23

Calculus of variations

23.1 Path-finding problems

Here are three classic problems whose solution requires us to extend differential calculus beyond ordinary variables:

(a) Brachistochrone problem

Given two points A, B in space, A higher than B, but not vertically above B, what shape of wire connecting A to B will have the property that a bead sliding smoothly along it under gravity gets from A to B in shortest time. (See Figure 23.1.)

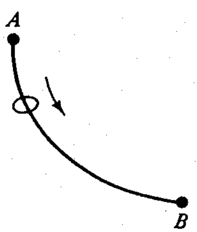

Fig. 23.1

(b) Geodesic problem

Given a surface in space, with equation $\phi(x, y, z) = 0$, and two points on it, A and B, find a path along the surface from A to B of shortest length (Figure 23.2).

(c) Isoperimetric problem

Among all plane curves of fixed length l find the one which encloses maximum area (Figure 23.3).

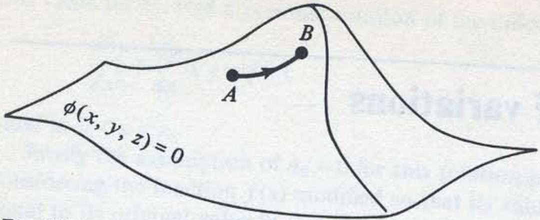

Fig. 23.2

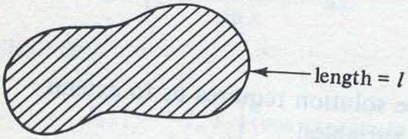

Fig. 23.3

The essential feature common to all three problems is that of finding some special curve Γ. In each case, too, a property $F(\Gamma)$ of curves is given: time of descent, length of curve, area enclosed by the curve; and we have to minimize/maximize $F(\Gamma)$ over all Γ. The variable of interest here is Γ and we need to know how F varies when Γ is varied.

It will be convenient to represent a curve Γ by an equation

$$x = x(t)$$

with

$$0 \leqslant t \leqslant T,$$

subject to $x(0) = A$

and

$$x(T) = B,$$

or perhaps by a vector equation, e.g.

$$\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t), x_3(t)) \quad (0 \leqslant t \leqslant T)$$

depending on context.

The property F of the 'curve x' will, more often than not, take the form

$$F(x) = \int_0^T f(x(t), \dot{x}(t), t)\,dt,$$

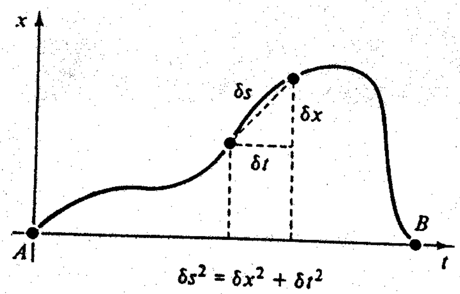

Fig. 23.4

where $\dot{x} \equiv dx/dt$. For example, curve length, in the case illustrated in Figure 23.4, is given by

$$\int_0^T \sqrt{1+\dot{x}^2}\,dt$$

and area below the curve is

$$\int_0^T x(t)dt.$$

We have purposefully used the letter x to denote a function rather than a real number, since we shall be interested in varying x. Since F assigns to each function x a real number, it is itself a function. We call a real-valued function acting on functions, a *functional*.

23.2 Variation of a functional

Suppose F is a functional and the function $\xi(t)$ $(0 \leqslant t \leqslant T)$ maximizes F. This means that for any other curve $x(t)$

$$F(x) \leqslant F(\xi).$$

Just as in calculus, we can try to compare $F(x)$ with $F(\xi)$ when x is 'close to ξ'. We can think of a function

$$x(t) = \xi(t) + h(t),$$

where $h(t)$ is also a function, as arising from an 'increment' h added to ξ. See Figure 23.5. We call h a *variation* of ξ. If also we want

$$x(0) = \xi(0) = A,$$
$$x(T) = \xi(T) = B,$$

then we require that $h(0) = h(T) = 0$. Thus not all variations will do.

