Summer 2011 examination

MA409

Continuous-time Optimisation
Half Unit

Suitable for all candidates

Instructions to candidates

Time allowed: 2 hours

This examination paper contains 6 questions. You may attempt as many questions as you wish, but only your best **4** questions will count towards the final mark. All questions carry equal numbers of marks.

Please write your answers in dark ink (preferably black or blue) only.

Calculators are **not** allowed in this exam.

You are supplied with: Maths Answer Booklet

1.

(a) State the Fundamental Lemma of the Calculus of Variations.

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function with (jointly) continuous partial derivatives with respect to the first and second argument.

Consider the problem of extremizing the functional

$$F(x) = \int_0^1 f(x(t), \dot{x}(t), t)\, dt$$

subject to the end-point restrictions $x(0) = a$, $x(1) = b$, where a, b are constants and the function $x : [0,1] \to \mathbb{R}$ is a continuously differentiable function.

Use the Fundamental Lemma to derive the Euler-Lagrange equation corresponding to this problem.

(c) Write down and solve the Euler-Lagrange equation in integrated form for the problem of finding the continuously differentiable function minimizing the functional

$$F(x) = \int_{-1}^1 x(t)\sqrt{1 + \dot{x}(t)^2}\, dt,$$

subject to $x(-1) = 0$ and $x(1) = 0$ and the constraint

$$\int_{-1}^1 \sqrt{1 + \dot{x}(t)^2} = 4.$$

Hint: *You may wish to note the solution to* $2a = \sinh a$ *is approximately* $a = 2.2$ *and the following indefinite integral*

$$\int \frac{dz}{\sqrt{z^2 - 1}} = \cosh^{-1}(z).$$

2.

(a) Let $\mathscr{X}$ be a normed vector space of continuously differentiable functions. Let the function $f : \mathbb{R}^3 \to \mathbb{R}$ have continuous partial derivatives.

Consider the problem of extremizing over $\mathscr{X}$

$$\begin{aligned} F(x) &= \int_0^1 f(x, \dot{x}, t)dt \\ \text{subject to } G(x) &= 0, \end{aligned}$$

where G is a differentiable function from $\mathscr{X}$ to a normed vector space $\mathscr{Y}$.

Suppose that the relative stationarity condition holds at $x = \xi$, that is, for each h in $\mathscr{X}$

$$DG(\xi)h = 0 \Rightarrow DF(\xi)h = 0.$$

(i) Deduce a Lagrange Multiplier Theorem.

(ii) Define any duality notions which you call upon in (i) above.

(b) Consider the problem of minimizing over the functions $u \in \mathscr{C}([0,2])$ the functional

$$J(u) := \int_0^2 (x_u(t) + u(t)^2)\, dt,$$

where $x_u(t)$ denotes the unique function in $\mathscr{C}^1([0,2])$ satisfying the equation

$$\dot{x} = -x + u, \text{ for } 0 \le t \le 2, \text{ with } x(0) = 1.$$

(i) Write down the Hamiltonian formulation of the necessary conditions to be satisfied by the minimizer, assuming one exists.

(ii) Hence find the minimizer, assuming one exists. You should identify both the choice $v(t)$ for the function $u(t)$ and the corresponding trajectory x_v.

(c) Suppose that in part (b) the further restriction that $0 \le u(t) \le 1$ is imposed on the problem. Suppose also that a minimizer $w(t)$ exist for this new problem.

(i) Use Pontryagin's Principle to deduce that for the control $w(t)$ the state $x_w(t)$ is constantly 1 on an initial time interval $[0, \tau]$, for some positive $\tau < 1$.

(ii) Hence find the corresponding trajectory x_w.

You may use the approximations: $e^2 = 7.4$, $\log_e 3 = 1.1$.

3.

A dynamical system has governing equation:

$$\ddot{x} = 3\dot{x} - 2x - u, \qquad |u| \leq 1,$$

with x real-valued.

(a) State the Pontryagin Principle in a form suited to deriving the minimum time trajectory taking the system from a given initial state to rest at the origin (i.e. $x = 0$, $\dot{x} = 0$).

(b) Use the Pontryagin Principle to show that the time-optimal control, if it exists, is of 'bang-bang' type and that at most one switch of control takes place.

(c) Find the singular points in the $(x, \dot{x})$ phase plane corresponding to the two constant controls $u = \pm 1$, and the linear trajectories through them. By considering the eigenvalues of the associated first-order formulation, say what shape of trajectories to expect in general.

(d) Sketch the trajectories $u = \pm 1$ which pass through the origin of the phase plane.

(e) Sketch the switching curve and indicate how it is used to characterize optimal trajectories, assuming they exist. Identify the controllable initial states $(x, \dot{x})$ in the phase plane.

4.

In this question $\mathscr{C}^1(S)$, with S either a finite interval $[0,T]$ or $\mathbb{R}_+$, denotes the family of real-valued continuously differentiable functions with domain S.

Consider the problem of finding

$$V(c,T) = \min\left\{ \int_0^T (9x^6 + x^4\dot{x}^2)\, dt : x \in \mathscr{C}^1[0,T] \text{ and } x(0) = c \right\}.$$

(a) Show that the Bellman equation for the problem is

$$0 = \min_{v\in\mathbb{R}} \left\{ (9c^6 + c^4v^2) + v\frac{\partial V}{\partial c} - \frac{\partial V}{\partial T} \right\}.$$

(b) Verify that $V(c,T) = c^6 G(T)$, where $G(T) = V(1,T)$, and hence find $G(T)$.

