1. MA409: 04/05 - Assignment 1

1.1. Elementary Differential Equations: Basic Types

Note: All the equations in this section are special versions of the equation: of progressively more complicated type.

Source books: A.O's Advanced mathematical Methods. P.O'Neil Advanced Engineering Mathematics.

1. Equation type:

$$\frac{dy}{dt} = \frac{P(t)}{Q(t)},$$

i.e $F(y,t) = -P(t)$ and $G(y,t) = Q(t)$. Solved by separating the variables

$$\frac{dy}{dt} = \frac{ty}{t^2+1}.$$

2. Equation type:

$$\frac{dy}{dt} + P(t)y = Q(t),$$

i.e. $F(y,t) = P(t)y - Q(t)$ and $G(y,t) = 1$.

(a) Use an integrating factor to solve the equation:

$$\frac{dy}{dt} + \frac{y}{t(t+1)} = t.$$

(b) Use the same method to solve the easier equation:

$$\frac{dy}{dt} + y = Q(t).$$

3. Equation type:

$$\frac{dy}{dt} + P(t)y = Q(t)y^{\alpha},$$

solved by *Bernouilli's substitution*, viz. $z = y^{1-\alpha}$ reducing it to the last type. Solve the equation:

$$\frac{dy}{dt} + yt = 3y^2 t.$$

4. Equation type: homogeneous in y, t, viz. for some λ and α:

$$F(\lambda y, \lambda t) = \lambda^\alpha F(y,t), \quad G(\lambda y, \lambda t) = \lambda^\alpha G(y,t)$$

which reduces to separation of variables type. Solve the equation:

$$\frac{dy}{dt} = \frac{y^2 + t\sqrt{t^2+y^2}}{yt}.$$

5. Exact equation type:

$$\frac{\partial F(y,t)}{\partial y} = \frac{\partial G(y,t)}{\partial t}$$

which condition is necessary and sufficient for the existence of a solution in the form $H(y,t) = 0$. Differentating implicitly one obtains:

$$\frac{\partial H(y,t)}{\partial t} + \frac{\partial H(y,t)}{\partial y} \cdot \frac{dy}{dt} = 0,$$

and this is compared against the form of (1). Check that the following equation is exact and hence solve it.

$$(3t^2y^2 + y^2\cos t) + (2t^3 y + 2y\sin t)\frac{dy}{dt} = 0.$$

Recall that you must find H by solving simultaneously the equations:

$$F(y,t) = \frac{\partial H(y,t)}{\partial t}, \quad G(y,t) = \frac{\partial H(y,t)}{\partial y}$$

and this involves "partial integration" (i.e. integrating w.r.t. one variable with the other held fixed) of each equation and comparing the two results.

1.2. Further Exercises

This section's exercises need trickery.

1. Solve by putting $z = x + y + 2$

$$\frac{dy}{dt} = (x + y + 2)^2$$

2. Solve by exchanging y and t

$$\frac{dy}{dt} = \frac{y}{(y + t)}.$$

3. Solve by putting $z = dy/dt$

$$3t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0.$$

4. Noting that $D(y^2) = 2y\dot{y}$ solve

$$\frac{d^2y}{dt^2} - 2y\frac{dy}{dt} = 0.$$

5. Solve

$$\frac{dy}{dt} \cdot \frac{d^2y}{dt^2} = t.$$

6. Solve by putting $z = ax + by + c$

$$\frac{dy}{dx} = \lambda + \frac{1}{ax + by + c}.$$

7. Use the substitution $y = Cx^\gamma$ to find a solution to the equation:

$$ax^2y'' + bxy' + cy = 0.$$

Also find a particular solution in the form of a polynomial of

$$ax^2y'' + bxy' + cy = Ax + B.$$

When can you combine the results of the first step with that of the second to find the general solution of the second differential equation?

8. Show that when n is an integer *Legendre's Equation*:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

has a polynomial solution of degree n. Find the solution subject to $y(1) = 1$when nn=0,1,2,3. With the boundary point restriction as just given these *Legendre polynomials* are denoted $P_n(x)$.

Note: Special functions like these are listed in Abramowicz M.A, Stegan I.A, Handbook of mathematical functions, Dover 1972.

1.3. Linear constant coefficients differential equations

Remark. *In what follows, I say that there is interference, when the exponential function appearing on the rhs is a member of the solution space of the homogeneous equation.*

Basic case

1. Find the general solution of

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0.$$

Find a particular solution of the form $y = Ae^{3t}$of the equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}.$$

What is the general solution?