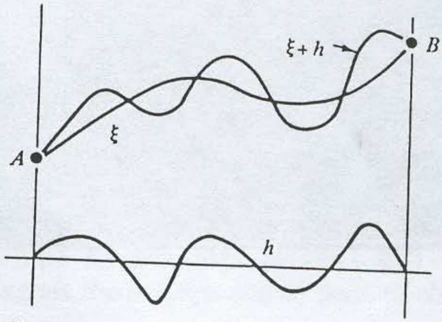

Fig. 23.5

Whenever we restrict our variations to fit in with the side conditions of a problem we will refer to them as *admissible* variations.

But now observe that in our problem sh is admissible for any real number s.

Thus the function

$$\psi(s) = F(\xi + sh)$$

has a maximum when $s = 0$. Hence, if ψ is differentiable in some interval round $s = 0$ we must have

$$\psi'(0) = 0,$$

i.e.

$$\frac{d}{ds} f(\xi + sh)|_{s=0} = 0.$$

Since this formula looks like a derivative in direction h we define

$$D_h F(x) = \lim_{s \to 0} \frac{F(x + sh) - F(x)}{s} = \frac{d}{ds} F(x + sh)|_{s=0}.$$

This is sometimes referred to as the weak derivative of F in direction h and sometimes as the *variation of* F relative to h.

Examples

(i) $F(x) = \int_0^T x(t)\,dt.$

Note for future purposes that

$$F(x + h) - F(x) = \int_0^T h(t)\,dt.$$

Also

$$D_hF(x) = \frac{d}{ds}\int_0^T (x(t) + sh(t))dt|_{s=0}$$

$$= \int_0^T \frac{d}{ds}(x(t) + sh(t))dt|_{s=0}$$

$$= \int_0^T h(t)dt.$$

(ii) $F(x) = \int_0^T \{x(t)\}^2 dt.$

Thus

$$F(x+sh) = \int_0^T \{x(t) + sh(t)\}^2 dt,$$

$$\frac{d}{ds}F(x+sh) = \int_0^T \frac{d}{ds}(x+sh)^2 dt = \int_0^T (2hx + 2sh^2)dt$$

so, setting $s = 0$

$$D_hF(x) = \int_0^T 2hx\, dt.$$

Note that

$$F(x+h) - F(x) = \int_0^T \{2hx + h^2 x\} dt.$$

(iii) $$F(x) = \int_0^T \sqrt{1 + \dot{x}^2}\, dt.$$

$$F(x+sh) = \int_0^T \sqrt{1 + (x+sh)^{\cdot 2}}\, dt = \int_0^T \sqrt{1 + (\dot{x} + s\dot{h})^2}\, dt.$$

$$\frac{d}{ds}F(x+sh) = \int_0^T \tfrac{1}{2}\{1 + (\dot{x} + s\dot{h})^2\}^{-1/2} . 2(\dot{x} + s\dot{h})\dot{h}\, dt.$$

Thus

$$D_hF(x) = \int_0^T \frac{\dot{x}\dot{h}}{\{1 + \dot{x}^2\}^{1/2}} dt.$$

Notice that in all our examples $D_hF(x)$, as a function of h (with x

fixed), is *linear*. Particularly interesting is example (iii) which we follow up:

$$D_{\alpha h_1+\beta h_2}F(x)=\int_0^T \frac{\dot{x}(\alpha \dot{h}_1+\beta \dot{h}_2)}{\sqrt{1+\dot{x}^2}}\,dt$$

$$=\int_0^T \frac{\dot{x}\alpha\dot{h}_1}{\sqrt{1+\dot{x}^2}}+\int_0^T \frac{\dot{x}\beta\dot{h}_2}{\sqrt{1+\dot{x}^2}}$$

$$=\alpha D_{h_1}F(x)+\beta D_{h_2}F(x).$$

It is often, though by no means always, also the case that $D_hF(x)$ is the *linear part* of $F(x+h)-F(x)$ in the sense that

$$F(x+h)-F(x)-D_hF(x)$$

is of higher order in h (compare Taylor's theorem).

23.3 The Euler–Lagrange equation

We now obtain a very useful equation that is necessarily satisfied by the function $\xi(t)$ which maximises/minimizes

$$F(x)\equiv\int_0^T f(x(t),\dot{x}(t),t)\,dt$$

subject to $x(0)=a$, $x(T)=b$.

