0.1. Example: Impulsive Dividend Skimming

Notes for a revision class in May 2006.

A similar analysis will help solve the problem of optimal dividend policy if we restrict attention to an impulse control. The firm's capital is modelled by a stochastic differential equation which refers to a term for the cummulative value to date t of (positive) dividend payments D_t. The cumulative dividend will in this example increase dis-continuously jumping at discrete moments of time when capital is taken away from the firm and distributed to the shareholders (following an optimal distribution mechanism). The equation is thus

$$dX_t = \mu dt + \sigma dz_t - dD_t$$

and the performance index, given that $X_t = x > 0$, is defined from the expected discounted dividend stream:

$$E_{t,x}[\int_t^\beta e^{-\rho t} dD_t],$$

where

$$\tau = \beta_X(x) = \inf\{s \geq t : X_s = 0\}.$$

That is, the firm goes out of business when its capital is reduced to zero.

We suppose that there is a fixed transaction cost γ associated with distributing any positive dividend, and if a dividend d_i is distributed at time τ_i where $0 \leq \tau_1 < \tau_2 <$ ...then the value function is

$$V(x,t) = E_{t,x}[\sum_{t \leq \tau_i < \beta} e^{-\rho \tau_i}(d_i - \gamma)].$$

We time shift. Set

$$\sigma_i = \tau_i - t$$

and $Y_s = X_{t+s}$ we obtain $Y_0 = x$ and

$$dY_s = \mu ds + \sigma dz_s - dD_s.$$

Put

$$\sigma = \sigma_Y(x) = \inf\{s > 0 : Y_s = 0\}.$$

Since the stochastic processes X and Y are Markovian (depend on the current state and not on the past) the distributions of S and of T are identical. Thus the associated value function for the shifted problem is $V(x,0)$ and of course

$$V(x,t) = e^{-\rho t} E_{0,x}[\sum_{0 \leq \sigma_i < \sigma} e^{-\rho \sigma_i}(d_i - \gamma)] = e^{-\rho t} V(x,0).$$

Hence, letting $W(x) = V(x,0)$, the Bellman equation which holds between dividend-distribution times takes the form

$$0 = -\rho W + \mu W'(x) + \frac{1}{2}\sigma^2 W''(x).$$

Thus as in the previous example, the auxiliary equation is

$$\frac{1}{2}\sigma^2 m^2 - \mu m - \rho = 0,$$

and the solution function takes the form

$$W(x) = Ae^{m_+ x} + Be^{m_- x},$$

where

$$m_\pm = \frac{\mu \pm \sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2} = \frac{\mu}{\sigma^2} \pm \Delta,$$

where

$$\sigma^2\Delta^2 = \mu^2 + 2\rho\sigma^2.$$

Note immediately that if $x = 0$ the firm is then already insolvent, and so $W(0) = 0$. Hence we must have

$$B = -A.$$

As for the dividend-distribution times, since the problem is time-homogeneous, beyond the initial starting point (when we may wish to reduce the capital with a one-off starting dividend) we will henceforth always wish to distribute dividend by reference to the same (optimally selected) threshold call it b,i.e. at the times

$$\tau_{i+1} = \inf\{t > \tau_i : X_{t-} = b\}$$

and the size of the dividend will always be d so that $a = b - d > 0$.

Thus for any pre-selected b we get from the Bellman optimality principle that

$$W(b) = \max_a\{W(a) + (b - a) - \gamma\} = \max_a\{W(a) - a\} + b - \gamma.$$

Assuming that $W''(x) < 0$ in the range $(0, b)$ the first-order condition for maximization gives a condition on a, namely

$$W'(a) = 1.$$

Selecting b optimally is a more delicate task. Let us regard ourselves as choosing a and b independently (though with $a < b$). Given b, if a has been selected optimally we have

$$W(a) - a = W(b) - b + \gamma.$$

It is apparently true that the right-hand side does not depend on a. One may be forgiven for rushing to the conclusion that in order that the left-hand side have maximal value, the value of b needs to be selected so as to maximize the right hand side. This would give the correct condition

$$W'(b) = 1,$$

but for the wrong reason. Indeed we will see later that $W''(b) > 0$.The extremizing condition just derived gives in fact a local minimum at b.Contradiction.

The correct view to take is that if the starting position of the firm had been at $x > b$ (with b optimal) then the optimal response is still to shift the capital down to a so that, just as at b. Denoting the value function for the range $[b, \infty)$ by W_+ we have

$$W_+(x) = W(a) + x - a - \gamma \text{ for } x \geq b.$$

This should be read as saying that the valuation is linear for $x \geq b$, and observe that here we have

$$W'_+(x) = 1.$$

The correct view is thus to smooth-paste the valuation function obtained by solving the ordinary differential equation in the range $(0, b)$ with the linear solution valid for $x > b$. This means that indeed we wish to have satisfied th condition

$$W'(b) = \lim_{x \searrow b} W'_+(x) = 1,$$

i.e.

$$W'(b) = 1,$$

but for a different reason.

In conclusion, we obtained four conditions on four unknowns: on the trigger values a, b and on the constant A, B of the solution function

$$W(x) = Ae^{\gamma_+ x} + Be^{\gamma_- x} = 2Ae^{\mu x/\sigma^2} \sinh(\Delta x)$$

These are

$$\begin{aligned} A + B &= 0, \\ W(a) - a &= W(b) - b + \gamma, \\ W'(a) &= 1, \\ W'(b) &= 1. \end{aligned}$$

It remains to check that these equations do actually have a solution. I won't do this here.

Our final comment, in preparation for a forthcoming example, is that as $\gamma \to 0$ the solution for a, b tends to $\bar{x}$, where

$$\tanh(\Delta x) = \frac{2\mu\sigma\Delta}{(\mu)^2 + \sigma^2\Delta^2} = \frac{\mu\sigma\Delta}{\mu^2 + \rho\sigma^2} = \frac{\mu\sigma\Delta}{\sigma^2\Delta^2 - \rho\sigma^2}.$$