The Euler-Lagrange Equation via multi-variable calculus

We offer an alternative argument for deriving (rigorously) the Euler-Lagrange equation on the basis of multivariable calculus. You will see it to be somewhat messy, although actually it is not far removed from the variational argument already given. The beauty of the earlier argument, is actually in how it manages to hide the rubbish under the carpet!

We assume that the optimal path for the problem of maximizing

$$\int_0^1 f(x(t), x'(t), t)dt$$

subject to

$$x(0) = z_0, \qquad x(1) = z_1$$

exists and is indeed differentiable. We will refer to it henceforth as $\bar{x}(t)$. We will need to assume that $\bar{x}''(t)$ exists and is continuous (bounded willl do!) throughout the interval (a, b). We will also make various differentiability assumptions on f. This is a weakness of the proof. However, in practice the assumption is not a drawback.

The maximizer may be found by considering a sequence of 'rectifying trajectories' (one for each partition of $[a, b]$) where a typical rectified trajectory is given by taking an equi-partition $0 = t_0 < t_1 < ... < t_n = 1$ where

$$t_i = t_0 + i\Delta t,$$

and for $i = 0..., n - 1$

$$x_{i+1} = x_i + y_i \Delta t.$$

We will eventually regard the y_i as free variables. Of course, we can equivalently think of the x_i as free variables with y_i defined through this equation by the x_i.

Note in particular that we can arrange to have $x_i = \bar{x}(t_i)$ by setting

$$y_i = \frac{x_{i+1} - x_i}{\Delta t},$$

and then

$$y_i = \frac{x_{i+1} - x_i}{\Delta t} \to \bar{x}'(t_i)$$

as $t_{i+1} \to t_i$. (Here think of t_i as fixed and t_{i+1} as variable).

The objective

$$I(x) = \int_0^1 f(x(t), x'(t), t)dt$$

is replaced by the approximating sum

$$S_n(\{x_i\}) = \sum_{i=0}^{n-1} f(x_i, y_i, t_i)\Delta t,$$

which can be made as close to the integral for *any* $x(t)$ as we wish with a fine enough partition with $x_i = \bar{x}(t_i)$, provided $\bar{x}\prime(t)$ is continuous (assuming continuity of f).

Comment

Our aim is to solve an associated problem: maximization of the $S_n(\{x_i\})$, assuming the associated maximiser $X_n = \{x_1, ..., x_n\}$ exists.

BUT: For fixed n , there is no outright guarantee that the maximiser of the associated problem is given by the values $x_i = \bar{x}(t_i)$, nor do we know that a sequence of rectifying curves X_n (associated with refinements of the time partition, i.e. as $n \to \infty$) will actually converge to a curve X.

On this latter point there is a natural way to guarantee convergence. For the problem with objective S_n one needs to require some kind of **uniformity conditions** to be observed: among these the requirement that for instance y_i and y_{i+1} differ by at most, say, 2^{-n}. This assumption guarantees that the limiting curve has a first derivative!

On the former point, we do **expect** proximity of the associated maximiser X_n to the original maximiser $\bar{x}(t)$ for small enough Δt.

Why? Well, we would **expect** that passage between the two objectives in either direction will create negligible errors.

In fact all we can say for *any* function $x(t)$ is that passage between integral and approximating sum may be made as small as any arbitrary $\varepsilon > 0$.

Passage from broken-line curves to smoothed-out versions we expect to create small errors (more precisely: small relative to ε) provided smoothing does not permit much variation in the ordinates and in the slopes; and this we have to include under the already required terms of the uniformity conditions.