We already know that the variation $D_hF(\xi)$ vanishes for all admissible h. We derive from this fact a differential equation to be satisfied by ξ. For this purpose we need to assume that $f(x,y,z)$ as a function of the *real* variables x,y,z has continuous partial derivatives $f_x(x,y,z)$ and $f_y(x,y,z)$. We have:

$$\frac{d}{ds}F(x+sh)=\int_0^T \left\{f_x(x+sh,\dot{x}+s\dot{h},t)\frac{d}{ds}(sh)\right.$$

$$\left.+f_y(x+sh,\dot{x}+s\dot{h},t)\frac{d}{ds}(s\dot{h})\right\}dt$$

(applying the chain rule!) and the latter equals

$$=\int_0^T\{f_x(x+sh,\dot{x}+s\dot{h},t)h+f_y(x+sh,\dot{x}+s\dot{h},t)\dot{h}\}\,dt.$$

Thus writing this out in full

$$D_hF(x)=\int_0^T\{f_x(x(t),\dot{x}(t),t)h(t)+f_y(x(t),\dot{x}(t),t)\dot{h}(t)\}\,dt.$$

Hence if $\xi(t)$ maximizes/minimizes F we have

$$D_h F(\xi) = 0,$$

or

$$\int_0^T \{f_x(\xi, \dot{\xi}, t)h + f_y(\xi, \dot{\xi}, t)\dot{h}\}\,dt = 0,$$

for all admissible h (i.e. for all h subject to $h(0) = h(T) = 0$).
Now we may integrate the last equation by parts to obtain

$$0 = \left[h(t) \int_0^t f_x(\xi(t), \dot{\xi}(t), t)\,dt \right]_0^T + \int_0^T \left\{ f_y - \int_0^t f_x \right\} \dot{h}\,dt.$$

Thus for all h:

$$\int_0^T \left\{ f_y - \int_0^t f_x \right\} \dot{h}\,dt = 0.$$

We claim that the curly brackets are constant over $[0, T]$.

Lemma

Suppose g is continuous on $[0, T]$ and that

$$\int_0^T g(t)\dot{h}(t)\,dt = 0$$

for all (continous) functions h with $h(0) = h(T) = 0$ such that $\dot{h}(t)$ is continuous; then g is constant.

Remark

This result is motivated by the observation that if g had been known to have a continuous derivative $\dot{g}$, then integrating by parts:

$$0 = \int_0^T g\dot{h}\,dt = [gh]_0^T - \int_0^T \dot{g}h\,dt,$$

so

$$\int_0^T \dot{g}h\,dt = 0 \quad \text{for all admissable } h, \text{ hence } \dot{g} = 0.$$

(Observe that $\dot{g}(t) > 0$ for some t, implies $\dot{g} > 0$ in a small interval round t; now take h zero outside this interval and positive inside the interval; for such an h the integral is positive.) In the present context however, all we know is that g is continuous.

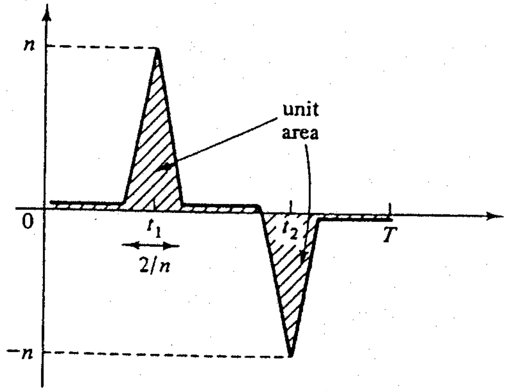

Fig. 23.6

Proof of the lemma. Let $0 < t_1 < t_2 < T$. We wish to prove $g(t_1) = g(t_2)$. For each n define the function $p_n(t)$ as illustrated in Figure 23.6. Take

$$h(t) = \int_0^t p_n(u)\,du.$$

Thus $h(0) = 0$ and $h(T) = 0$ (counterbalancing areas). By continuity of g at t_1 and t_2, g is approximately equal to $g(t_1)$ in the interval round t_1, and is equal approximately to $g(t_2)$ in the small interval round t_2. Hence

