1. Control under uncertainty: finite horizon

We turn to the dynamical programming approach to deal with a choice of control under uncertainty. Recall that in earlier lectures the change in the state of the system after the lapse of time Δt was of the form $\Delta x = f_1(x,u)\Delta t + o(\Delta t)$ where $o(\Delta t)/\Delta t \to o$ as $\Delta t \to 0$. We now assume that

$$\Delta x = a(x, u(t))\Delta t + b(x, u(t))\Delta z_t,$$

where $\Delta z_t = z(t+\Delta t) - z(t)$ is an increment in a standard Wiener process $z(t)$. (See Section ??)

The term a is said to measure *drift*, while the term b is known as the *volatility*. For simplicity we assume these to be continuous functions. To avoid ambiguity of notation we will write $X(t)$ for the state at time t which state is now a random variable, reserving the letter x for a known state. Thus we will write

$$\Delta X = a(X(t), u(t), t)\Delta t + b(X(t), u(t), t)\Delta z.$$

We are concerned again with the minimization or maximization of an expected value

$$C(x,t) = E_{t,x}[\int_t^T f_0(X(s), u(s))ds],$$

where the expectation symbol $E_{t,x}$ denotes expectations conditional on knowing that the realization of $X(t)$ is x and the process followed by X is as above.

Under appropriate smoothness assumptions, Itô's Formula (see section ??) now allows us to assert that:

$$E_{t,x}[C(x+\Delta X, t+dt)] = C(x,t) + \left(\frac{\partial C}{\partial t} + a\frac{\partial C}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 C}{\partial x^2}\right)\cdot \Delta t.$$

It will be helpful to review the very small amount of stochastic calculus which we need for the following analysis. Speaking crudely, we are concerned with increments ΔX in a variable $X(t)$, where $\Delta X = a\Delta t + b\Delta z_t$, in which the time increment and the coefficient functions $a(\cdot,\cdot), b(\cdot,\cdot)$ are deterministic (though with non-deterministic inputs!) and the stochastic increment Δz_t behaves so that when we apply the expectations operator $E = E_t$ (i.e. expectation given information available at time t) we may proceed intuitively and make the substitutions $E[\Delta z] = 0$ and $E[(\Delta z)^2] = \Delta t$ (as the lapse of time is Δt).

Consider a possible value of ΔX namely Δx. Then provided enough smoothness is assumed of F we have

$$F(x+\Delta x) = F(x) + F'(x)\Delta x + \frac{1}{2}F''(x)(\Delta x)^2 + o((\Delta x)^3)$$

where the last term is $o((\Delta x)^3)$, i.e. has the property that $o((\Delta x)^3)/(\Delta x)^3 \to 0$ as $\Delta x \to 0$.Now ΔX must be regarded as a random variable which may take various possible values Δx. So we may substitute for ΔX in the Taylor expansion:

$$F(x+\Delta X) = F(x) + F'(x)\Delta X + \frac{1}{2}F''(x)(\Delta X)^2 + error$$

provided we refer to a norm in the vector space to which ΔX belongs and regard the last term *error* to be $o((\Delta X)^3)$ in the sense that $||error||/(||\Delta X||)^3 \to 0$ as $||\Delta X|| \to 0$.

Finally, we obtain that conditional on the realization of $X(t)$ being known at time t to be x, then

$$\begin{aligned} & F(x+\Delta X) \\ = \; & F(x) + F'(x)\{a\Delta t + b\Delta z\} + 1/2F''(x)\{a\Delta t + b\Delta z\}^2 + o((\Delta X)^3) \\ = \; & F(x) + aF'(x)\Delta t + \frac{1}{2}b^2 (\Delta z)^2 F''(x) + b\Delta z F'(x) \\ & + ab\Delta t\Delta z + \frac{1}{2}a^2 F''(x) (\Delta t)^2 + o((\Delta X)^3). \end{aligned}$$

Applying the expectations operator and the substitutions mentioned above we obtain the expansion:

$$E_{t,x}[F(x+\Delta X)] = F(x) + \Delta t\{aF'(x) + \frac{1}{2}b^2 F''(x)\} + o((\Delta t)^2),$$

where now the final error term has the property that $o((\Delta t)^2)/o((\Delta t)^2) \to 0$ as $\Delta t \to 0$. We customarily drop this error term. In what follows below we assume tacitly all the necessary smoothness.

We are now able to argue from the Optimality Principle that for any $\Delta t > 0$:

$$C(x,t) = \max_u E_{t,x}[\int_t^{t+\Delta t} f_0(X(s),u(s))ds + C(x+\Delta X, t+\Delta t)],$$

where $u = u(s)$ is a control function defined for $t \le s \le t + \Delta t$ and ΔX is defined from $u(t)$ as at the beginning of the section.

Evidently, invoking the assumed continuity of the process X we conclude (as usual) that

$$E_{t,x}[\int_t^{t+\Delta t} f_0(X(s),u)ds] = f_0(x,u)\Delta t + o(\Delta t).$$

Now using the law of iterated conditional expectation we observe that the expression being maximised is equal to:

$$f_0(x,u)\Delta t + E_{t,x}E_{t+\Delta t, X(t+\Delta t)}[C(x+\Delta X, t+\Delta t)] + o(\Delta t).$$

And now we invoke Itô's Formula (in a suitable form to include time t) to obtain:

$$\begin{aligned} & E_{t+\Delta t, X(t+\Delta t)}[C(x+\Delta X, t+\Delta t)] \\ = \; & C(x,t) + \left(\frac{\partial C}{\partial t} + a\frac{\partial C}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 C}{\partial x^2}\right) \cdot \Delta t, \end{aligned}$$

and so

$$C(x,t) = \min_u\{f_0(x,u)\Delta t + E_{t,x}[C(x,t) + \left(\frac{\partial C}{\partial t} + a\frac{\partial C}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 C}{\partial x^2}\right) \cdot \Delta t]$$

But $E_{t,x}[C(x,t)] = C(x,t)$, so we are thus led to the Bellman Equation:

$$0 = \min_u[f_0(x,u) + \frac{\partial C}{\partial t} + a\frac{\partial C}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 C}{\partial x^2}].$$

2. Bellman Equation with an Infinite Horizon

An interesting simplification occurs with a time-discounted performance index such as

$$J(u,x,s) = E_x[\int_s^\infty e^{-\rho t} f_0(X_t, u(t))dt] \text{ where } X_s = x$$

and time does not enter explicitly other than through the discount factor.