Likewise we expect to create 'relatively small' errors passsing from smooth curves to their rectifications, and this last point is fine if for example $\bar{x}''(t)$ is assumed continuous (provided Δt is taken small enough). This latter intuition can be justified by reference to an assumed boundedness on the partials f_x, f_y and f_t . Indeed, say a bound is M, then we can justify the following inequality (by invoking the Mean

Value Theorem of Integration)

$$
\begin{aligned}
|I(x) - S(\{x_i\})| &= |\sum_{i=0}^{n-1} [f(x_i, y_i, t_i)\Delta t - \int_{t_{i-1}}^{t_i} f(x(t), x'(t), t)dt]| \\
&= |\sum_{i=0}^{n-1} [f(x_i, y_i, t_i)\Delta t - f(x(t_i^*), x'(t_i^*), t_i^*)\Delta t]| \\
&\leq \sum_{i=0}^{n-1} \{f_x \cdot (x_i - x(t_i^*)) + f_y \cdot (y_i - x'(t_i^*)) + f_t \cdot (t_i - t_i^*)\}\Delta t \\
&\leq M\Delta t \cdot \sum_{i=0}^{n-1} \{(x_i - x(t_i^*)) + (y_i - x'(t_i^*)) + (t_i - t_i^*)\}.
\end{aligned}
$$

The last term can be made small under the rectification process provided $x''(t)$ is assumed continuous.

Solution of the associated problem

We will regard the collection of variables y_i as free variables but will eventually regard one of the positions t_i as fixed.

We aim to maximize

$$\sum_{i=0}^{n-1} f(x_{i-1} + y_{i-1}\Delta t, y_i, t_i)\Delta t$$

subject to the initial constraint that (i)

$$x_0 = z_0,$$

and (ii) the terminal endpoint constraint, which now reads

$$z_1 - z_0 = x_n - x_0 = \sum_{i=1}^{n}(x_i - x_{i-1}) = \Delta t(y_0 + ... + y_{n-1}).$$

The Lagrangian of the problem needs to be written in a specific form for the purposes that we will identify in a moment.

$$\begin{aligned} L(y_0, .., y_{n-1}, \lambda) &= f(z_0, y_0, t_0)\Delta t + \sum_{i=1}^{n-1} f(x_{i-1} + y_{i-1}\Delta t, y_i, t_i)\Delta t \\ &\quad -\lambda[\Delta t\,(y_0 + ..y_{n-1}) - (z_1 - z_0)]. \end{aligned}$$

Indeed (i) has already been incorporated in the first term here, and the penalty term incorporates (ii). Note that explicitly we have

$$f(z_0, y_0, t_0)\Delta t + f(z_0 + y_0\Delta t, y_1, t_1)\Delta t + f(x_1 + y_1\Delta t, y_2, t_2)\Delta t + ...$$

and this identifies where the choices of y_i are made (contingent on choices of variables earlier in the list).

The FOC in y_{i-1} for $i = 1...n$ is thus seen to be

$$f_y(x_{i-1}, y_{i-1}, t_{i-1})\Delta t + f_x(x_{i-1} + y_{i-1}\Delta t, y_i, t_i)(\Delta t)^2 = \lambda\Delta t,$$

or, simply:

$$f_y(x_{i-1}, y_{i-1}, t_{i-1}) + f_x(x_i, y_i, t_i)\Delta t = \lambda.$$

Eliminating λ between any two successive equations yields

$$f_y(x_{i-1}, y_{i-1}, t_{i-1}) + f_x(x_i, y_i, t_i)\Delta t = f_y(x_i, y_i, t_i) + f_x(x_{i+1}, y_{i+1}, t_{i+1})\Delta t.$$

To get identical second terms on each side, assuming continuity of f and closeness of x_i and x_{i+1} as well as of y_i and y_{i+1} we write

$$f_y(x_{i-1}, y_{i-1}, t_{i-1}) + f_x(x_i, y_i, t_i)\Delta t = f_y(x_i, y_i, t_i) + f_x(x_i, y_i, t_i)\Delta t + \varepsilon\Delta t,$$

where ε may be made as small as we please by taking Δt small enough.

