0.1 Example on Impulse control: Interest rate pegging

These notes extend the comments at the start of Section 3 of the final set of Notes (section entitled "Examples requiring Optimal Timing"). For the sake of clarity I have changed the notation so that α is replaced by u and β is replaced by v. That is when the process reaches $X = a$ it is reset to take the value u, and when it reaches b it is reset to take the value v.You should interpret the example as pegging an exchange rate to lie between a and b. The pegs a and b are selected by exogenous agents, but the resetting positions u and v are selected by optimality considerations (to reduce costs of resetting).

Our analysis begins by observing that for times t satisfying $\tau_i < t < \tau_{i+1}$ the Bellman equation for $W(x) = V(x, 0)$ reads

$$0 = -\rho W + \mu W'(x) + \frac{1}{2}\sigma^2 W''(x),$$

a constant coefficient second order equation with auxiliary

$$\frac{1}{2}\sigma^2\gamma^2 + \mu\gamma - \rho = 0,$$

whose roots are $\gamma_+ > 0$ and $\gamma_- < 0$ where

$$\gamma_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2}.$$

The solution is thus of the form

$$W(x) = Ae^{\gamma_+ x} + Be^{\gamma_- x}$$

Fot $t = \tau = \tau_i$ we have two cases according as $X_{\tau-} = a$ or b. We take the lower bound a first and observe that the Bellman equation is now

$$V(\tau, a) = e^{-\rho\tau}W(a) = \min_u\{e^{-\rho\tau}(\gamma + c(u - a)) + e^{-\rho\tau}W(u)\}$$

since $X_{\tau-} = a$ and the fundamental variable shifts to $X_\tau = u$. Thus

$$W(a) = \min_u\{\gamma + c(u - a) + W(u)\}$$

and the first order condition for u reads

$$c + W'(u) = 0.$$

Thus we have for the optimal value of u that

$$\begin{aligned} W(a) &= \gamma + c(u - a) + W(u), \\ W'(u) &= -c, \end{aligned}$$

(notice this is like smooth pasting at $x = u$ rather than at a since a and b are given, and u and v are being selected optimally).

Similarly if $X_{\tau-} = b$ we have if $X_\tau = v$ that

$$W(b) = \min_v \{\gamma + c(b - v) + W(v)\}$$

and so

$$-c + W'(v) = 0,$$

(in effect smooth pasting at $x = v$, rather than at b, since as was said above, a and b are given and u and v are being selected optimally).

Thus we have for the optimal value of v that

$$\begin{aligned} W(b) &= \gamma + c(b - v) + W(v), \\ W'(v) &= c. \end{aligned}$$

Now for states x between a and b we have

$$W(x) = Ae^{\gamma_+ x} + Be^{\gamma_- x}.$$

There are thus four equations for the four unknowns u, v, A, B, namely

$$\begin{aligned} Ae^{\gamma_+ a} + Be^{\gamma_- a} &= \gamma + c(u - a) + Ae^{\gamma_+ u} + Be^{\gamma_- u}, \\ A\gamma_+ e^{\gamma_+ u} + B\gamma_- e^{\gamma_- u} &= -c, \\ Ae^{\gamma_+ b} + Be^{\gamma_- b} &= \gamma + c(b - v) + Ae^{\gamma_+ v} + Be^{\gamma_- v}, \\ A\gamma_+ e^{\gamma_+ v} + B\gamma_- e^{\gamma_- v} &= c. \end{aligned}$$

It is not immediately clear that these four equations can be solved simultaneously with the restriction.

$$a < u < v < b.$$

Note that

$$A\gamma_+ \{e^{\gamma_+ v} e^{\gamma_- u} - e^{\gamma_+ u} e^{\gamma_- v}\} = c\{e^{\gamma_- u} + e^{\gamma_- v}\}$$

so that if $u < v$ then $A > 0$. Similarly $B > 0$.Hence $W''(x) > 0$.

Indeed they can be. We refer the interested reader to the excellent paper by M.Jeanblanc-Pique for a proof. See Math. Finance Vol 3. No.2 (April 1993), 161-177. The proof begins with a change of variables to render the system in a more canonical form as a preliminary to reducing it to two equations in two unknowns.