Summer 2006 examination

MA409

Continuous-time Optimisation Half Unit

Suitable for all candidates

Instructions to candidates

Time allowed: 2 hours

This examination paper contains 6 questions. You may attempt as many questions as you wish, but only your best **4** questions will count towards the final mark. All questions carry equal numbers of marks.

Please write your answers in dark ink (preferably black or blue) only.

Calculators are **not** allowed in this exam.

You are supplied with: Maths Answer Booklet

1.

Derive the Euler-Lagrange equation corresponding to the problem of extremizing

$$F(x) = \int_{-1}^{1} f(x(t), \dot{x}(t),\, t)\, dt$$

subject to $x(-1) = a, \quad x(1) = b$, where a, b are constants and $x : [-1, 1] \to R$ is a continuously differentiable function. You may appeal to any lemmas, provided you state them correctly.

Use the equation to find the trajectory which minimizes

$$\int_{-1}^{1} x(t)\sqrt{1+\dot{x}^2}\, dt$$

subject to $x(-1) = x(+1) = 0$ and the isoperimetric constraint

$$\int_{-1}^{1} \sqrt{1+\dot{x}^2}\, dt = 4.$$

Hint: *You may wish to note that the solution to* $2a = \sinh a$ *is about* $a = 2.2$ *and that*

$$\int \frac{dz}{\sqrt{z^2-1}} = \cosh^{-1}(z) + \text{constant}.$$

2.

Let $\mathscr{C}[0,1]$ denote the space of real continuous functions on $[0,1]$ with supremum norm.

(a) Solve the unrestricted control problem of minimizing $F(x,u)$ over $\mathscr{C}[0,1] \times \mathscr{C}[0,1]$, where

$$F(x,u) = \int_0^1 (x^2 + u^2)dt,$$

subject to the state equation $\dot{x} = u - x$ and to $x(0) = x_0, \quad x(1) = 0$.

(b) Prove the following variant of the Fundamental Lemma. Suppose that $g(t)$ is continuous at all points of $[0,1]$ except possibly at $t = \frac{1}{2}$. Suppose further that

$$\int_0^1 g(t)\dot{h}(t)dt = 0 \qquad (*)$$

for all continuously differentiable functions $h(t)$ such that $h(0) = 0$ and $h(1) = 0$. Deduce that for some constant k it is the case that $g(t) = k$ for all $t \neq \frac{1}{2}$.

(c) Represent the state equation of (a) in the form $G(x,u) = 0$ with G defined on $\mathscr{C}[0,1] \times \mathscr{C}[0,1]$.

(d) Suppose $\lambda(t)$ represents the Lagrange Multiplier corresponding to the maximization of $F(x,u)$ subject to the constraint $G(x,u) = 0$. Put

$$f(x,u,\dot{x},t) = (x^2 + u^2) + \lambda(t)(u - x) - \dot{x}\lambda(t),$$

Why is the following functional stationary when (x,u) solves the problem of part (a)?

$$I(x,u) = \int_0^1 f(x,u,\dot{x},t)dt.$$

(e) Suppose that $\lambda(t)$ is continuous at all points of $[0,1]$ except possibly at $t = \frac{1}{2}$. Deduce that for some constant c

$$\lambda(t) = \int_0^t [2x(\theta) - \lambda(\theta)]d\theta + c,$$

for all $t \neq \frac{1}{2}$.

Hint: *If (x,u) solves the problem of part (a), you may assume (1) holds as in (b) for* $g(t) = f_{\dot{x}}(x,u,\dot{x},t) - \int_0^t f_x(x,u,\dot{x},\theta)d\theta$.

(f) Conclude that in fact $\lambda(t)$ is continuous at $t = \frac{1}{2}$ and so is differentiable in the interval $0 < t < 1$.

3.

A dynamical system has governing equation:

$$\ddot{x} = -\dot{x} + 2x + u, \qquad |u| \leq 1.$$

(i) State the Pontryagin Principle in a form suited to deriving the minimum time trajectory taking the system from a given initial state to rest at the origin (i.e. $x = 0, \quad \dot{x} = 0$).

(ii) Use the Pontryagin Principle to show that the time-optimal control is of 'bang-bang type' and that at most one switch of control takes place.

(iii) Find the singular points in the $(x, \dot{x})$ phase plane corresponding to the two constant controls $u = \pm 1$, and the linear trajectories through them. By considering the eigenvalues of the associated first-order formulation, say what shape of trajectories to expect in general.

(iv) Sketch the trajectories $u = \pm 1$ which pass through the origin of the phase plane.

(v) Sketch the switching curve and indicate how it is used to obtain optimal trajectories from controllable initial states $(x, \dot{x})$ in the phase plane.

4.

(i) Show that the Bellman equation for the problem of finding

$$V(c,T) = \max\{F_0(x) : x(0) = c \text{ and } x(t) \text{ is continuously differentiable on } [0,T]\}$$

where

$$F_s(x) = \int_s^T \left(4x^2 + \dot{x}^2\right) dt$$

is

$$0 = \max_w \{\left(4c^2 + w^2\right) + w\frac{\partial V}{\partial c} - \frac{\partial V}{\partial T}\}.$$

(ii) Check that $V(c,T) = c^2 G(T)$ where $G(T) = V(1,T)$, and hence find $V(c,T)$.

