Homework on Control under uncertainty (Using the Hamilton-Jacobi-Bellman equation)

Exercise 1. Solve the Bellman equation for the valuation problem

$$V(x) = \max_{\mu(t)} E[\int_0^\infty e^{-rs}(X_s - \mu_s^2)ds]$$

subject to $dX_t = \mu_t dt + \sigma dz$ with $X_0 = x$ and σ constant. The integral may be interpreted as income from a cash flow X_t less quadratic cost μ_t^2 of a manager who drives the stochastic asset value with a chosen effort level μ_t . Assuming a linear dependence of $V(x)$ on x deduce that μ_t is constant.

In the following model we use the notation

$$\delta_t X = X(t) - X(t-) = X(t) - \lim_{\theta \nearrow t} X(\theta),$$

for the amount the process jumps as it approaches time t through earlier times. (It is thus also a forward looking differential.)

Exercise 2 (Insightful!). In the absence of any cash payout (=dividend), the cash reserves of a firm follow the equation

$$dX = u_t \mu dt + u_t \sigma dz_t, \text{ for } 0 \leq t < \beta,$$

where the initial state is $X_0 = x_0$, the control u_t (reducing variance at the cost of the drift) is selected by management and satisfies

$$0 \leq u_t \leq 1,$$

whereas β is the bankruptcy time, here defined to be the first time the reserves reach zero:

$$\beta = \inf\{t > 0 : X_t = 0\}.$$

Suppose now that at times $0 \leq \tau_1 < \tau_2 ...$ management makes a payout of ξ_n at time τ_n. Then the state is described as satisfying both

$$\delta_{\tau_n} X = \xi_n,$$

and

$$dX_t = u_t \mu dt + u_t \sigma dz_t, \text{ for } t \notin \{\tau_1, \tau_2, ...\}.$$

Comment: *This is taken to mean that for* $t < \beta$

$$X(t) = x + \mu \int_0^t u_\theta d\theta + \sigma \int_0^t u_\theta dz_\theta - \sum_{n=1}^{\infty} \xi_n \mathbf{1}_{[\tau_n < t]}.$$

Management seek to maximize the expected receipts to the shareholders modelled as

$$V(x) = E_x \left[\sum_{n=1}^{\infty} e^{-\rho \tau_n} (k\xi_n - K) 1_{[\tau_n < \beta]} \right],$$

where the constant K represents a fixed dividend payment cost to the firm, and the constant tax rate which the firm must apply is $1 - k$ (with $0 < k \leq 1$).

Solve the optimization problem by assuming that the dividend payment is triggered in the same way at each time τ_n, i.e. by reference to a cash reserve level x_+ so that the n-th time of dividend payment is τ_n where

$$\tau_n = \inf\{t > \tau_{n-1} : X_t = x_+\}.$$

(Here $\tau_0 = 0$). Suppose that x_+ is selected optimally, and the payment made by the firm is such that the reserve is drawn down at time τ_n (with $n > 0$) always to the same level x_-. This means that

$$\xi_n = x_+ - x_-$$

unless β is reached earlier.

(a) Show that the Bellman equation

$$\max_u \frac{1}{2}\sigma^2 u^2 V'' + u\mu V' - \rho V = 0$$

leads to two possibilities for maximization: (i) with $u = 1$ (a constant coefficients case), and (ii) with

$$u = -\frac{\mu V'}{\sigma^2 V''}.$$

(b) Show that in the case (ii) a power solution Cx^γ is available for some γ which you should find. Deduce that the range of validity for this solution is $0 \leq x \leq \bar{x}$, where

$$\bar{x} = \frac{\sigma^2(1-\gamma)}{\mu}.$$

(c) Write down the smooth-pasting and value-matching conditions at $\bar{x}$ for transition from (i) to (ii).

(d) Assuming x_+ is selected so that $x_+ \geq \bar{x}$, suggest why the following linear solution is valid for $x \geq x_+$:

$$\begin{aligned} V(x) &= V(x_-) + k(x - x_-) - K \\ &= V(x_+) + k(x - x_+). \end{aligned}$$

[Hint: If $x = x_+$ then dividend payment calls to adjusts the state down to x_-.]

What smooth-pasting conditions at $x = x_+$ does this in turn suggest?

(e) What is the number of unknowns in relation to piecewise fitting the three solution formats?

(Remember to include in your count x_+ and x_-.) Suggest why the missing condition is

$$V'(x_-) = k.$$

(f) Show that the equations in (d) imply that

$$\int_{x_-}^{x_+} [1 - V'(x)/k] = K/k.$$

Noting the dependance of V on K, k by way of $V(x; k, K)$ show that

$$V(x; k, K) = k \cdot V(x; 1, K/k).$$

Comment. It remains to be verified that all the equations derived above can be solved simultaneously. One established approach to checking this, is to eliminate as many free variables as possible and then considering a geometric interpretation of the equation

$$\int_{x_-}^{x_+} [1 - C \cdot v'(x)] = K/k$$

(where C is the only free variables and $v'(x)$ does not depend on C and $x_\pm$ locates where $Cv' = k$.).