Acting on the intuition of time-homogeneity, one argues that since the time to termination is infinite, the optimization problem faced at any moment later in time is identical except for the value being the discounting of value through the passage of time. For this intuition to be valid the coefficients α, β in the stochastic differential equation

$$dX_t = \alpha dt + \beta dz_t$$

must in turn not depend on time explicitly. Working in current-value terms rather than in terms of present-values one thus expects that.

$$e^{\rho t} V(x,t) = V(x,0).$$

Evidently if

$$V(x,t) = e^{-\rho t} V(x,0)$$

then

$$\frac{\partial}{\partial t} V(x,t) = \frac{\partial}{\partial t}\left(e^{-\rho t} V(x,0)\right) = -\rho e^{-\rho t} V(x,0) = -\rho V(x,t),$$

so that the Bellman Equation simplifies to

$$0 = \min_u [e^{-\rho t} f_0 - \rho V + \alpha \frac{\partial V}{\partial x} + \frac{1}{2}\beta^2 \frac{\partial^2 V}{\partial x^2}]$$

for all t.Now $V(x,t) = e^{-\rho t} V(x,0)$ so putting

$$v(x) = V(x,0)$$

the Bellman equation reduces to the ordinary differential equation

$$0 = f_0 - rv + \alpha v' + \frac{1}{2}\beta^2 v''.$$

2.1. Note on Feed-Back Control

Note that in the time homogenous situation in which $C(x,t) = e^{-\rho t} C(x,0)$ the Bellman equation reads

$$\rho C(x,0) = \min_u [f_0(x,u) + a\frac{\partial C(x,0)}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 C(x,0)}{\partial x^2}].$$

In this case the minimizing choice for $u = \hat{u}$ is in general some function of the state x and so of the form

$$\hat{u} = U(x)$$

for some function U. As a result we have for each time t

$$\hat{u}(t) = U(\hat{x}(t)).$$

That is, the optimal control is obtained by applying the same function U at each moment in time to the current state.

2.2. Example 1

Find the optimal control for

$$C(x,t)=\min_u E[\int_t^\infty e^{-\rho t}(aX(s)^2+bu(s)^2)ds$$

with

$$dX_s=uds+\sigma X dz_s,\ \text{and}\ X(t)=x,$$

where a,b,σ are positive constants. We have

$$\rho C(x,0)=\min_u[ax^2+bu^2+u\frac{\partial C(x,0)}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 C(x,0)}{\partial x^2}].$$

Minimization over u gives

$$u=-C_x/2b$$

and so

$$\rho C=ax^2-\frac{1}{4b}{C_x}^2+\frac{1}{2}\sigma^2x^2C_{xx}.$$

We try a solution of the form $C=Ax^\gamma$ and obtain

$$\rho Ax^\gamma=ax^2-\frac{1}{4b}A^2\gamma^2x^{2\gamma-2}+\frac{1}{2}\sigma^2x^\gamma A\gamma(\gamma-1).$$

This will be an identity if $\gamma=2$ and A satisfies

$$\rho A=a-\frac{1}{b}A^2+\sigma^2A,$$

i.e.

$$A^2+(\rho-\sigma^2)bA-ab=0.$$

There are two roots $A_\pm$, one positive, one negative. Thus since

$$u=-Ax/b$$

and we seek a minimum, we must take $A=A_+$ in order to reduce $X(t+\Delta t)$. Thus

$$u(t)=-A_+x(t)/b.$$

Note that $C(x,t)=e^{-\rho t}C(x,0)=e^{-\rho t}A_+x^2$.

In our next two examples we need to introduce a stopping time τ defined by reference to the underlying stochastic process X_t reaching a prescribed level u to be chosen optimally. We let

$$\tau_u=\inf\{t\geq 0: X_t=u\}$$

where we suppose that $X_0\neq u$.It is important to note that this definition is such that the event $\tau_u=t$ does not make use of any information that is unavailable at time t.

3. Examples requiring optimal timing: Machine Abandonment and the Perpetual Put

Our next examples take the argument presented so far a little further. The performance index can be more general. For example the terminal time can be made to depend on the first time τ when the stochastic process reaches a pre-selected level:

$$\max_{u,\tau} E_x[\int_0^\tau e^{-\rho t} f_0(X_t, u)dt].$$

This gives rise to what is called a 'stopping time' problem. It is possible also to include a terminal time contribution (referred to as 'bequest' term):

$$\max_{u,\tau} E_x[\int_0^\tau e^{-\rho t} f_0(X_t, u)dt + e^{-\rho\tau} g(X_\tau) \cdot 1_{\tau<\infty}].$$

Note the presence of an indicator function which removes the said contribution if the stopping time is infinite. It is of course possible that $f_0 = 0$ identically as in the case of the put option of Example ?

A more complicated example of zero running costs is provided by permitting controls to be applied only at discrete moments. We offer an example of this by considering impulse control of the exchange rate or some transform of it, which we might term the fundamental variable. In our example the controlling bank intervenes whenever the fundamental variable, modelled by a drifting Brownian process hits a lower boundary $a$say or an upper boundary $b$say. Thus the intervention times are defined inductively by

$$\tau_{i+1} = \inf\{t > \tau_i : \text{ either } X_{t-} = a \text{ or } X_{t-} = b\},$$

where $\tau_0 = 0$ and X_{t-} denotes the limit from below the time t, that is $\lim_{s\nearrow t} X_s$. The intervention at time τ resets (shifts) the fundamental either up to a new position say from $X_{\tau-} = a$ to $X_\tau = \alpha$ or down to a new position β from b with

$$a < \alpha < \beta < b$$

at a cost

$$\gamma + c|X_\tau - X_{\tau-}|$$

which comprises a fixed intervention cost γ and a variable cost proportional to the absolute amount of the shift. The performance index is thus

$$\min_{\alpha,\beta} E_{x,0}[\sum_i e^{-\rho\tau_i} f_0(X_{\tau_i -}, \alpha, \beta)]$$

a sum of bequest terms.

3.1. First of the two examples

(Dixit & Pindyck. page 110) We take the state X to be the profit flow and assume that

$$dX_t = -adt + bdz_t,$$

where a, b are constants with $a > 0$ and $b > 0$, thus representing a tendency for the profit flow to decrease with time. Rather than use numerical tools needed to solve the Bellman partial differential equation, we assume that the horizon is infinite. Thus we seek the optimal control u^* which maximizes the expected profit flow given a lay-off level u; in so doing we find that

$$V(x,0) = \max_u E_x[\int_0^\tau e^{-\rho t} X_t dt],$$

where

$$\tau = \tau(x) = \inf\{t \geq 0 : X_t = u\}$$

and we assume that the process starts at $x > u$. (If the profit flow is too low, the flow is cut-off!)