Distinct roots + interference

2. Find the general solution of

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0.$$

Find a particular solution of the form $y = Ate^{2t}$of the equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{2t}.$$

What is the general solution?

Double root + interference

3. Find the general solution of

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0.$$

Find a particular solution of the form $y = At^2e^{3t}$of the equation

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 2e^{3t}.$$

What is the general solution?

Double root + interference

4. Find the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

Find a particular solution of the form $y = At^2e^{-t}$of the equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}.$$

What is the general solution?

Double root + interference

5. Find the general solution of

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0.$$

Find a particular solution of the form $y = At^2e^{3t}$of the equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = e^{2t}.$$

What is the general solution?

Distinct roots + polynomial × exponential

6. Find the general solution of

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0.$$

Find a particular solution of the form $y = (At + B)e^{3t}$of the equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = te^{3t}.$$

What is the general solution?

Complex roots

7. Find the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0.$$

Complex roots + interference

8. Find the general solution of

$$\frac{d^2y}{dt^2} + y = 0.$$

Find a particular solution of the form $y = t(A\cos t + B\sin t)$ of the equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = \cos t.$$

What is the general solution?

2. Simultaneous Systems & Diagonalization

Note that the exercises revise specific terms from linear algebra. If you are rusty on these, you may consult A.O's Advanced Mathematical Methods,.Chapter 1-5.

1. Show that the second-order equation

$$\ddot{x} = ax + b\dot{x},$$

is equivalent to the simultaneous equation system

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x},$$

where $x_1 = x$ and $x_2 = \dot{x}$. It is implicit in the notation that $x_1 = x_1(t)$ etc.

2. For the matrix A of question 1 check that the *characteristic equation*, i.e.. $\det(A - \mu I) = 0$, is

$$\mu^2 - b\mu - a = 0.$$

That is, it is identical to the auxiliary equation of the equation $\ddot{x} = ax + b\dot{x}$.

If the *eigenvalues* are distinct and equal to μ_1 and μ_2, show that two *independent* eigenvectors of A are

$$\begin{bmatrix} 1 \\ \mu_1 \end{bmatrix}, \begin{bmatrix} 1 \\ \mu_2 \end{bmatrix}.$$

What are the slopes of the lines joining these vectors to the origin? (That is, regard the vectors as points in the plane $\mathbb{R}^2$.)

3. For μ_1 and μ_2, distinct find the *inverse* of

$$P = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}.$$

(Note that this is a 'Van-der Monde' matrix of order 2.)

Check that if μ_1 and μ_2 are the distinct eigenvalues of A, then

$$P^{-1}AP = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}.$$

Thus, the matrix P *diagonalizes* A.

4. Show that the change of variables given by $\mathbf{X} = P^{-1}\mathbf{x}$ transforms the equation system $\dot{\mathbf{x}} = A\mathbf{x}$ to the uncoupled form:

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = D\mathbf{X},$$

with solution

$$X_1 = Ke^{\mu_1 t}, \qquad X_2 = Le^{\mu_2 t},$$

where $K = X_1(0)$ and $L = X_2(0)$ specify the *initial conditions* of the system.

5. A *trajectory* of the system $\dot{\mathbf{X}} = D\mathbf{X}$ of question 4 is the locus of a point $P_t = (Ke^{\mu_1 t}, Le^{\mu_2 t})$ as t progresses from $t = 0$ to $+\infty$. Check that for certain initial conditions the trajectory is a *line*, but that in general the trajectory takes the form

$$X_1^{\mu_2} = \text{const } X_2^{\mu_1}.$$

Sketch trajectories in the cases: (i) $\mu_2 = 1, \mu_1 = 2$, (ii) $\mu_2 = 1, \mu_1 = -1$.

From a geometric perspective these two cases are canonical examples. Suggest why.

6. Use all of the previous questions to deduce that a general trajectory of the system $\dot{\mathbf{x}} = A\mathbf{x}$,

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x},$$

is given by the equation

$$(\mu_2 x_1 - x_2)^{\mu_2} = \text{const } (\mu_1 x_1 - x_2)^{\mu_1}.$$

How does this equation relate to the equations of the two linear trajectories?

Hint: Consider the situation when const $= 0$ and when const $= \infty$ (whatever that means!).

7. How should the final equation of question 6 be modified when the system of equations is of the form

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\mathbf{x} + u \cdot \mathbf{b},$$

where u is a constant. Assume that $a \neq 0$.

Hint: Consider the change of variables corresponding to a shift of origin:

$$y_1 = x_1 + u/a, \qquad y_2 = x_2.$$