(c) Denoting the optimal trajectory by x_T, show that $\dot{x}_T(0)/x_T(0) = -3G(T)$ and, more generally, that

$$\frac{dx_T}{x_T} = \frac{\dot{x}_T(t)}{x_T(t)} = -3G(T-t). \qquad (*)$$

(d) Deduce that

$$\lim_{T\to\infty} G(T) = 1, \quad \text{and hence} \quad \lim_{T\to\infty} \frac{\dot{x}_T(0)}{x_T(0)} = -3.$$

(e) Interpreting (*) as a differential equation, and integrating from a fixed t to $T > t$, show that

$$\lim_{T\to\infty} \frac{x_T(T)}{x_T(t)} = c,$$

for some constant c (i.e. independent of t).

Hint: *Note that* $\int (e^x/(1+e^x))dx = \log(1+e^x)$.

5.

An economic activity is modeled as follows. A cost I dollars is incurred at the time of initiation of the activity, irrespective of timing. If initiated at a future time t, it is assumed that the activity will provide an income stream of value v_t. Suppose v_t is modeled for $t > 0$ by the stochastic differential equation

$$dv_t = \alpha v_t dt + \beta v_t dz_t,$$

where $v_0 = v$ may be arbitrary, α, β are positive constants, and z_t is a standard Wiener process.

A function C is defined by

$$C(v) = \max_u E[e^{-\rho\tau(u,v)}(v_{\tau(u,v)} - I)^+],$$

where $\rho > \alpha$ is a positive constant discount rate and

$$\tau(u,v) := \inf\{t > 0 : v_t = u\}, \text{ for } u > 0.$$

Assume that the function C is twice continuously differentiable, and that the optimal time of initiating activity is of the form $\tau(u,v)$ for an appropriate u.

(a) Show that, for $v < u$, C satisfies the Hamilton-Jacobi-Bellman equation

$$\frac{1}{2}\beta^2 v^2 C'' + \alpha v C' - \rho C = 0.$$

(b) Find the value function of the optimization problem, assuming it is of the form $C(v) = Av^\gamma$ for some A and γ. You should identify A and γ. You may assume that smooth-pasting conditions apply.

(c) Find the optimal value of u when $\rho = 2\alpha$ and $\beta^2 = 2\rho = 4\alpha$.

(d) Sketch the value function in this case.

6.

In this question $\mathscr{C}[0,1]$ denotes the normed vector space of real-valued continuous functions on $[0,1]$; $\mathscr{C}^1[0,1]$ denotes the subspace consisting of functions x whose derivative $\dot{x}$ exists and is continuous. $\mathscr{C}[0,1]$ is equipped with the maximum norm $||x||_\infty$, whereas $\mathscr{C}^1[0,1]$ is equipped with the following norm:

$$||x||_1 := ||x||_\infty + ||\dot{x}||_\infty.$$

(a) In the context of these two norms, for L a continuous linear function from $\mathscr{C}^1[0,1]$ to $\mathscr{C}[0,1]$, define $||L||$, the norm of L.

(b) Let $a,b \in \mathbb{R}$ and $L(x) := ax(1) + b\dot{x}(1)$, for $x \in \mathscr{C}^1[0,1]$. Here the real number on the right is interpreted as a function that is constant.

(i) Show that L is linear and that $||L|| \leq \max\{|a|,|b|\}$.

(ii) For the case that $b = 0$, show that $||L|| = |a|$.

(c) For $S : \mathscr{C}^1[0,1] \to \mathscr{C}[0,1]$ and $T : \mathscr{C}[0,1] \to \mathbb{R}$ linear and continuous, show that

(i) the composition map TS, i.e. $T \circ S(x) := T(S(x))$, is continuous, and that

(ii) $||TS|| \leq ||T|| \cdot ||S||$.

(d) Let F be a function from $\mathscr{C}^1[0,1]$ to $\mathscr{C}[0,1]$.

(i) Define what is meant by saying that $F(x)$ is $o(||x||)$.

(ii) For L a continuous linear function from $\mathscr{C}[0,1]$ to $\mathbb{R}$, show that $L \circ F(x)$ is $o(||x||)$ if $F(x)$ is $o(||x||)$.

(e) Let F be a function from $\mathscr{C}^1[0,1]$ to $\mathscr{C}[0,1]$.

(i) Define the *Gateaux derivative $D_hF(x)$ in direction h*, and the *Fréchet derivative $DF(x)$ at x*.

(ii) Suppose now that the Fréchet derivative $DF(x)$ exists at x. Show that

$$D_hF(x) = DF(x)h, \text{ for all } h \text{ in } \mathscr{C}^1[0,1].$$

(iii) Suppose also that $G : \mathscr{C}[0,1] \to \mathbb{R}$ is Fréchet differentiable at every point in its domain. Show that $G \circ F(x)$ is differentiable at x with derivative $DG(y) \circ DF(x)$, where $y = F(x)$.

(f) A function M is defined on $\mathscr{C}^1[0,1]$ by

$$M(x) = x(1)\dot{x}(1),$$

and is regarded as mapping into $\mathscr{C}[0,1]$. Show that:

(i) $D_hM(x) = h(1)\dot{x}(1) - x(1)\dot{h}(1)$, and

(ii) if $||h||_1 \leq \varepsilon$, then

$$||M(x+h) - M(x) - D_hM(x)||_\infty \leq \varepsilon ||h||_1.$$

Conclude that the function $M(x)$ is Fréchet differentiable at each $x \in \mathscr{C}^1[0,1]$.