Bellman's Equation then states that for $x > u^*$ (when u^* is optimally selected) the value function $V(x,0)$ satisfies:

$$\rho V(x,0) = x - aV'(x,0) + \frac{1}{2}b^2 V''(x,0),$$

and that $V(u^*, 0) = 0$.

Let us compare this to the derivation of the maximum height problem: there the cut-off is the optimal height H. Therefore one considers state x such that $x < H$ (note the reversal in that example).

Returning to our machine abandonment problem, for Δt small enough we have $X(\Delta t) > u$ (since $X(0) > u$) and we may argue that if u has been selected optimally then

$$V(x) = E_x[x\Delta t + \int_{\Delta t}^{\tau(x)} e^{-\rho t} X_t dt].$$

Setting

$$s = t - \Delta t, Y(s) = X(s + \Delta t)$$

and noting that

$$\tau(X(\Delta t)) + \Delta t = \tau(x)$$

and also that

$$dY_s = -ads + bdz_s,$$

we have

$$\begin{aligned} V(x) &= E_x[x\Delta t + \int_0^{\tau(x)-\Delta t} X(s+\Delta t)e^{-\rho[s+\Delta t]}ds + error] \\ &= x\Delta t + E_x E_{\Delta t, X(\Delta t)}[e^{-\rho\Delta t}\int_0^{\tau(X(\Delta t))} Y(s)e^{-\rho s}ds + error] \\ &= x\Delta t + E_x(1-\rho\Delta t)E_{\Delta t, X(\Delta t)}[\int_0^{\tau(X(\Delta t))} Y(s)e^{-\rho s}ds] + error \\ &= x\Delta t + E_x[(1-\rho\Delta t)[V(X(\Delta t)) + error] \\ &= x\Delta t + E_x[(1-\rho\Delta t)[V(x) - aV'(x) + \frac{1}{2}b^2V''(x)]\Delta t + V'(x)b\Delta z_t + error] \\ &= x\Delta t + (1-\rho\Delta t)[V(x) - aV'(x) + \frac{1}{2}b^2V''(x)]\Delta t + error] \\ &= x\Delta t + [V(x) - aV'(x) + \frac{1}{2}b^2V''(x)]\Delta t - \rho\Delta t[V(x) + ..] + error], \end{aligned}$$

where the error is $o(\Delta t)$. Cancelling the common term we obtain

$$0 = x\Delta t + [-aV'(x) + \frac{1}{2}b^2V''(x)]\Delta t - r\Delta t[V(x) + ..] + error].$$

Dividing by Δt and passing to the limit we obtain the equation

$$0 = x - aV'(x) + \frac{1}{2}b^2V''(x)] - rV(x).$$

Remark. In general consider the following performance index, denominated at time s in **present-value terms** (values as of time $t = 0$),namely:

$$V(y,s) = \max_u E_{s,y}[\int_s^\tau e^{-\rho t} f_0(X_t, u)dt].$$

Here $X_s = y$ and $X_0 = x$,and

$$\tau = \tau_s(y) = \inf\{t > s : X_t = u\}$$

and it is assumed that $s < \tau(y)$.i.e. $y \neq u$.

Put $\theta = t - s$ so that $t = s$ corresponds to $\theta = 0$, let $Y_\theta = X_{\theta+s}$. Then $dY_\theta = -ad\theta + bdw_\theta$, where $w_\theta = z_{\theta+s}$. Conditional on $Y_0 = X_s = y$ we have

$$\bar{\tau} = \tau(y) = \inf\{\theta \geq 0 : Y_\theta = u\} = \tau - s,$$

and so we have

$$\begin{aligned} V(y,s) &= \max_u E_x[\int_s^\tau e^{-\rho t} f_0(X(t), u)dt] = \max_u E_x[\int_0^{\tau - s} e^{-\rho(\theta+s)} f_0(X(\theta+s), u)d\theta] \\ &= e^{-\rho s}\max_u E_x[\int_0^{\bar{\tau}} e^{-\rho\theta} f_0(Y(\theta), u)d\theta] = e^{-\rho s}V(y,0). \end{aligned}$$

The conclusion is that

$$V(y,s) = e^{-\rho s} V(y,0),$$

when V is denominated in present-value terms.

Thus the problem is time-homogeneous, that is the value function is the same despite the passage of a time s, but only in the sense that the performance index is adjusted for discounting relative to the passage of a time interval s.

If the value function $U(y,s)$ is defined by reference to current-values (i.e. values as of time s), then evidently:

$$\begin{aligned} U(y,s) &= \max_u E_x[\int_s^\tau e^{-\rho(t-s)} f_0(X_t,u)dt] \\ &= e^{\rho s} V(y,s). \end{aligned}$$

(Loosely speaking: "current values are up-graded relative to present-values, as the latter are lower".)

In particular, setting $s = 0$ we obtain:

$$U(x,0) = e^0 V(x,0).$$

(the unique moment where current and present time coincides). But then the earlier derived equation

$$V(y,s) = e^{-\rho s} V(y,0),$$

may be re-stated as

$$e^{-\rho s} U(y,s) = e^{-\rho s} U(y,0).$$

That is,

$$U(x,s) = U(x,0).$$

Thus when denominated in current value terms the value function is independent of time. It is this last statement which justifies the assertion that the problem is 'time-homogeneous'.

It is worth pointing out, for clarification's sake, that if a function $V(x,t)$ satisfies, for all λ(resp. all $\lambda > 0$) the relation

$$V(\lambda x, t) = \lambda^d V(x,t),$$

where d is a constant, then the function is said to be 'homogenous of degree d' in x (resp. positively homogeneous).

Returning to our problem, write $v = V(x,0)$, we are to solve

$$\frac{1}{2} b^2 v'' - a v' - \rho v = -x.$$

Step 1: General Form of the solution.