$$0 = \int_0^T g(t)\dot{h}(t)\,dt = \int_0^T g(t)p_n(t)\,dt$$

$$\approx g(t_1)\int_{t_1 - 1/n}^{t_1 + 1/n} p_n(t)\,dt + g(t_2)\int_{t_2 - 1/n}^{t_2 + 1/n} p_n(t)\,dt$$

$$= g(t_1)\cdot\{1\} + g(t_2)\cdot\{-1\}.$$

The error committed in the second line may be made as small as we please provided n is large enough. Hence, in the limit as n tends to infinity,

$$g(t_1) - g(t_2) = 0,$$

as required. Thus g is constant in value throughout $(0, T)$ and hence also throughout $[0, T]$ (by continuity).

We conclude that in our present context

$$f_y(\xi(t), \dot{\xi}(t), t) = \int_0^t f_x(\xi(t), \dot{\xi}(t), t)\,dt + \text{const}.$$

But the right-hand side is differentiable, consequently:

$$\frac{d}{dt}\{f_y(\xi(t),\dot{\xi}(t),t)\} = f_x(\xi(t),\dot{\xi}(t),t).$$

This is known as the Euler–Lagrange equation. It is sometimes written in the easily memorized forms

$$\frac{d}{dt}(f_{\dot{x}}) = f_x, \quad \text{or} \quad \frac{d}{dt}\left(\frac{df}{d\dot{x}}\right) = \frac{df}{dx}.$$

Before we attempt to solve some problems let us observe a special form of the Euler–Lagrange equation. Suppose we are maximizing/minimizing

$$F(x) \equiv \int_0^T f(x(t),\dot{x}(t))dt$$

with

$$x(0) = x_0, \quad x(T) = x_1.$$

Here the integrand f does not explicitly depend on t, that is f is a function of only *two* real variables $f(x,y)$ and there is *no third variable*. On the assumption that the optimal curve $\xi(t)$ possesses a second derivative $\xi''(t)$ the Euler–Lagrange equation is equivalent to

$$f(\xi,\xi') - \xi' f_y(\xi,\xi') = \text{const}.$$

Indeed by the chain rule

$$\frac{d}{dt}\{f - \xi' f_y\} = f_x(\xi,\xi')\xi' + f_y(\xi,\xi')\xi''$$

$$- \xi'' f_y - \xi' \frac{d}{dt} f_y$$

$$= \xi'\{f_x - \frac{d}{dt} f_y\} = 0.$$

Integrating this equation leads to the desired result.

23.4 Example. The brachistochrone problem

Choose axes through A as origin, measuring x downwards vertically and s horizontally, so that $B = (1, 1)$ (cf. Figure 23.7). The equation of motion

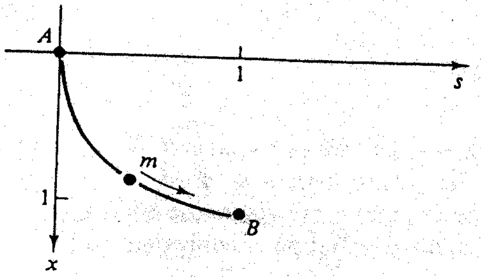

Fig. 23.7

(in the absence of friction) of a bead of mass m along the smooth wire is:

kinetic energy gained = potential energy lost

i.e.

$$\tfrac{1}{2}m(\dot{x}^2+\dot{s}^2)=mgx,$$

or

$$\frac{ds}{dt}\left(\left(\frac{\dot{x}}{\dot{s}}\right)^{1/2}+1\right)=\sqrt{(2gx)}.$$

Hence

$$\text{time taken}=T=\int_0^T dt=\int_0^1 k\left(\frac{1+(\dot{x}/\dot{s})^2}{x}\right)^{1/2}ds$$