Discussion Exercise 3. Suppose (H_t, K_t) is a time t indexed trading strategy in a risky asset priced S_t and riskless cash K_t which enables a consumption C_t. The asset follows the usual geometric Brownian model $dS_t = \mu S dt + \sigma S dz_t$, and initially the asset is priced $S_0 = S$, and the initial endowment is $H_0 = H, K_0 = K$. The portfolio behaves according to the following book-keeping identities:

$$\begin{aligned} dK_t &= (rK_t - C_t)dt - S_t(1+b)p_t dt + S_t(1-s)n_t dt, \\ dH_t &= (H_t + p - n)dt \end{aligned}$$

where $p_t, n_t \geq 0$ are respectively the purchase and selling rates and there is a constraint that $H_t, K_t \geq 0$. Here we think of a valuation for the asset but with $S(1+b)$ being the buying price and $S(1-s)$ the selling price. The objective is

$$\sup_{C_t} E\left[\int_0^\infty e^{-\beta t} U(C_t)dt\right],$$

with the controls being $p_t, n_t \geq 0$ as well as $C_t \geq 0$. Let

$$V(H, K, S) = \sup_{C_t p_t n_t} E\left[\int_0^\infty e^{-\beta t} U(C_t)dt\right]$$

subject to $H_0 = H, K_0 = K, S_0 = S$. Show that the Bellman equation reads:

$$\begin{aligned} 0 &= \max\left\{\left[\frac{\partial V}{\partial H} - (1+b)\frac{\partial V}{\partial K}\right] p\right\} + \max\left\{\left[-\frac{\partial V}{\partial H} + (1-s)\frac{\partial V}{\partial K}\right] n\right\} \\ &\quad + \max_C \left[U(C) - C\frac{\partial V}{\partial K}\right] + rK\frac{\partial V}{\partial K} + HS\frac{\partial V}{\partial H} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \beta V. \end{aligned}$$

[Hint: Assume that $0 \leq p, n \leq \kappa$.]

Discussion Exercise 4. (Capital investment in a firm: adapted from Abel and Eberly) The following homogeneous function of order one $\pi(x, k) = x^\gamma k^{1-\gamma}$ is used to model the instantaneous rate at which operating profit (=revenue minus cost of non-capital inputs) is generated by a firm which has a quantity k of capital and faces market demand conditions that are given by a market-index value x. The market-index $X(t)$ follows a geometric Brownian motion

$$\frac{dX}{X} = \mu dt + \sigma dz_t.$$

The firm may buy capital at a constant price B and sell capital for S where $S < B$. The capital stock $K(t)$ at time t depreciates at a constant rate δ. Denoting by $U(t)$ the total of purchases of capital in the period $[0, t]$ and by $W(t)$ the total of sales of capital, so that both are (weakly) increasing functions, the change in capital is

$$dK = dU - dW - \delta K dt.$$

If the firm maximises the present value of cash flows, it may be valued at time t when capital stock is k and the market-index is x by the formula

$$V(k,x,t) = \max_{U,W} E[\int_t^\infty e^{-r\theta}\pi(X_\theta, K_\theta)d\theta - \int_t^\infty Be^{-r\theta}dU(\theta) + \int_t^\infty Se^{-r\theta}dW(\theta)].$$

(This requires $r > \mu$.) Letting $s = \theta - t$ and noting that $V(k,x,t)e^{rt} = V(k,x,0) =_{def} V(k,x)$ we see that

$$V(k,x) = \max_{U,W} E[\int_0^\infty e^{-rs}\pi(X_{t+s}, K_{t+s})ds - \int_0^\infty Be^{-rs}du(t+s) + \int_0^\infty Se^{-rs}dv(t+s)].$$

(i) Explain why V is homogeneous of degree one and so V_k is homogeneous of degree zero.

Let $y = x/k$ and $q(y) = V_k(1,y)$. Verify that $q(y) = V_k(k,ky) = V_k(k,x)$. Show that

$$\mu x V_{kx} - \delta k V_{kk} = (\mu+\delta) y q'(y)$$

[Hint: By Euler's theorem $kV_{kk} = -xV_{kx}$.]

Why does $Y = X(t)/K(t)$ follow a geometric Brownian motion with drift $\mu + \delta$ and standard deviation σ?

(ii) On the optimal path explain why $V_k(K(t), X(t)) = B$ whenever $dU(t) > 0$ and similarly $V_k(K(t), X(t)) = S$ whenever $dW(t) > 0$. Thus $dU = dW = 0$ when $S < V_k(K(t), X(t)) < B$.

(iii) Use (ii) to show that the Bellman equation is

$$rV(k,x) = x^\gamma k^{1-\gamma} - \delta k V_k(k,x) + \mu x V_x(k,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}(k,x).$$

Differentiate this equation with respect to k to obtain

$$rV_k(k,x) = (1-\gamma)x^\gamma k^{-\gamma} - \delta[V_k(k,x) + kV_{kk}] + \mu x V_{xk}(k,x) + \frac{1}{2}\sigma^2 x^2 V_{xxk}(k,x)$$

and explain why this is equivalent to

$$(r+\delta)q(y) = (1-\gamma)y^\gamma + (\mu+\delta)yq'(y) + \frac{1}{2}\sigma^2 y^2 q''(y).$$

(iv) Show that the last equation has a solution in the form

$$q(y) = ay^{\alpha} + by^{\beta} + cy^{\gamma}$$

for some constants a, b, c. You should identify α, β, c.

(v) Define y_B and y_S by $q(y_B) = B, q(y_S) = S$. Explain why $y_S < y_B$. [Hint: $y = x/k$ and a low capital position implies purchase.] Since $q'(y) = 0$ in the interval $y_S < y < y_B$ we must take

$$q'(y_B) = 0 = q'(y_S).$$

This enables the identification of a and b.