Consider first the complementary equation

$$\frac{1}{2} b^2 v'' - a v' - \rho v = 0.$$

Assume a complementary function of the form $v(x) = e^{\gamma x}$, then the auxiliary equation for γ is

$$\frac{1}{2}b^2\gamma^2 - a\gamma - \rho = 0,$$

with roots $\gamma = \gamma_\pm = (a \pm \sqrt{a^2 + 2b^2\rho})/b^2$, one positive and one negative (since the product of the roots is $-2\rho/b^2$). A particular solution is $v_0 = Ax + B$ with A, B constants provided

$$-Aa - \rho(Ax + B) = -x.$$

Comparing coefficients of x on both sides we have $A = 1/\rho$ and so $B = -Aa/\rho = -a/\rho^2$. So the general solution is

$$v(x) = Ke^{x\gamma_+} + Le^{x\gamma_-} + \frac{x}{\rho} - \frac{a}{\rho^2}.$$

Step2: Boundary behaviour of the value function

Evidently at the optimal layoff level $x = u$ we have $V(u) = 0$. (If the value were positive the lay-off may be shifted to the left; if the value were negative the continued activity at $x = u$ is loss-making and not warranted).

Next we examine the behaviour of the value function in its dependence on the starting state.

Now $X(t) = x - at + bz(t)$.

In regard to the first two terms we may easily compute that

$$\int_0^\infty e^{-\rho t}(x - at)dt = \frac{x}{\rho} - \frac{a}{\rho^2}.$$

For a fixed time T we can also compute that

$$E[\int_0^T z_t d(e^{-\rho t})] = e^{-\rho T}E[z_T] - E[\int_0^T e^{-\rho t}dz_t] = 0.$$

This suggests that for large x we should have $v(x)$ approximately x/π.

To see the intuition for this last integral calculation note the identity that

$$\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u\Delta v.$$

Now take $u = e^{-\rho t}$ and $v = z_t$. Thus $\Delta u = -\rho e^{\rho t}\Delta t$ and $\Delta v = \Delta z$. Evidently in this example $\Delta u\Delta v$ is small. Finally note that

$$\int_0^T d(uv) = (uv)|_{t=T} - (uv)|_{t=0}.$$

Thus

$$e^{-\rho T}z_T - 1 \cdot z_0 = \int_0^T d(uv) = \int_0^T e^{-\rho t}dz_t + \int_0^T z_t d(e^{-\rho t})$$

or

$$e^{-\rho T}z_T = \int_0^T e^{-\rho t}dz_t - \rho\int_0^T z_t e^{-\rho t}dt.$$

Now that we have a feeling for what is going on, let us consider the expected value of

$$\begin{aligned}\int_0^\tau e^{-\rho t} X_t dt &= \int_0^\tau e^{-\rho t}(x - at + bz(t))dt \\ &= \int_0^\tau e^{-\rho t}(x - at)dt + b\int_0^\tau e^{-\rho t} z(t)dt \\ &= \frac{x}{\rho} - \frac{a}{\rho^2} - \frac{b}{\rho}\left[e^{-\rho\tau} z(\tau) - \int_0^\tau e^{-\rho t} dz_t\right].\end{aligned}$$

It is intuitively clear that the final integral has zero expectation (again as a limiting sum of weighted increments all with zero expectation). However, this intuition needs some technical support: since τ can in principle take arbitrary values (meaning that it is not a bounded random variable) we need to know whether the integral $I_T = \int_0^T e^{-\rho t} dz_t$ converges as $T \to \infty$. We can see that this is the case by computing for $n = 0, 1, 2, ...$that

$$\int_n^{n+1} e^{-\rho t} dz_t = e^{-\rho n}\int_0^1 e^{-\rho\theta} dz_{\theta+n} = e^{-\rho n} Y_n, \text{ say,}$$

where the random variables Y_n are i.i.d. : independent and identically distributed; they all have expected value zero, and finite variance, say s^2. Their arbitrary finite sums therefore have zero expectation and have variance (additive from the independence) always bounded by

$$s^2 \sum_{n=0}^{\infty} e^{-2\rho n} = \frac{s^2}{1 - e^{-2\rho}}.$$

(Actually, I_t is a martingale, and we are proving an instance of the Optional Stopping Theorem, which requires the presence of uniform integrability to deduce that $E[I_\tau] = I_0 = 0.$).

So we need to consider the expected value of $e^{-\rho\tau} z(\tau)$. But,

$$e^{-\rho\tau} z(\tau) = e^{-\rho\tau}(u - x + a\tau),$$

by definition of τ. Now, for arbitrary $t \geq 0$, the values of $te^{-\rho t}$ and $e^{-\rho t}$ are both bounded (by unity). Thus the expected value of the item in question takes the form

$$\begin{aligned} E[|e^{-\rho\tau} z(\tau)|] &\leq xE[e^{-\rho\tau(x)}] + \text{quantity bounded as } x \text{ tends to infinity,} \\ &\leq x + \text{quantity bounded as } x \text{ tends to infinity,}\end{aligned}$$

(since $e^{-\rho t} \leq 1$).

In conclusion, supposing $z_0 = 0$ for simplicity only, we may use the triangle inequality to write

$$\begin{aligned}
\left|E\left[\int_0^\tau e^{-\rho t}X_t dt\right]\right| &= \left|E\left[\int_0^\tau e^{-\rho t}(x-at)dt + b\int_0^\tau e^{-\rho t}z(t)dt\right]\right| \\
&= \left|E\left[\int_0^\tau e^{-\rho t}(x-at)dt - \frac{b}{\rho}e^{-\rho\tau}z(\tau)\right] + \frac{b}{\rho}E\left[\int_0^\tau e^{-\rho t}dz_t\right]\right| \\
&= \left|E\left[\int_0^\tau e^{-\rho t}(x-at)dt - \frac{b}{\rho}e^{-\rho\tau}z(\tau)\right] + \frac{b}{\rho}E\left[\int_0^\tau e^{-\rho t}dz_t\right]\right| \\
&= \left|E\left[\int_0^\tau e^{-\rho t}(x-at)dt - \frac{b}{\rho}e^{-\rho\tau}z(\tau)\right]\right| \\
&\leq E\left[\int_0^\tau \left|e^{-\rho t}(x-at)\right| dt + \frac{b}{\rho}\left|e^{-\rho\tau}z(\tau)\right|\right] \\
&\leq E\left[\int_0^\tau e^{-\rho t}(x+at)dt\right] + \frac{b}{\rho}E[\left|e^{-\rho\tau}z(\tau)\right|] \\
&\leq \int_0^\infty e^{-\rho t}(x+at)dt + \frac{b}{\rho}E[\left|e^{-\rho\tau}z(\tau)\right|] \\
&\leq \frac{x}{\rho} + \frac{a}{\rho^2} + \frac{b}{\rho}\{x + \text{quantity bounded as } x \text{ tends to infinity}\}.
\end{aligned}$$

Our estimate says that for large x that

$$v(x) \leq \frac{b+1}{\rho}x + \text{const.}$$

This rules out exponential growth and hence we take $K = 0$. (Otherwise, the term $Ke^{x\gamma_+}$ introduces unbounded behaviour stronger than estimated).