(where $k=1/\sqrt{(2g)}$). But $\dot{x}/\dot{s}=dx/ds$, hence we have to find the curve $x=x(s)$ which minimizes

$$k\cdot\int_0^1 f\left(x(s),\frac{dx}{ds}\right)ds$$

with

$$x(0)=0 \quad\text{and}\quad x(1)=1,$$

where

$$f(x,y)=((1+y^2)/x)^{1/2}.$$

Hence the Euler–Lagrange equation in integrated form reads (with s for t)

$$\left(\frac{1+\left(\frac{dx}{ds}\right)^2}{x}\right)^{1/2}-\frac{dx}{ds}\cdot\frac{\frac{dx}{ds}}{\sqrt{x}\left(1+\left(\frac{dx}{ds}\right)^2\right)^{1/2}}=c=\text{const.},$$

since

$$f_y = \frac{1}{\sqrt{x}}\frac{1}{2}(1+y^2)^{-1/2}\cdot 2y$$

$$= \frac{y}{\sqrt{\{x(1+y^2)\}}}.$$

Thus

$$\left\{1+\left(\frac{dx}{ds}\right)^2\right\} - \left(\frac{dx}{ds}\right)^2 = c\sqrt{x}\left(1+\left(\frac{dx}{ds}\right)^2\right)^{1/2},$$

hence

$$1 = c^2x\left\{1+\left(\frac{dx}{ds}\right)^2\right\},$$

or

$$\left(\frac{dx}{ds}\right)^2 = \frac{1}{c^2x} - 1 = \frac{1-c^2x}{c^2x},$$

so

$$\int ds = \int\left(\frac{c^2x}{1-c^2x}\right)^{1/2} dx.$$

Put $x = (1/c^2)\sin^2\theta$.
Integrating we obtain:

$$s - B = \int \frac{2}{c^2}\frac{\sin\theta}{\cos\theta}\sin\theta\cos\theta\, d\theta$$

$$= \frac{1}{c^2}\int(1-\cos 2\theta)d\theta = A(\theta - \tfrac{1}{2}\sin 2\theta),$$

where $A = (1/c^2)$ and B is a constant.

But we have the implications

$$\theta = 0 \Rightarrow s = 0.$$

So $B = 0$.

Writing $2\theta = \phi$ and $a = A/2$ we obtain the parametric representation:

$$\left.\begin{aligned} s &= a(\phi - \sin\phi) \\ x &= a(1-\cos\phi) \end{aligned}\right\}$$

of a curve known as a *cycloid*. Note the geometric interpretation: x is traced by a fixed point P on the rim of a wheel rolling along the s-axis. See Figure 23.8.

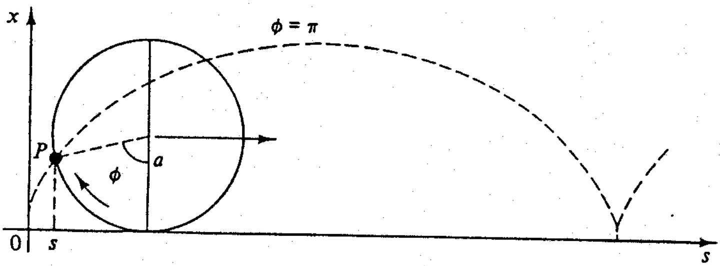

Fig. 23.8

23.5 Extension to vector-valued functions

We have considered so far only problems involving a curve

$$x = x(t) \quad 0 \leqslant t \leqslant T.$$

Problems involving curves in space will involve looking at functions

$$x(t) = (x_1(t), x_2(t), x_3(t)) \quad 0 \leqslant t \leqslant T,$$

or more generally

$$x(t) = (x_1(t), \dots x_n(t)).$$

Typically, we then deal with

$$F(x) = \int_0^T f(x_1(t), \dots x_n(t), \dot{x}_1(t), \dots \dot{x}_n(t), t)\,dt$$

subject to

$$x_1(0) = x_1^0, \dots, x_n(0) = x_n^0$$
$$x_1(T) = x_1^1, \dots, x_n(T) = x_n^1.$$

If $\xi(t) = (\xi_1(t), \dots, \xi_n(t))$ solves the problem, then clearly the function $\xi_1(t)$ solves the problem of maximising/minimising:

$$\int_0^T f(x_1(t), \xi_2(t), \dots, \xi_n(t), \dot{x}_1(t), \xi_2(t) \dots \xi_n(t), t)\,dt$$

subject to

$$x_1(0) = x_1^0, \quad x_1(T) = x_1^1.$$

Hence a necessary condition for ξ to be optimal, is that for each $i = 1, 2, \dots n$ the Euler–Lagrange equation

$$\frac{d}{dt}(f_{\dot{x}_i}) = f_{x_i}$$

must be satisfied.

