# 1. Applications 2: Consumption and Investment under uncertainty (the Merton Model)

We consider an individual whose wealth derives entirely from an initial wealth $W_0$ which is part consumed, part invested in the bank and part invested in a single risky asset (e.g a unit trust). It is assumed that the asset follows the stochastic equation

$$dS = \mu S dt + \sigma S dz.$$

There will be two control variables. One is inevitably consumption, the other is the ratio of the wealth to be held in the risky asset. So consider a time $t$ portfolio $\Pi(t) = (w(t), 1 - w(t))$, divided between a risky asset and a bank deposit in proportions comprising $w(t)$ risky and $(1 - w(t))$. We examine the growth in the associated wealth $W(t)$ after an elapse of time $\Delta t$. Now the riskless part of the portfolio is initially in the bank deposit and worth

$$B = (1 - w)W$$

and this part grows so that

$$dB = Brdt = r(1 - w)Wdt$$

where we have assumed a constant long-term riskless interest rate of $r$ compounded continuously. The other part is initially in $\nu$ units of asset priced $S$ so has value

$$wW = \nu S$$

so that part grows by

$$\begin{aligned} \nu dS &= \mu\nu S dt + \sigma\nu S dz \\ &= \mu w W dt + \sigma w W dz. \end{aligned}$$

Thus the overall change in value of the wealth after an amount $cdt$ has been consumed is

$$dW = [r(1 - w)W + \mu w W - c]dt + W\sigma w dz.$$

Subject to this modelling assumption and an initial wealth holding

$$W(0) = W_0,$$

we wish to maximise expected utility in the usual way, viz

$$\max_{c(t),w(t)} \ \mathrm{E}\int_0^\infty e^{-\delta t}u(c)dt.$$

Using the Bellman approach we introduce the value function

$$V(W,t) = \max_{c(\tau),w(\tau)} \ \mathrm{E}\int_t^\infty e^{-\delta\tau}u(c(\tau))d\tau$$

subject to the stochastic equation obeyed by the wealth process which starts at time $t$ from a value $W$. The Bellman equation - given the infinite horizon - will thus read as previously

$$0 = \max_{c,w}\left\{u(c) - \delta V + \frac{\partial V}{\partial W}[r(1-w)W + \mu wW - c] + \frac{1}{2}[W\sigma w]^2\frac{\partial^2 V}{\partial W^2}\right\}$$

where as usual $c, w$ are the values to be selected at the time $t$. Performing the $c$ optimisation we obtain that either $c = 0$ or the first-order condition holds, i.e.

$$u'(c) = \frac{\partial V}{\partial W}.$$

Similarly either $w = 0$ or

$$\frac{\partial V}{\partial W}[(\mu - r)W] + wW^2\sigma^2\frac{\partial^2 V}{\partial W^2} = 0.$$

The latter condition implies that either $W = 0$ or

$$\frac{\partial V}{\partial W}(\mu - r) = wW\sigma^2\frac{\partial^2 V}{\partial W^2},$$

or

$$w = \frac{(r-\mu)}{W\sigma^2}\cdot\frac{\frac{\partial V}{\partial W}}{\frac{\partial^2 V}{\partial W^2}} = \frac{(r-\mu)}{W\sigma^2}\cdot\frac{V'}{V''}$$

where we have used the simplified notation of dashes to imply differentiation with respect to wealth.

Let us seek an internal solution at all times and let us specialise the utility to

$$u(c) = \frac{c^{1-\eta}}{1-\eta},$$

so that

$$u'(c) = c^{-\eta}.$$

Thus

$$\begin{aligned}\delta V(W) &= \frac{1}{1-\eta}(V')^{(\eta-1)/\eta} + V'[(\mu-r)W\cdot\frac{(r-\mu)}{W\sigma^2}\cdot\frac{V'}{V''} + rW - (V')^{-1/\eta}] + \\ &\quad + \frac{(r-\mu)^2}{2\sigma^2}\left(\frac{V'}{V''}\right)^2\cdot V'',\end{aligned}$$

or

$$\delta V(W) = \frac{\eta}{1-\eta}(V')^{(\eta-1)/\eta} + [-\frac{(r-\mu)^2}{\sigma^2}\cdot\frac{(V')^2}{V''} + rWV'] + \frac{(r-\mu)^2}{2\sigma^2}\frac{(V')^2}{V''},$$

i.e.

$$\delta V(W) = \frac{\eta}{1-\eta}(V')^{(\eta-1)/\eta} + [-\frac{(r-\mu)^2}{\sigma^2}\cdot\frac{(V')^2}{V''} + rWV'] + \frac{(r-\mu)^2}{2\sigma^2}\frac{(V')^2}{V''},$$

or finally

$$\delta V(W) = \frac{\eta}{1-\eta}(V')^{(\eta-1)/\eta} + rWV' - \frac{(r-\mu)^2}{2\sigma^2}\frac{(V')^2}{V''}.$$

We have to guess a solution. So we try a function like the utility function that we chose for the model, i.e. we try

$$V = \frac{K}{1-\eta}W^{1-\eta}.$$

Then we have

$$V' = KW^{-\eta} \text{ and } V'' = -\eta KW^{-\eta-1}$$

so

$$\frac{K\delta}{1-\eta}W^{1-\eta} = \frac{\eta}{1-\eta}(K)^{(\eta-1)/\eta}W^{1-\eta} + rKW^{1-\eta} + \frac{(r-\mu)^2}{2\sigma^2}\frac{K^2W^{-2\eta}}{\eta KW^{-\eta-1}},$$

or

$$\frac{K\delta}{1-\eta} = \frac{\eta}{1-\eta}(K)^{1-1/\eta} + rK + \frac{(r-\mu)^2}{2\sigma^2}\frac{K}{\eta}$$

so that $K$ must satisfy

$$K^{-1/\eta} = \frac{1-\eta}{\eta}\cdot\left(\frac{\delta}{1-\eta} - r - \frac{(r-\mu)^2}{2\eta\sigma^2}\right)$$

Given this we obtain

$$c^{-\eta} = KW^{-\eta} \text{ i.e. } c = WK^{-1/\eta}$$

and

$$\begin{aligned} w &= \frac{(r-\mu)}{W\sigma^2} \cdot \frac{V'}{V''} = \frac{(r-\mu)}{W\sigma^2} \cdot \frac{KW^{-\eta}}{-\eta K W^{-\eta-1}} \\ &= \frac{(\mu - r)}{\eta\sigma^2}. \end{aligned}$$

Thus the optimal division between risky and riskless assets is constant. The greater the expected growth in the risky asset, the greater is the allocation to it; but the more volatile it is, the less is allocated. A constant fraction of the wealth is consumed at each moment.