For any u now choose $L = L(u)$ such that the corresponding function $v(x)$ satisfies $v(x) = 0$ at $x = u$. This means that

$$L(u)e^{u\gamma_-} + \frac{u}{\rho} - \frac{a}{\rho^2} = 0,$$

so that when u is arbitrarily selected we have that:

$$v(x) = (-\frac{u}{\rho} + \frac{a}{\rho^2})e^{(x-u)\gamma_-} + \frac{x}{\rho} - \frac{a}{\rho^2}.$$

Thus the value function solving the Bellman equation is dependent on the choice of u. The diagram shows three example graphs of $v(x,u)$ for different choices of u; two of these, those corresponding to $u = -1$ and $u = -3$ are blatantly not the answer we want, since they offer negative values for x to the right of their corresponding cut-offs u.

We need one further condition to identify u uniquely.

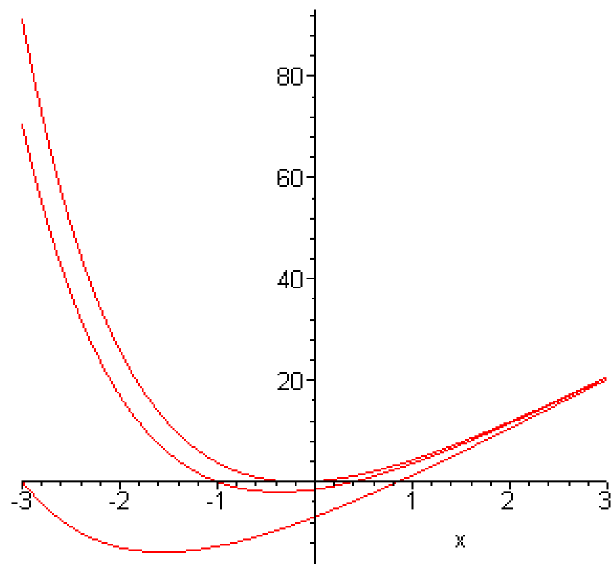


Figure: Three different choices of u give rise to alternatives for the true value function.

Step 3. Smooth-pasting condition

Let us make the dependence on u explicit now and write $v(x, u)$. At any state x it must be the case that the optimal u maximizes $v(x, u)$. So for any $x > u$,

$$\frac{\partial v}{\partial u}(x, u) = 0.$$

In view of the continuity of this partial (check this in our case!), we thus also have

$$\frac{\partial v}{\partial u}(u, u) = 0.$$

But it is likewise true that the optimal level of u must maximize the value of $v(u, u)$. Hence we must have

$$0 = \frac{dv}{du} = \frac{\partial v}{\partial x}(u, u) + \frac{\partial v}{\partial u}(u, u) = \frac{\partial v}{\partial x}(u, u).$$

In summary, to find the optimal lay-off value $x = u^*$ we must satisfy the boundary conditions

$$v(u^*) = 0, \; v'(u^*) = 0,$$

where v' denotes differentiation with respect to x.

The second condition is known as the **smooth-pasting condition**; we ask that the value function smoothly connects to the constant valuation $v(x) = 0$ for $x \leq u$. (You might call this smooth transition from one formula to the formula $v = 0$.)

Thus $x = u^*$ satisfies

$$\begin{aligned} Le^{u^*\gamma_-} + \frac{u^*}{\rho} - \frac{a}{\rho^2} &= 0, \\ L\gamma_- e^{u^*\gamma_-} + \frac{1}{\rho} &= 0. \end{aligned}$$

Eliminating the exponential term gives the optimal control a value independent of x (as assumed) equal to:

$$u^* = \frac{a}{\rho} + \frac{1}{\gamma_-}.$$

Worked Example: If $a = 0.1, b = 0.2, \rho = 0.1$ then $u^* = -0.17082$. Note also that

$$L = -\frac{e^{-u^*\gamma_-}}{\rho\gamma_-} > 0$$

hence the value function is as indicated in the Figure. (Recall that $\gamma_\pm = (a \pm \sqrt{a^2 + 2b^2\rho})/b^2$).

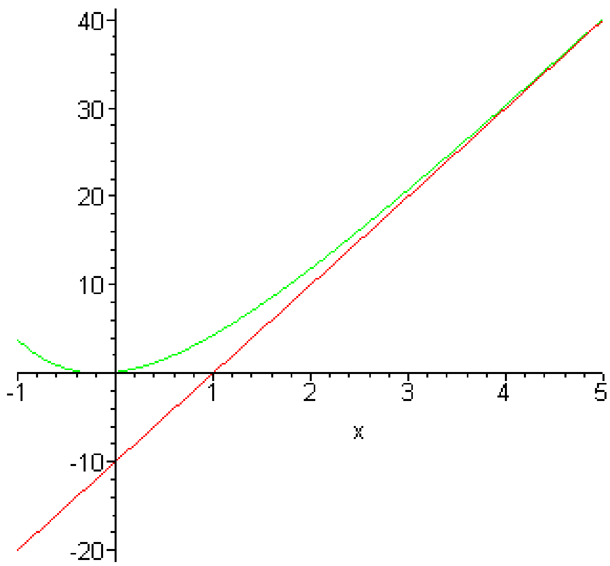


Figure. The value function and its linear asymptote.

4. The Black-Scholes Equation

Note: *You will not be examined on the derivation of the B-S equation. You should however, be aware that Finance problems may be restated as continuous-time optimization problems in which the drift terms is set equal to the riskless rate of interest. The purpose of this section is to justify that kind of modelling.*

The asset price is assumed to be modelled by the equation

$$dS_t = \mu S_t dt + \sigma S_t dz_t.$$

We value the contract which at time $t = T$ pays

$$\max\{S_T - K, 0\}.$$

This is known as a '**call option**' contract for the supply of one unit of asset at a price K. The grounds for this interpretation are that when the asset price is above K the payoff supplies enough money to the holder of the contract to subsidize his payment S_T for a unit of asset down to K : the effective bill is down to K.

Officially an option bestows on one party to the contract the right to buy at price K (one unit of asset) from the issuer of the contract but does not require a transaction at K to be effected. Thus an optimizing agent will transact iff $S_T > K$ and so derive zero value from the contract when $S_T \leq K$. Hence the payoff function indicated at the start.

