \dot{x}^{2m})\, dt$$

subject to $x(0) = c$, where m is an integer. You should derive the equation

$$0 = \min_{v \in R} \{(c^{2m} + v^{2m}) + vV'(c)\}$$

for the optimal cost $V(c)$ given by:

$$V(c) = \min_{x(0)=c} \int_0^\infty (x^{2m} + \dot{x}^{2m})\, dt.$$

Show that $V(c) = c^{2m}V(1)$ and hence find $V(1)$ assuming this to be finite. Deduce that on the optimal trajectory $\dot{x}$ is proportional to $-x$ and hence find the trajectory assuming $x(0) = 1$.

Question 5. It costs I to enter an economic activity which will provide revenue at a rate v_t at time t. The market conditions are such that v_t is modelled for $t > 0$ by

$$dv_t = \alpha v_t dt + \beta v_t dz_t, v_0 = v.$$

with α, β positive constants.

Show that to maximize profit, the value function defined by

$$C(v) = \max_u E_{t,v}[e^{-\rho\tau}(v_\tau - I)],$$

where

$$\tau = \inf\{t > 0 : v_t = u\},$$

satisfies the Hamilton-Jacobi-Bellman equation

$$\frac{1}{2}\beta^2 v^2 C'' + \alpha v C' - \rho C = 0.$$

Find the value function by taking $C(v) = Av^\gamma$ for some A. You may assume smooth-pasting conditions apply.

Question 6. If F is a function with values in $\mathcal{C}[0,1]$ and takes as argument continuously differentiable functions $x : [0,1] \to \mathbf{R}$, what is the *derivative in direction* h, D_h, and what is the (strong) *Fréchet derivative* $DF(x)$? Show that

$$DF(x)h = D_h F(x),$$

provided the strong derivative exists. Compute the directional derivative when

$$F(x)(t) = \int_0^t \{x(u)\}^3\, du.$$

(b) When is a real valued function $f(h)$ said to be $o(|h|)$ and how is this term extended to cover $o(||h||)$ for vector h? What is the *supremum norm* $||x||_\infty$ on $\mathcal{C}[0,1]$? For the function $F(x)$ in part (a) show that

$$||F(x+h) - F(x) - D_h F(x)||_\infty \leq K||h||_\infty^2,$$

where the norm $||h||\infty$ is the supremum norm on $\mathcal{C}[0,1]$ and $K = 3||x|| + 1$. Conclude that the functional $F(x)$ has a strong derivative.

(c) For the problem of extremizing

$$F(x) = \int_0^1 f(x, \dot{x}, t)dt,$$

subject to

$$G(x) = 0,$$

where $G : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ is a normed vector space of continuously differentiable functions and $\mathcal{Y}$ is a normed vector space, suppose that the relative stationarity condition holds at $x = \xi$:

$$DG(\xi)h = 0 \Rightarrow DF(\xi)h = 0.$$

Deduce a Lagrange Multiplier Theorem. You should define any duality notions which you call upon.