

## Notes: Smooth-pasting and the Envelope theorem.

Let  $W(x)$  be some given function, differentiable in an interval  $I$ , say with  $I = (a, b)$  for  $a < b$ , where the endpoints may be finite or infinite (i.e. we permit  $a = -\infty$  and/or  $b = +\infty$ ). The function  $W(x)$  may perhaps be a specific solution to some differential equation, or just some naturally occurring (valuation) function, like an abandonment valuation, or the value at exercise of a put option. A more general circumstance (associated with an optimally determined regime switch) envisages  $W(x)$  as selected (optimally, of course) from a one-parameter family of value functions  $w(x, c)$  for some choice of  $c$ .

Next suppose given a one-parameter family of functions  $V(x, c)$  that is sufficiently smooth, i.e. satisfies conditions including (C1) and (TD) below (*in addition* to the usual requirements on  $V_x$  and  $V_{xx}$  for a HJB equation), defined for  $x > a$  and solving a HJB equation in  $x$ . Suppose, more specifically, that  $V(x, c)$  corresponds to a time-homogeneous value maximization problem to one side of  $c$ , for definiteness say on the right, i.e. for  $x \geq c$ , with  $c$  a parameter value in  $I$ , with asymptotic behaviour specified as  $x \rightarrow +\infty$  (consistently with the right-sided formulation just adopted), and subject to the value matching condition:

$$V(x, c) = W(x) \text{ at } x = c.$$

In particular, the functions  $V(x, c)$  satisfy

$$V(c, c) = W(c).$$

Given the second-order nature of the HJB equation and the two boundary conditions, we are indeed justified in interpreting the family  $V(x, c)$  as parametrized by the free choice of the location  $x = c$  for ‘contact’ between  $V(x, c)$  and  $W(x)$ , subject to:

$$a < c < b,$$

and possibly also allowing any finite end-points.

Assume that the optimal choice  $c = \hat{c}$  is internal, i.e.  $a < \hat{c} < b$ . (N.B. Optimality of finite end-point locations are not ruled out theoretically.)

For the optimal choice of  $c = \hat{c}$ , the value  $V(x, \hat{c} + h)$ , at any  $x > \hat{c}$ , should be maximized by the choice  $h = 0$ . So assume that

$$V_c(x, c) \text{ is continuous at } (x, \hat{c}). \tag{C1}$$

Taking limits, as  $x \rightarrow \hat{c}+$ , we obtain the first-order condition

$$V_c(\hat{c}, \hat{c}) = 0.$$

Next, assume that the map

$$c \rightarrow V(c, c) \text{ is differentiable for } a < c < b. \quad (\text{TD})$$

Since  $W(c) = V(c, c)$  for  $c$  in  $(a, b)$  we obtain, from the chain rule and the first-order condition above,

$$W'(\hat{c}) = V_x(\hat{c}, \hat{c}) + V_c(\hat{c}, \hat{c}) = V_x(\hat{c}, \hat{c}). \quad (\text{Env})$$

We thus arrive at the **smooth-contact condition** (asserting continuity of slope):

$$W'(x) = V_x(x, c) \text{ at } (x, c) = (\hat{c}, \hat{c}). \quad (\text{SP})$$

**Remarks.**

1. The term **smooth-pasting** is used for the condition (SP) in the context where a valuation  $U(x)$  is obtained by cases thus:

$$U(x) = \begin{cases} V(c, x), & x \geq c, \\ W(x), & x \leq c, \end{cases}$$

subject to  $V(c, c) = W(c)$ . Thus  $U(x)$  is said to be obtained by *pasting* the two functions  $V(x, c)$  and  $W(x)$  together at  $x = c$  *smoothly*.

2. The conclusion in the equation (Env) here recalls the Envelope Theorem of microeconomics. There one is concerned with the problem of studying the maximized value

$$M(p) = \max_y V(y, p),$$

e.g. the ‘indirect profit’ arising when for instance choosing output  $y$  when facing input price  $p$  and maximizing profits, so that  $V(y, p) = f(y) - py$ ; here  $f$  is strictly concave. Assume there is a maximizer function  $y^* = y^*(p)$  for  $p$ . (Thus, in the example,  $y^*(p)$  is related to the so-called Fenchel dual of  $f$ .) Now  $M(p) = V(y^*(p), p)$  traces dependence of ‘indirect profit’ on price levels, and the Envelope Theorem asserts that

$$\begin{aligned} M'(p) &= V_y(y^*(p), p) \frac{dy^*}{dp} + V_p(y^*(p), p) \\ &= V_p(y^*(p), p), \end{aligned}$$

again by the chain rule, of course, and because  $V_y(y^*(p), p) = 0$ . Here, however,  $y$  is the instrument (choice) variable corresponding to our instrument variable  $c$ , the contact location  $c$ .

3. Two classic examples which we study are: Dixit and Pindyck's *machine abandonment model* and the Black-Scholes model of a *perpetual put*.

4. If  $W(x)$  is replaced by  $W(x, c)$  corresponding to a maximization problem for  $x < c$ , we deduce that under similar assumptions (mutatis mutandis) on  $W$  that

$$W_x(x, c) = V_x(x, c) \text{ at } (x, c) = (\hat{c}, \hat{c}).$$

This is used in cases like Bhattacharya et al. who model bank audits (two regimes occur), for which see: *Bank Capital Regulation with Random Audits*, Journal of Economic Dynamics and Control, Volume 26.7 (2002), 1301-1321.

5. An even more sophisticated example of smooth pasting appears in relation to 'dividend creaming' with a multiple regime switching production company examined by R. Radner and L.A. Shepp in: *Risk vs. Profit Potential: A Model for Corporate Strategy*, Journal. of Economic Dynamics and Control, 20 (1996), 1373-1393.

6. For yet another example of smooth pasting with two solutions taken from two HJB equations, see A. J. Ostaszewski, '*Equity smirks' and embedded options: the shape of a firm's value function*, Accounting & Business Research, 2004 (34 No. 4), 301-331 (and compare Footnote 3 page 11 on smooth-pasting). This latter example is in the literature inspired by the Burgstahler-Dichev empirical study identifying a measure of convexity in the earnings to equity relationship.