Let the function $V(S,t)$ satisfy the equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV,$$

with boundary condition

$$V(S,T) = (S-K)^+.$$

Note the Bellman Equation has rS substituted for μS.

One way to interpret the situation is to say that valuation by way of the expectations procedure in dynamical programming refers by way of the Bellman equation to a 'synthetic asset price' process X_t governed by the equation:

$$dX_t = rX_t dt + \sigma X_t dz_t \text{ with } X_0 = S_0.$$

Another way to interpret the situation is to say that the valuation procedure is as though the asset has the same expected rate of return as a riskless asset.

We will not discuss here the connection between the true asset process and the synthetic asset process, except to note a very satisfying conclusion from the fact that, since X_t has a drift of r, its discounted process $e^{-rt}X_t$ has drift zero. The conclusion is that at any time t the future value $e^{-rs}X_{t+s}$ is in expectation X_t.

(The underlying reason is the need to adjust the probability law to take into account certain portfolio actions which render the holder of the option insured from risk in a way which we are about to describe below.)

We will show that $V(S_0, 0)$ is the value of the option by way of an inductive argument.

Suppose the option is worth $V_t = V(S_t, t)$ at time t when the asset is worth S_t, and that a 'self-financing' portfolio strategy has been constructed up to the time t whose value is V_t. This means that on liquidation (assuming zero transactions costs) the holder has V_t amount of cash.

Consider a portfolio (H_0, H_1) in cash and asset respectively, to be created from the cash V_t so that

$$H_0 + H_1 S_t = V_t.$$

After a time $\Delta t > 0$ this portfolio is worth

$$H_0(1 + r\Delta t) + H_1(S_t + \Delta S_t)$$

and we wish to replicate the new value of the expression $V_t + \Delta V_t = V(S_t + \Delta S_t, t + \Delta t)$.

The value matching requirement is thus:

$$H_0(1 + r\Delta t) + H_1(S_t + \Delta S_t) = V_t + \Delta V_t.$$

Subtraction yields

$$H_0 r\Delta t + H_1 \Delta S_t = \Delta V_t.$$

By Itô's Rule

$$\Delta V_t = \{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\}\Delta t + \frac{\partial V}{\partial S}\sigma S \Delta z$$

(to first order in Δt), so

$$H_0 r\Delta t + H_1(\mu\Delta t + \sigma S\Delta z_t) = \{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\}\Delta t + \frac{\partial V}{\partial S}\sigma S\Delta z.$$

The uncertain terms will therefore agree provided

$$H_1 = \frac{\partial V}{\partial S}.$$

In this case we have

$$H_0 r\Delta t + \frac{\partial V}{\partial S}(\mu\Delta t) = \{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\}\Delta t.$$

Note the cancellation of the terms in μS.

Now $H_0 = V_t - H_1 S_t$ so

$$\{V_t - \frac{\partial V}{\partial S}S_t\}r\Delta t = \{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\}\Delta t.$$

Dividing by Δt and passing to the limit we obtain

$$\{V_t - \frac{\partial V}{\partial S}S_t\}r = \{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\}.$$

But the function $V(S,t)$ satisfies the equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV,$$

so the value matching condition is indeed met.

Finally at time $t = T$ the value matching condition ensures that the portfolio has value $V(S_T, T)$ which thus agrees with $(S_T - K)^+$. Hence $V(S_0, 0)$ is the value of the option at time $t = 0$.

5. The Feynman-Kac Formula (to be done in class)

By this stage you should realize that if a process X_t obeys a stochastic differential equation

$$dX = a(X_t)dt + b(X_t)dz_t$$

[NB: no explicit time reference in $a(.)$ and $b(,)$] and you consider the value function defined by the so-called Feynman-Kac formula:

$$V(x,T) = E_x[\int_0^T e^{-\rho t}f(X(t))dt + e^{-\rho T}g(X_T)],$$

where $T$is a fixed (deterministic) time. Then since $X(t)$ is assumed continuous we have

$$\begin{aligned}
V(x,T) &= f(x) + E_x[\int_{\Delta t}^{T} e^{-\rho t} f(X(t))dt + e^{-\rho T} g(X_T)] \\
&= E_x[\int_0^{T-\Delta t} e^{-\rho(s+\Delta t)} f(Y(s))ds + e^{-\rho T} g(X_T)] \\
&= e^{-\rho \Delta t} E_x[\int_0^{T-\Delta t} e^{-\rho(s+\Delta t)} f(Y(s))ds + e^{-\rho(T-\Delta t)} g(X_{T-\Delta t}) + error] \\
&= (1-\rho\Delta t) E_x E_{x+\Delta X}[V(x+\Delta X, T-\Delta t) + error] \\
&= (1-\rho\Delta t)[V(x,T) + [-\frac{\partial V}{\partial T} + a\frac{\partial V}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial x^2}]\Delta t + error]
\end{aligned}$$

Hence $V(x)$ satisfies

$$0 = f(x) + a\frac{\partial V}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial T} - \rho V.$$

Notice that T may be interpreted as measuring time left to the bequest date.

Now the Black-Scholes equation asserts

$$-\frac{\partial V}{\partial T} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV,$$

so taking

$$f(x) = 0, g(x) = (x-K)^+, a = rS, b = \sigma S, \rho = r$$

enables the Feynman-Kac formula to solve the Black-Scholes equation.

In turn we see why the adjustment of μ to r is required, if the expectations operation is to yield a valuation of the call option.

6. The American perpetual put option

It is instructive to consider the perpetual put option: the holder may exercise at any time of their choosing. This is a problem requiring the optimal-timing exercise of an contract to sell an asset at a fixed price K, that is

$$\max[e^{-r\tau}(K - S_t)^+]$$

We choose the optimal time for exercise by using a trigger value for the synthetic asset price X_t reaching the value $\hat{S}$. The time of exercise is thus

$$\tau = \tau(S_0) = \inf\{t : S_t = \hat{S}\},$$

given that $S_0 = x < \hat{S}$. Thus the threshold-value notation u of the previous examples has $u = \hat{S}$ here.

