0.1 Example on Impulse control: Interest rate pegging

These notes extend the comments at the start of Section 3 of the final set of Notes (section entitled “Examples requiring Optimal Timing”). For the sake of clarity I have changed the notation so that $\alpha$ is replaced by $u$ and $\beta$ is replaced by $v$. That is when the process reaches $X = a$ it is reset to take the value $u$, and when it reaches $b$ it is reset to take the value $v$. You should interpret the example as pegging an exchange rate to lie between $a$ and $b$. The pegs $a$ and $b$ are selected by exogenous agents, but the resetting positions $u$ and $v$ are selected by optimality considerations (to reduce costs of resetting).

Our analysis begins by observing that for times $t$ satisfying $\tau_i < t < \tau_{i+1}$ the Bellman equation for $W(x) = V(x, 0)$ reads

$$0 = -\rho W + \mu W'(x) + \frac{1}{2}\sigma^2 W''(x),$$

a constant coefficient second order equation with auxiliary

$$\frac{1}{2}\sigma^2 \gamma^2 + \mu \gamma - \rho = 0,$$

whose roots are $\gamma_+ > 0$ and $\gamma_- < 0$ where

$$\gamma_{\pm} = \frac{-\mu \pm \sqrt{\mu^2 + 2\sigma^2 \rho}}{\sigma^2}.$$

The solution is thus of the form

$$W(x) = Ae^{\gamma_+x} + Be^{\gamma_-x}.$$ 

For $t = \tau = \tau_i$ we have two cases according as $X_{\tau_-} = a$ or $b$. We take the lower bound $a$ first and observe that the Bellman equation is now

$$V(\tau, a) = e^{-\rho \tau} W(a) = \min_u \{e^{-\rho \tau} (\gamma + c(u - a)) + e^{-\rho \tau} W(u)\}$$

since $X_{\tau_-} = a$ and the fundamental variable shifts to $X_\tau = u$. Thus

$$W(a) = \min_u \{\gamma + c(u - a) + W(u)\}$$

and the first order condition for $u$ reads

$$c + W'(u) = 0.$$

Thus we have for the optimal value of $u$ that

$$W(a) = \gamma + c(u - a) + W(u),$$

$$W'(u) = -c,$$

(notice this is like smooth pasting at $x = u$ rather than at $a$ since $a$ and $b$ are given, and $u$ and $v$ are being selected optimally).
Similarly if $X_\gamma = b$ we have if $X_\tau = v$ that

$$W(b) = \min_v \{ \gamma + c(b - v) + W(v) \}$$

and so

$$-c + W'(v) = 0,$$

(in effect smooth pasting at $x = v$, rather than at $b$, since as was said above, $a$ and $b$ are given and $u$ and $v$ are being selected optimally).

Thus we have for the optimal value of $v$ that

$$W(b) = \gamma + c(b - v) + W(v),$$

$$W'(v) = c.$$

Now for states $x$ between $a$ and $b$ we have

$$W(x) = Ae^{\gamma+u} + Be^{\gamma-u}.$$ 

There are thus four equations for the four unknowns $u, v, A, B$, namely

$$Ae^{\gamma+u} + Be^{\gamma-u} = \gamma + c(u - a) + Ae^{\gamma+u} + Be^{\gamma-u},$$

$$A\gamma + e^{\gamma+u} + B\gamma e^{\gamma-u} = -c,$$

$$Ae^{\gamma+v} + Be^{\gamma-b} = \gamma + c(b - v) + Ae^{\gamma+v} + Be^{\gamma-v},$$

$$A\gamma + e^{\gamma+v} + B\gamma e^{\gamma-v} = c.$$ 

It is not immediately clear that these four equations can be solved simultaneously with the restriction.

$$a < u < v < b.$$ 

Note that

$$A\gamma + \{ e^{\gamma+v}e^{\gamma-u} - e^{\gamma+u}e^{\gamma-v} \} = c(e^{\gamma-u} + e^{\gamma-v})$$

so that if $u < v$ then $A > 0$. Similarly $B > 0$. Hence $W''(x) > 0$.

Indeed they can be. We refer the interested reader to the excellent paper by M.Jeanblanc-Pique for a proof. See Math. Finance Vol 3. No.2 (April 1993), 161-177. The proof begins with a change of variables to render the system in a more canonical form as a preliminary to reducing it to two equations in two unknowns.