23.6 Conditional maximum or minimum

Suppose, as in the geodesic problem, that we require to find $x(t) = (x_1(t), \ldots x_n(t))$ which maximises or minimises a functional

$$F(x) = \int_0^T f(x_1, \ldots x_n, \dot{x}_1, \ldots \dot{x}_n, t)\,dt$$

subject to a number of constraints of the form

$$\left.\begin{aligned} \phi_1(x_1(t), \ldots x_n(t), t) &= 0 \\ \phi_2(x_1(t), \ldots x_n(t), t) &= 0 \\ &\vdots \\ \phi_m(x_1(t), \ldots x_n(t), t) &= 0 \end{aligned}\right\},$$

where $m < n$.

To apply our previous technique we do the same as in *ordinary* calculus. We introduce Lagrange multipliers. The only difference is that here Lagrange multipliers, not surprisingly, become functions $\lambda_1(t), \ldots, \lambda_m(t)$ and we then find a stationary point of

$$\int_0^T \{f(x, \dot{x}, t) + \sum \lambda_i(t)\phi_i(x(t))\}\,dt.$$

Thus writing

$$l(x, \dot{x}, \lambda) \equiv f(x, \dot{x}, t) + \sum \lambda_i \phi_i(x)$$

we seek x, λ to satisfy

$$\frac{d}{dt}(l_{\dot{x}_i}(x, \dot{x}, \lambda)) = l_{x_i}(x, \dot{x}, \lambda)$$

together with

$$\phi_j(x(t), t) = 0 \quad (j = 1, \ldots m)$$

subject as usual to

$$x(0) = (x_1^0, \ldots x_n^0),$$
$$x(T) = (x_1^1, \ldots x_n^1).$$

Generally speaking the $m + n$ equations for $x_1, \ldots x_n, \lambda_1, \ldots, \lambda_m$ are sufficient to determine these functions and the boundary conditions will (if non-contradictory) determine the $2n$ arbitrary constants arising from the differential equations. More precisely, it is necessary to assume that the constraints $\phi_1 = 0, \ldots, \phi_m = 0$ are independent, that is that the

Jacobian:

$$\begin{bmatrix} \dfrac{\partial \phi_1}{\partial x_1}, \ldots, \dfrac{\partial \phi_1}{\partial x_n} \\ \cdots \\ \dfrac{\partial \phi_m}{\partial x_1}, \ldots, \dfrac{\partial \phi_m}{\partial x_n} \end{bmatrix}$$

should have rank n. We brush such niceties aside (leaving them to courses on functional analysis).

23.7 Examples

1 Geodesic problem

If $A = (x^0, y^0, z^0)$ and $B = (x^1, y^1, z^1)$ lie on the surface $\phi(x, y, z) = 0$, find the shortest path from A to B lying on the surface (cf. Figure 23.9). Thus we seek to minimise

$$\int_0^T \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

subject to

$$\phi(x(t), y(t), z(t)) = 0 \quad \text{for } 0 \leqslant t \leqslant T,$$

with

$$(x(0), y(0), z(0)) = A \quad \text{and} \quad (x(T), y(T), z(T)) = B.$$

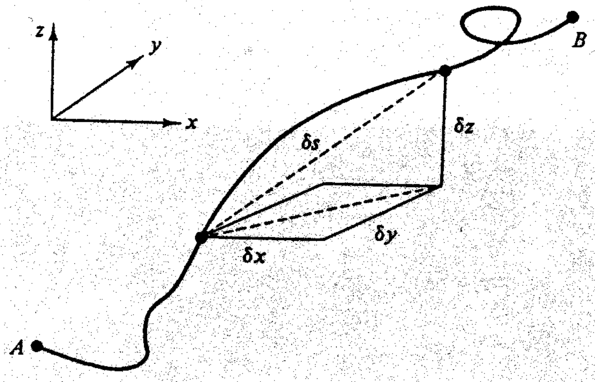

Fig. 23.9. $(\delta s)^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2$

We form $l(x, \dot{x}, \lambda) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} + \lambda(t)\phi(x(t), y(t), z(t))$. The Euler Lagrange equations give

