Exercises on computing the derivative

1. Compute

$$DF(x,\tau)(h,\sigma)^T$$

when

$$F(x,\tau) = \int_a^\tau f(x(t),\dot{x}(t),t)dt.$$

2. Let $q(t)$ be a continuously differentiable function. Compute

$$DG(x,\tau)(h,\sigma)^T$$

when

$$G(x,\tau) = \begin{pmatrix} x(a) - x_0 \\ x(\tau) - q(\tau) \end{pmatrix} \in \mathbf{R}^2.$$

Comment: If the objective is to minimize $F(x,\tau)$ subject to $x(a) = x_0$ and $x(\tau) = q(\tau)$ the constraint can be represented as $G(x,\tau) = 0$.

3. In the last question, what is the connection between h and σ when $DG(x,\tau)(h,\sigma)^T = 0$? Use your result to draw the following conclusions.

(i) Deduce the Euler-Lagrange Equation by setting $h(1) = 0$.

(ii) Use (i) to show that if $DF(x,\tau)(h,\sigma)^T = 0$, then a consequence of $DG(x,\tau)(h,\sigma)^T = 0$ is that

$$0 = f(x(\tau),\dot{x}(\tau),\tau)\sigma + \int_a^\tau f_x h dt + \Big[f_{\dot{x}} h(t) \Big]_a^\tau - \int_a^\tau \frac{d}{dt}(f_{\dot{x}})h(t)dt.$$

(iii) From the last displayed equation deduce that the optimizer in the minimization problem described in the Comment, satisfies the **transversality condition:**

$$f - (\dot{x} - \dot{q})f_{\dot{x}} = 0 \quad \text{when} \quad t = \tau.$$

Careful: This last condition holds only at the time instant $t = \tau$. Of course if $q(t) = x_1 = const$, then we recover an instantaneous version of the equation

$$f - \dot{x}f_{\dot{x}} = c,$$

but with $c = 0$.