1. Exercises on Itô

Exercise

$$\int_0^T z(t)dz(t) = \frac{1}{2}(z(T)^2 - z(0)^2) - \frac{1}{2}T.$$

$$\int_0^T z^2(t)dz(t) = \frac{1}{3}(z(T)^3 - z(0)^3) - \int_0^T z(t)dt.$$

Exercise Show that

$$\int_0^T tdz_t = Tz_T - \int_0^T z_t dt.$$

Example. If $z_0 = 0$, use the last two exercises to deduce that $z_T^3 - 3\int_0^T z^2(t)dz(t)$ has zero drift.

Example. If the process η_t satisfies

$$d\eta_t = r\eta_t dt + \sigma\sqrt{\eta_t}dz_t,$$

find $d(\sqrt{\eta_t})$.

Example. If the process y_t satisfies

$$dy_t = ry_t dt + \sigma dz_t,$$

find $d(y_t^2)$.

Exercise Show that if $z_0 = 0$ then

$$\int_0^T g(t)dz_t = g(T)z_T - \int_0^T g'(t)z_t dt.$$

Exercise Generalize this result when $f(t, z) = g(t)h(z)$.

Exercise The stochastic differential equation for the rate of inflation I is given by

$$dI = \mu I dt + \sigma I dz_t.$$

Find the equation followed by the real interest rate R defined by $R = B/I$ where $B = e^{rt}$. Show that

$$E[\frac{\Delta R}{R}] = r - \mu + \sigma^2.$$

Hint: Consider $f(x, t) = e^{rt}/x$.

Exercise. If $v_t = \exp(-w_t)$ and w_t is given by

$$w_t = \gamma z_t + \frac{1}{2}\gamma^2 t$$

i.e.

$$dw_t = \frac{1}{2}\gamma^2 dt + \gamma dz_t,$$

where z_t is standard Brownian motion, show that

$$dv_t = -\gamma v_t dz_t.$$

Example. Verify that if

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

where z_t is say P-Brownian motion and

$$dS_t^* = (\mu - r) S_t^* dt + \sigma S_t^* dz_t$$

then with $\gamma = (\mu - r)/\sigma$

$$X_t^* = v_t S_t^*$$

is a zero drift Brownian motion (under P).

Exercise. A risky asset has price S_t modeled by the equation

$$dS_t = \mu S_t dt + \sigma S_t dz_t.$$

If $H(t) = (h(t), k(t))$ is the portfolio of $h(t)$ units of risky asset and $k(t)$ units of cash then its total value is $V(t) = hS_t + kB_t$, where $B_t = e^{rt}$. Show that if the portfolio is self-financing, i.e. $dV = hdS + kdB$, then we have

$$dV = [rV + hS(\mu - r)]dt + \sigma hSdz_t].$$

Exercise. With the notation of the last two exercise, show that $X^* = v_t V^*$ has zero drift and satisfies

$$dX^* = v_t[\sigma hS^* - \gamma V^*]dz_t.$$

[Hint: First find dV^*.]