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\dot{x}(t)}{\sqrt{\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2\}}}\right) &= \lambda(t)\phi_x(x(t), y(t), z(t))\\ \frac{d}{dt}\left(\frac{\dot{y}(t)}{\sqrt{\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2\}}}\right) &= \lambda(t)\phi_y(x(t), y(t), z(t))\\ \frac{d}{dt}\left(\frac{\dot{z}(t)}{\sqrt{\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2\}}}\right) &= \lambda(t)\phi_z(x(t), y(t), z(t))\end{aligned}\right\} .$$

Sometimes, however, the problem simplifies down.

2 Geodesics on a cylinder

The two points A and B lie on $\Phi(x, y, z) \equiv x^2 + y^2 - R^2 = 0$. Introducing cylindrical polar co-ordinates (cf. Figure 23.10)

$$(x, y, z) = (r\cos\phi, r\sin\phi, z)$$

we have on the surface that $r = R$, hence

$$\dot{x}(t) = -R\sin\phi\cdot\dot{\phi}, \quad \dot{y} = +R\cos\phi\cdot\dot{\phi}.$$

Thus we are to minimize

$$\int\sqrt{\{R^2\sin^2\phi\cdot\dot{\phi}^2 + R\cos^2\phi\cdot\dot{\phi}^2 + \dot{z}^2\}}\,dt = \int\sqrt{\{R^2\dot{\phi}^2 + \dot{z}^2\}}\,dt.$$

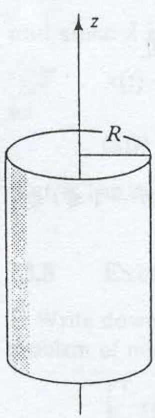

Fig. 23.10

The change of variables decreases the number of variables (the problem is now essentially two-dimensional); moreover ϕ and z are unconstrained. We are thus left with an unconstrained problem for which the Euler–Lagrange equations read

$$\frac{d}{dt}\left\{\frac{\dot{\phi}}{\sqrt{\{R^2\dot{\phi}^2+\dot{z}^2\}}}\right\}=0, \quad \frac{d}{dt}\left\{\frac{\dot{z}}{\sqrt{\{R^2\dot{\phi}^2+\dot{x}^2\}}}\right\}=0$$

hence for some constants A and B

$$\dot{\phi}=A\sqrt{\{R^2\dot{\phi}^2+\dot{z}^2\}} \quad \text{and} \quad \dot{z}=B\sqrt{\{R^2\dot{\phi}^2+\dot{z}^2\}},$$

so

$$\frac{dz}{d\phi}=\frac{\dot{z}}{\dot{\phi}}=\text{const}=a,$$ say. Thus $x=a\phi+b$ a *spiral.*

3 Isoperimetric problem

We take the problem in the form (cf. Figure 23.11): find $x(t)$ for $-1\leqslant t\leqslant 1$ with $x(-1)=x(1)=0$ so as to maximise

$$\int_{-1}^{1} x(t)\,dt$$

subject to

$$\ell=\int_{-1}^{1}\sqrt{\{1+\dot{x}^2\}}\,dt \qquad \text{(fixed arc-length).}$$

To turn this problem into the kind considered above define

$$y(t)=\int_{-1}^{t}\sqrt{\{1+\dot{x}^2\}}\,dt.$$

Thus

$$\dot{y}(t)=\sqrt{\{1+\dot{x}^2\}}, \qquad y(-1)=0 \quad \text{and} \quad y(1)=\ell.$$

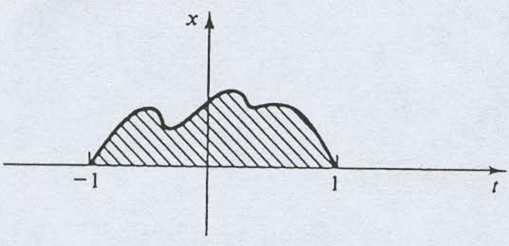

Fig. 23.11

We take $\phi(x, y, \dot{x}, \dot{y}) \equiv \dot{y} - \sqrt{\{1 + \dot{x}^2\}} = 0$ and introduce a Lagrange multiplier $\lambda(t)$.