(iii) Use the Euler-Lagrange equation and the natural boundary condition at $t = T$ to verify the solution obtained in part (i).

5.

Suppose that a bank's asset value X_t at time t is modeled 'risk-neutrally' by the geometric Brownian model:

$$dX_t = X_t(r-\delta)dt + \sigma X_t dz_t, \qquad X_0 = x,$$

where δ is the dividend rate (paid by the bank to its shareholders).

Let b be some constant. As long as $b < X_t$, the bank rewards its depositors paying them at a rate $C \geq 0$, i.e. it makes payments of size $C\Delta t$ in the interval $[t, t+\Delta t]$. Define β by

$$\beta = \beta(x) = \inf\{t : X_t = b\}.$$

The state regulator audits the bank randomly. The probability is $\lambda \Delta t$ that an audit occurs in the time interval $[t, t+\Delta t]$. Conditional on such an inspection occurring at time $t < \beta$, the regulator *closes* the bank iff $X_t < a$, and distributes the value of the bank's assets X_t to the depositors. Here $a > b$ is a choice variable for the regulator. Define the closure time α to be

$$\alpha = \alpha(x) = \inf\{t : \text{audit occurs at time } t \text{ and } b \leq X_t < a\}.$$

Let $\tau = \min\{\alpha, \beta\}$. The regulator maximizes the value of the bank to its depositors, namely

$$J(x,a) = E_x\left[\int_0^\tau e^{-rt} C dt + e^{-r\tau} X_\tau \cdot 1_{\tau<\infty}\right].$$

(i) Deduce that the Bellman equation satisfied by $D(x) = \max_a J(x,a)$ takes the form:

$$\begin{aligned} \frac{1}{2}\sigma^2 x^2 D_{xx} + (r-\delta)xD_x - rD &= -C \text{ if } x > a, \\ \frac{1}{2}\sigma^2 x^2 D_{xx} + (r-\delta)xD_x - (r+\lambda)D &= -C - \lambda x \text{ if } b < x < a. \end{aligned}$$

Hint: *The continuation value is discounted by the probability of no audit occurring; the bequest term arises with the probability of audit occurring.*

(ii) Suggest why the following boundary conditions should be satisfied:

$$\lim_{x\to\infty} D(x) = \int_0^\infty e^{-rt} C dt = \frac{C}{r}, \qquad D(b) = b.$$

(iii) Solve the Bellman equations in the two intervals $[b,a]$ and $[a,\infty)$ using the above two observations and the smooth-pasting requirement that

$$D(a-) = D(a+), \qquad D_x(a-) = D_x(a+).$$

when $2C = 2r = 2\delta = \lambda = 2\sigma^2$ on the assumption that $a = 1, b = 1/2$. Comment on the shape of the function $D(x)$.

Hint: *Show the problem reduces to solving the simultaneous system (with solution:* $L = -82/387$, $M = -68/387$, $N = -14/387$ *which you do not have to check):*

$$M + N - L = 0, \qquad 9M - 6N + 3L = -2, \qquad 3M + 96N = -4.$$

6.

(a) If F is a function taking as argument continuous functions $x : [0,1] \to \mathbf{R}$ with values in $\mathscr{C}[0,1]$, define (i) the *Gateaux derivative* $D_h F(x)$ (the directional derivative) *in direction* h, and (ii) the *Fréchet derivative* (the strong derivative) $DF(x)$. Show that if the strong derivative exists it is the case that

$$D_h F(x) = DF(x)h.$$

(b) What is the *supremum norm* $||x||_\infty$ on $\mathscr{C}[0,1]$?

For the function $S : \mathscr{C}[0,1] \to \mathscr{C}[0,1]$ defined by

$$S(x)(t) = x(t)^2,$$

explain why the Gateaux derivative $D_h S(x)$ is the function y such that $y(t) = 2x(t)h(t)$.

Show further that

$$||S(x+h) - S(x) - D_h S(x)||_\infty \leq ||h||_\infty^2,$$

where the norm $||h||_\infty$ is the supremum norm on $\mathscr{C}[0,1]$. Conclude that the function $S(x)$ has a strong derivative.

(c) Let $\mathscr{C}(R)$ denote the bounded continuous functions defined on R with the supremum norm. A functional $F : \mathscr{C}(R) \times \mathscr{C}(R) \times R \to R$ is defined by

$$F(x) = \int_0^\tau (y^2 + \dot{y}^2)dt + \int_\tau^1 (z^2 + \dot{z}^2)dt,$$

where $x = (y, z, \tau) \in \mathscr{C}[0,1] \times \mathscr{C}[0,1] \times R$.

Find the Gateaux derivative $D_h F(x)$ in direction $h = (u, v, \sigma)$.