We have seen that the Black-Scholes equation holds up to the time of exercise and that it has the form of the Bellman equation. The context of the maximization problem has us considering

$$V(S) = \max_{\hat{S}} E[e^{-r\tau}(K - X_\tau)^+]$$

where the starting price at time $t = 0$ of the asset is S and the synthetic asset price follows

$$dX_t = rX_t dt + \sigma X_t dz_t, \text{ with } X_0 = S.$$

{We regard the Feynman-Kac formula as telling us that to optimize the value of holding the put we are maximizing $E[e^{-r\tau}(K-X_\tau)^+]$.}

The Bellman equation here is

$$rV = rxV' + \frac{1}{2}\sigma^2 x^2 V''.$$

We look for a power function solution $V = x^\gamma$.This yields the auxiliary equation as being

$$\frac{1}{2}\sigma^2\gamma(\gamma-1) + r\gamma - r = 0.$$

Put

$$\rho = \frac{2r}{\sigma^2}$$

and this reads

$$\gamma(\gamma-1) + \rho(\gamma-1) = 0$$

or

$$(\gamma+\rho)(\gamma-1) = 0.$$

The general solution is thus

$$V(x) = Ax + Bx^{-\rho}.$$

As in the last example of Section ?? we begin by examining boundary behaviour. Clearly the put value satisfies

$$\lim_{x\to\infty} V(x) = 0,$$

so $A = 0$. To find B we get at $x = \hat{S}$

$$B\hat{S}^{-\rho} = K - \hat{S}$$

so

$$B = \hat{S}^{\rho}(K - \hat{S})$$

and thus

$$V(S) = \hat{S}^{\rho}(K-\hat{S})S^{-\rho},$$

and thus the value function is now dependent on S and also on the threshold level $u = \hat{S}$.

This time we note directly from the formula for V that for fixed S, the value function as a function of its threshold $\hat{S}$ takes one of the forms indicated in the figure (local convexity/concavity at the origin depends on the magnitude of ρ).

Whatever the behaviour near the origin, the value is maximized in the interval $(0, K)$ with $\hat{S}$ satisfying the first-order condition

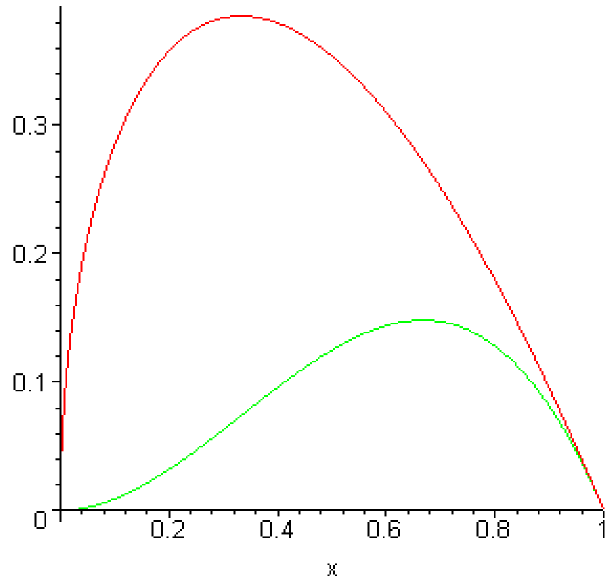
$$K\rho\hat{S}^{\rho-1} - (\rho+1)\hat{S} = 0,$$

so that

$$\hat{S} = \frac{\rho}{\rho+1}K.$$

For instance if $\rho = 1$we obtain

$$\hat{S} = \frac{1}{2}K.$$



We note that

$$\frac{dV}{dS} = -\rho B S^{-\rho-1}$$

so

$$\left.\frac{dV}{dS}\right|_{S=\hat{S}} = -\rho B \hat{S}^{-\rho-1} = -\rho \hat{S}^{\rho}(K - \hat{S}^{\rho})\hat{S}^{-\rho-1} = -\rho(K\hat{S}^{-1} - 1) = -\rho(\frac{\rho+1}{\rho} - 1) = -1.$$

Thus for the optimal trigger it is true that the value function

$$V(S) = \frac{K}{1+\rho}(\hat{S}/S)^{\rho}$$

is tangential to the so-called 'intrinsic value'

$$K - S.$$

Evidently, the **smooth-pasting condition** holds (no surprise, since the explicit reasoning here is consistent with the function approach of our earlier example). An example graph is shown below.

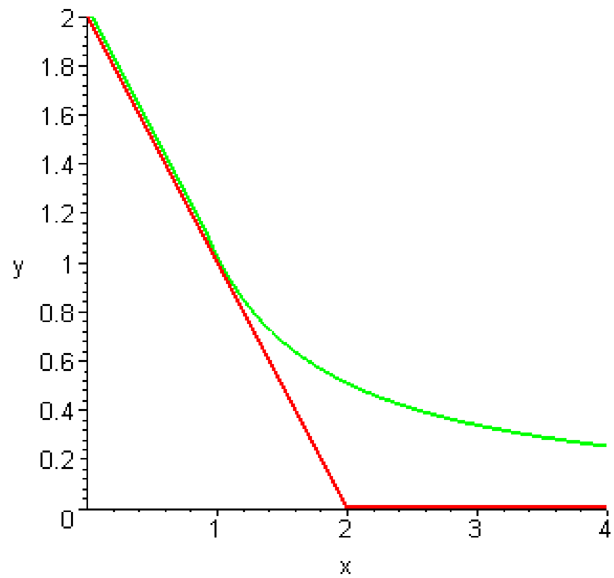


Figure. Intrinsic value and perpetual put-option value.

Remark. The two examples are extreme cases of a performance index

$$J(u) = E[\int_0^\tau L(t, X_t, u(t))dt + \Psi(\tau, X_\tau)],$$

comprising a running cost and terminal cost.

6.1. The second of two examples: When to invest

(D&P page 140)

This example is reminiscent of the American perpetual call option. But its structure is more like a call option.

It costs I to enter an economic activity which will provide revenue at a rate v_t at time t. The market conditions are such that v_t is modelled for $t > 0$ by

$$dv_t = \alpha v_t dt + \beta v_t dz_t,$$

with $\alpha > 0, \beta > 0$ constants. The time discount factor is $\rho > \alpha$ (to ensure convergent value function). At what time should one enter?

Here

$$v_t = v_0 \exp(\alpha t + \beta z_t).$$

The value function as at time s is here defined by

$$C(v, s) = \max_u E_{s,v}[e^{-\rho\tau}(v_\tau - I)],$$

where

$$\tau = \inf\{t \geq s : v_t = u\}.$$

Obviously if $v - I < 0$ it does not pay to invest. So $u > I$. But if $v - I > 0$, although it is surely already profitable to invest, one should ask whether to wait until the circumstances are even more favourable. That is we expect that the optimal stopping threshold u will satisfy $u > I$. Observe therefore that in effect the underlying payoff is similar to the call payoff: $(v_t - I)^+$.