Collecting like terms and dividing by Δt we obtain

$$\frac{f_y(x_i, y_i, t_i) - f_y(x_{i-1}, y_{i-1}, t_{i-1})}{\Delta t} = f_x(x_{i-1}, y_{i-1}, t_{i-1}) - \varepsilon,$$

or

$$\frac{f_y(x + \Delta x, y + \Delta y, t + \Delta t) - f_y(x, y, t)}{\Delta t} = f_x(x, y, t) - \varepsilon.$$

This is almost what we need to get the E-L equation (though not exactly). So let's be careful!

Let us assume that the limiting curve made out of the rectifying curves does in fact yield $\bar{x}(t)$.

Fix the location $t = t_{i-1}$ and the values $x_{i-1} = \bar{x}(t_{i-1})$ and $x_i = \bar{x}(t_i)$, where $\bar{x}(t)$ is the optimal curve of the original problem. Select y_i corresponding to the rectification of the originating problem. Now note that by the Mean Value Theorem

$$y_{i-1} = \frac{x_i - x_{i-1}}{\Delta t} = \bar{x}'(t'_{i-1})$$

for some t'_{i-1} between t_{i-1} and t_t. In fact the assumption of two-fold differentiability of $\bar{x}(t)$ enables us to write the Taylor Expansion as

$$\bar{x}(t_{i-1} + \Delta t) = \bar{x}(t_{i-1}) + \bar{x}'(t_{i-1})\Delta t + \bar{x}''(\hat{t}_{i-1})(\Delta t)^2$$

for some point $\hat{t}_{i-1}$. Thus

$$\frac{x_i - x_{i-1}}{\Delta t} = \bar{x}'(t_{i-1}) + \varepsilon' \Delta t,$$

where ε' may be made arbitrarily small by taking Δt small enough.

The assumption that the optimal trajectory is twice differentiable thus yields

$$\Delta y_i = y_i - y_{i-1} = \frac{x_{i+1} - x_i}{\Delta t} - \frac{x_i - x_{i-1}}{\Delta t} = \bar{x}'(t_{i-1} + \Delta t) - \bar{x}'(t_{i-1}) + \varepsilon'' \Delta t$$

where ε'' may be made arbitrarily small by taking Δt small enough. In consequence, assuming f_{yy} is continuous we obtain

$$f_y(., y + \Delta y, .) = f_y(., \bar{x}'(t_i), .) + \varepsilon'' f_{yy}(...)\Delta t,$$

where $\varepsilon'' f_{yy}(...)$ may be made arbitrarily small by taking Δt small enough.

Thus recalling that $y = \bar{x}'(t)$ we have

$$\frac{f_y(x + \Delta x, \bar{x}'(t + \Delta t), t + \Delta t) - f_y(x, y, t)}{\Delta t} = f_x(x, y, t) - \varepsilon - \varepsilon'' f_{yy}.$$

We may thus validly pass to the limit as $\Delta t \to 0$ to obtain on the optimal path that

$$\frac{d}{dt}\{f_y(x(t), x'(t), t)\} = f_x(x(t), x'(t), t).$$

Closing comments

The argument above assumes enough continuity that various approximations may validly be made. Many of the same assumptions quietly exist in the background when we take differentiation ‘through the integral sign’ in the ‘variational proof’ in the earlier Lectures. The assumption of uniformity conditions on the ‘broken lines’ resurfaces when we place a proper norm structure on the domain of our continuous-time problem (namely the space of ‘admissible functions’).

Please note that most textbooks avoid this historic perspective, or just ride roughshod over it, e.g. Gelfand & Fomin, *Calculus of variations*, Dover 1991, page 28, as also in the text cited in my Reading List: L.Elsgolts, *Differential Equations and the Calculus of Variations,* page 411.

A careful analysis was given by L.C. Young in the 1930's. A modern reworking of Young in the language of 'infinitessimal calculus', afforded through modern mathematical logic, is given by Curtis Tuckey in:

Non-standard methods in the Calculus of Variations, Longman Scientific & Technical, Pitman Research Notes in Mathematics Series, Vol 297.