Now we seek to maximise

$$\int_{-1}^{1} (x + \lambda(t)\{\dot{y} - \sqrt{[1 + \dot{x}^2]}\})dt.$$

The Euler–Lagrange equations are thus

$$\frac{d}{dt}\left\{\frac{-\lambda\dot{x}}{\sqrt{\{1 + \dot{x}^2\}}}\right\} = 1,$$

$$\frac{d}{dt}\{\lambda\} = 0.$$

Hence $\lambda(t)$ is in fact a constant. So we solve the first equation:

$$\frac{-\lambda\dot{x}}{\sqrt{(1 + \dot{x}^2)}} = t + c,$$

whence

$$\frac{\dot{x}^2}{1 + \dot{x}^2} = \frac{(t + c)^2}{\lambda^2}$$

and so

$$\dot{x}^2 = \frac{(t + c)^2}{\lambda^2 - (t + c)^2}.$$

Thus

$$\int dx = \int \frac{(t + c)\,dt}{\sqrt{\{\lambda^2 - (t + c)^2\}}}$$

and since λ is a constant

$$x(t) - a = \sqrt{\{\lambda^2 - (t + c)^2\}},$$

so

$$(x(t) - a)^2 + (t + c)^2 = \lambda^2,$$

that is the curve is part of a circle.

23.8 Exercises

1. Write down and solve the Euler–Lagrange equation corresponding to the problem of maximising/minimising

$$\int_0^T f(x(t), \dot{x}(t), t)\,dt$$

subject to boundary conditions when $f(x, y, z)$ is

(i) $4xz - y^2$,

(ii) $xy - 2y^2$,

(iii) $\frac{1}{x}\sqrt{\{1 - y^2\}}$,

(iv) $x^2 - 6xz$,

(v) $-y^2z^{-3}$.

2. Find the curve $x(t)$ with endpoints A, B so that the area of the surface of revolution (generated by rotating the curve round the t-axis) is minimised. See Figure 23.12.

3. At time $t = 0$ a man possesses £s. His total satisfaction over the time interval $[0, T]$ is assumed to be

$$\int_0^T e^{-\beta t} U(r(t))dt$$

where $r(t)$ is his rate of expenditure and $U(r) = \log(1 + r)$. Let $x(t)$ be his capital at time t thus

$$\dot{x}(t) = \alpha x(t) - r(t),$$

where the constant α is the interest rate. Find his optimal $x(t)$ if he seeks to maximise total satisfaction subject to

$$x(T) = 0$$

(no inheritors!).

4. A cable of *fixed* length l hangs between two supports in the shape of a curve $x(t)$ parametrised by t with $0 \leqslant t \leqslant T$. If the cable hangs so as to minimise the potential energy

$$V = mg\int_0^T x\sqrt{\{1 + \dot{x}^2\}}\,dt,$$

show that $x(t) = A\cosh((t + B)/A) + C$.

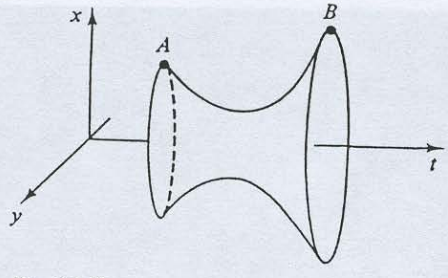

Fig. 23.12

5. Show by checking the equations in Section 23.1 that the line

$$\left.\begin{aligned} x &= 1 - t, \\ y &= 1 + t, \\ z &= t, \end{aligned}\right\}$$

is a geodesic joining the points $(1, 1, 0)$ and $(0, 2, 1)$ on the surface $(x + z)(y - z) = 1$.