

1. Hint for q2 (on the surface of revolution)

Hint: A vertical slice of the surface of revolution between positions t and $t + \Delta t$ is approximately a cylinder of radius $x(t)$, (that's easy!) and height Δs , where Δs is the increment in curve length (that's less obvious, but it certainly isn't Δt). Its surface area is thus $2\pi x \Delta s$.

In conclusion the surface of revolution is given by the integral

$$\int_a^b 2\pi x \sqrt{1 + \dot{x}^2} dt.$$

To understand the argument more fully, follow through these items:

1. Revolve a line segment of length r emanating from the origin and at an angle α to the horizontal. The line segment lies along the line $y = (\tan \alpha)x$ and creates a cone whose base has perimeter $2\pi h$ when $h = r \sin \alpha$ is the height of the other extremity of the line segment above the horizontal axis. Unfolded, the surface of revolution is of course a sector of circle of radius r and angle θ such that

$$r\theta = 2\pi h = 2\pi r \sin \alpha.$$

The area of the sector is $\frac{1}{2}r^2\theta$. (Check: if $\theta = 2\pi$ then what?). If r is extended to $r + \Delta s$ the increment in area is (to first order) given by

$$\frac{1}{2}\theta[(r + \Delta s)^2 - r^2] \approx r\theta\Delta s = 2\pi h\Delta s = 2\pi(r \sin \alpha)\Delta s$$

This says that the incremental strip of the sector is roughly a rectangle of width Δs and length $r\theta$.

2. Consider a rectifying chord on the curve $x(t)$ of length Δs corresponding to an increment Δt in the independent variable. Thus to first order

$$\Delta s = \sqrt{1 + \dot{x}^2} \Delta t.$$

Extending this short line segment back to its intersection with the horizontal axis creates the same geometry as in item 1 above.

The line segment at its extremity is a height $h = x$ above the horizontal axis.

3. Conclusion the surface of revolution is

$$\int_a^b 2\pi x \sqrt{1 + \dot{x}^2} dt.$$