Evidently, shifting the time so that time s is the new time origin, we put

$$w_t = v_{s+t}, w_0 = v_s = w$$

so we have that

$$dw_t = \alpha w_t dt + \beta w_t dz_t, w_0 = w,$$

and

$$\bar{\tau} = \inf\{t \geq 0 : w_t = u\} = \tau - s,$$

we have

$$C(v, s) = e^{-s} \max_u E_{s,v}[e^{-\rho\bar{\tau}}(w_{\bar{\tau}} - I)] = e^{-\rho s} C(v, 0).$$

Thus $c(v) = C(v, 0)$ satisfies the HJB equation

$$\frac{1}{2}\beta^2 v^2 c'' + \alpha v c' - \rho c = 0.$$

A solution of this second order, constant coefficients ordinary differential equation is provided by power functions like $c(v) = v^\gamma$ on condition that

$$\begin{aligned}\frac{1}{2}\beta^2\gamma(\gamma-1) + \alpha\gamma - \rho &= 0, \\ \beta^2\gamma^2 + (2\alpha - \beta^2)\gamma - 2\rho &= 0.\end{aligned}$$

The auxiliary equation has one root positive, one root negative (as their product is negative) and are given by

$$\gamma_\pm = [(\beta^2 - 2\alpha) \pm \sqrt{(\beta^2 - 2\alpha)^2 + 8\rho\beta^2}]/2\beta^2.$$

For more information on the roots consider

$$\phi(\gamma) = \beta^2\gamma^2 + (2\alpha - \beta^2)\gamma - 2\rho$$

and note that for $\gamma = 1$ we have

$$\phi(1) = \beta^2 + (2\alpha - \beta^2) - 2\rho = 2(\alpha - \rho) < 0$$

assuming that $\alpha < \rho$. Thus $\gamma_+ > 1$ since $\phi(\infty) = \infty$.

The general solution of the equation is thus of the form

$$c(v) = Kv^{\gamma_+} + Lv^{\gamma_-},$$

with K, L constants.

We again consider boundary behaviour. Now $c(0) = 0$, since the value of the project remains at zero if $v_0 = 0$. So we take $L = 0$.

Next, we satisfy the value-matching condition at entry into the investment: $c(u) = u - I$ which makes the resulting solution function take the form $c(v) = v^{\gamma_+}[(u - I)/u]^{1/\gamma_+}$

To choose u optimally we assume as before that the **smooth-pasting condition** is to hold at the entry threshold (i.e. we demand smooth transition from $c(v)$ to the function $v - I$). Indeed if $v > u$ the if it pays to enter at u it pays to enter at v.Thus for all $v \geq u$ we have

$$c(v) = v - I.$$

In summary we require of u^* that

$$c(u^*) = u^* - I \text{ and } c'(u^*) = 1.$$

Hence

$$\begin{aligned} K(u^*)^{\gamma_+} &= u^* - I, \\ K\gamma_+(u^*)^{\gamma_+ - 1} &= 1. \end{aligned}$$

Eliminating the exponential term, we have

$$u^* - I = \frac{u^*}{\gamma_+},$$

i.e. investment should occur when the market yields a revenue rate at least equal to

$$u^* = \frac{\gamma_+}{\gamma_+ - 1} I > 0.$$

Note that

$$K = \frac{(u^*)^{1-\gamma_+}}{\gamma_+}.$$

The value function is thus representable also in the form:

$$c(v) = Kv^{\gamma_+} = \frac{u^*}{\gamma_+}(v/u^*)^{\gamma_+}, \text{ for } 0 < x < u^*.$$

Comment. Without the smooth-pasting condition, we could infer from the value matching condition:

$$Ku^{\gamma_+} = u - I$$

that

$$K = K(u) = u^{1-\gamma_+} - Iu^{-\gamma_+}$$

so that

$$\begin{aligned} c(v) &= c(v; u) = K(u)v^{\gamma_+} \\ &= (u^{1-\gamma_+} - Iu^{-\gamma_+})v^{\gamma_+}, \end{aligned}$$

where we are treating u as a free parameter to be selected optimally.

But, for any v it must be the case the $c(v; u)$ should be maximized over all u.Hence

$$0 = \frac{\partial c(v; u)}{\partial u} = \left[(1 - \gamma_+)u^{-\gamma_+} + \gamma_+ I u^{-1-\gamma_+})\right] v^{\gamma_+}.$$

Hence

$$(1 - \gamma_+)u^{-\gamma_+} + \gamma_+ I u^{-1-\gamma_+} = 0$$

or

$$(\gamma_+ - 1)u = \gamma_+ I,$$

that is, as before

$$u = \frac{\gamma_+}{\gamma_+ - 1} I.$$

You can work back from this equation to verify that the smooth-pasting condition holds. Compare this with the "First of two examples" section.

Worked Example: Take $\alpha = 0.02, \beta = 0.3, \rho = 0.04, I = 1$.

Then $\gamma_+ = 1.260655890$ and $u^* = 4.836475746$.

The figure below identifies the value function $c(v)$ and the cost function $v - I$ to which it is smoothly pasted.

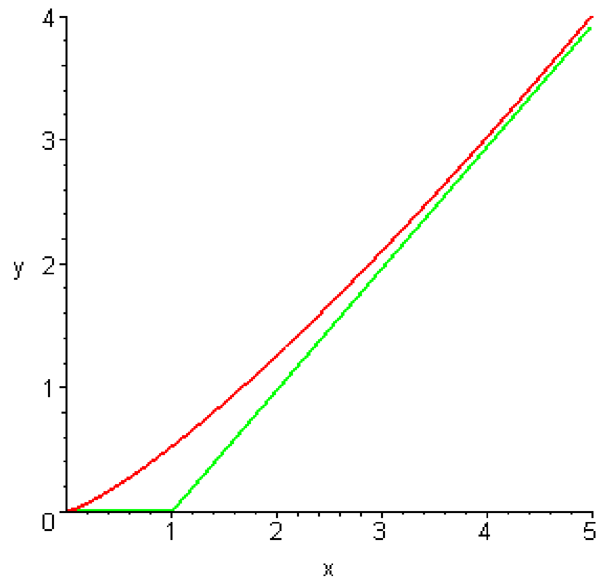


Figure. Value of an Investment, with optimal entry at $v = u^*$.

It is clear from the shape that we are in effect dealing with a perpetual call format.