4.1 Control Problems

The simplest control problem takes the form

$$\min_{u(t)} \quad \int_o^T f_0(x(t), u(t))dt$$

subject to

$$\dot{x} \;=\; f_1(x, u), \tag{1}$$

$$x(0) \;=\; x_0, \quad x(T) \;=\; x_1.$$

The function $u(t)$ is known as the **control** function, whereas $x(t)$ is the **state** function since its intended meaning is the description of a dynamical system whose evolution over time is determined by equation (1) from the initial condition $x(0) = x_0$ and the control u. Equation (1) is therefore called the **state equation** (or the kinetic equation).

Remark The fact that x_0 and $u(t)$ determine $x(t)$ uniquely over time is a theorem in the theory of differential equations, given continuity assumptions on f_1.

Example Minimize $\displaystyle\int_0^1 (x^2 + u^2)dt$

subject to $\dot{x} \;=\; -x \;+\; u$

$$x(0) \;=\; x_0, \quad x(T) \;=\; 0.$$

We shall return to this example after some useful preliminaries.

Remark Note the absence of $\dot{x}$ in $f_0(x, u)$.

4.2 Introducing the Hamiltonian We can turn the control problem into a constrained optimization problem in the Calculus of Variations provided we see ourselves as selecting a pair (x, u) in $\mathcal{X} \times \mathcal{U}$ where $\mathcal{X}$ is a vector space of differentiable functions and $\mathcal{U}$ is a vector space of appropriate control functions. In applications $\mathcal{U}$ is often - though not always as we see later - a vector space of continuous functions. In this formulation we seek to minimize

$$F(x, u) \;=\; \int_o^T f_0(x(t), u(t))dt$$

subject to the constraint equation

$$G(x,u) \;\equiv\; x(t) - x_0 - \int_0^t f_1(x(s),u(s)))ds \;=\; 0$$

and $x(T) \;=\; x_1$. As regards the terminal value specification, it is possible to relax the problem and not demand that the system be driven to a predetermined state. In general, it will not be clear, however, whether the system is 'controllable', i.e. can be driven from a given state x_0 to an arbitrary terminal state $x(T)$, though we might assume it.

Note that $G : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ where $\mathcal{Y}$ is again a vector space of continuous functions (if, as we will wish to assume, $f_1(.,.)$ is continuous). Observe that we have eliminated $\dot{x}$ by integrating.

The Lagrangian is thus

$$L(x,u,\lambda) \;=\; F(x,u) \;+\; \lambda^* G(x,u),$$

where $\lambda^* \in \mathcal{Y}^*$.

Let us suppose $\mathcal{Y} \;=\; \mathcal{C}[0,T]$ with $\|x\|_\infty$ norm. Then $\mathcal{Y}^*$ is the space $BV[0,T]$ of functions of bounded variation and

$$\lambda^*(y) \;=\; \int_0^T y(t)d\alpha(t)$$

for some $\alpha \in BV[0,T]$.

We thus have

$$L \;=\; \int_0^T f_0(x,u)dt \;+\; \int_0^T \left(x(t) - x_0 - \int_0^t f_i(x(s),u(s))ds \right) d\alpha(t),$$

so, integrating by parts, we get

$$L = \int_0^T f_0(x,u)dt \;+\; \left[\alpha(t)\{x(t) - x_0 - \int_0^t f_1(x(s),u(s))ds\} \right]_0^T - \;\int_0^T \alpha(t)\{\dot{x}(t) - f_1(x(t),u(t))\}dt.$$

$$L = \int_0^T (f_0 + \alpha f_1 - \alpha \dot{x})dt + \alpha(T)\{x(T) - x_0 - \int_0^T f_1(x(t),u(t))dt.$$

We can of course select $\alpha(T) = 0$ and dispose of the last term. Alternatively we may regroup the terms of the equation and, noting that $x(T) - x(0) = \int_0^T \dot{x}\; dt$, consider $\lambda(t) = \alpha(t) - \alpha(T)$. In either case the Langrangian then takes the form:

$$L = \int_0^T (f_0 + \lambda f_1 - \lambda \dot{x})dt.$$

It is, of course, possible that the control $u(t)$ employed already ensures that the contents of the braces are equal to zero, in which case we need not have worked so hard. We note that although $\lambda(T) = 0$, it is not clear that $\lambda(t)$ is continuous at $t = T$; it is differentiable in the open interval$(0,T)$ by the Euler-Lagrange equations. See below (Section 4.3). The integrand here, denoted $l(x,u,\lambda,\dot{x})$ has a very simple dependence on $\dot{x}$. We therefore let

$$\mathcal{H} = f_0 + \lambda f_1$$

and call this the **Hamiltonian**. Remembering that $l(x, u, \lambda, \dot{x}) = \mathcal{H}(x, u, \lambda) - \lambda\dot{x}$, we may write the Euler-Lagrange equations in the following form:

$$\frac{d}{dt}(l_{\dot{x}}) = l_x, \quad \text{in other words } \frac{d\lambda}{dt} = -\frac{\partial\mathcal{H}}{\partial x}, \tag{2}$$

$$\frac{d}{dt}(l_{\dot{u}}) = l_u, \quad \text{which reduces to } 0 = -\frac{\partial\mathcal{H}}{\partial x}, \tag{3}$$

and of course we also have:

$$\frac{dx}{dt} = \frac{\partial\mathcal{H}}{\partial\lambda}. \tag{4}$$

Note the remarkable similarity between (2) and (4). The Lagrange multiplier is often called the **dual variable** of x (as it corresponds to the constraint placed on x). It is also called the **co-state** variable and equation (2) is referred to as the co-state equation.

Equation (3) tells us that the Hamiltonian is stationary at the optimal x, u, λ in respect of x, u. In a later chapter we will see that in fact the Hamiltonian is minimized/maximized, depending on context. We stress that this first order condition is available to us on the assumption that the control variable u is numerically unrestricted. If for instance $-1 \le u(t) \le 1$ is required, then the Pontryagin theory (see the next chapter) will lead us to check for both internal minima (where $\frac{\partial\mathcal{H}}{\partial u} = 0$), and boundary minima $u = \pm 1$.

We are now in a position to solve the example of section 4.1. **Example** (of 4.1) Evidently

$\mathcal{H} = x^2 + u^2 + \lambda(u - x)$.

Thus we have

$$\dot{\lambda} = -\frac{\partial\mathcal{H}}{\partial x} = -(2x - \lambda), 0 = \frac{\partial\mathcal{H}}{\partial u} = 2u + \lambda,$$

and $\dot{x} = -x + u$.

We thus have three equations in three unknowns. Now

$$\dot{\lambda} = -2(\dot{x} + x),$$

so $\dot{\lambda} = -2(\dot{x} + \ddot{x})$, hence

$$+2(\dot{x} + \ddot{x}) = 2x + 2(\dot{x} + x), \dot{x} + \ddot{x} = x + \dot{x} + x,$$

i.e. $\ddot{x} = 2x$.

Roots of the auxiliary equation $t^2 - 2 = 0$ are $t = \pm\sqrt{2}$, so

$$x(t) = Ae^{t\sqrt{2}} + Be^{-t\sqrt{2}}.$$

But $x(0) = x_0$ and $x(T) = 0$, so we calculate as follows.

$$x_0 = A + B0 = Ae^{T\sqrt{2}} + (x_0 - A)e^{-T\sqrt{2}}0 = A(e^{T\sqrt{2}} - e^{-T\sqrt{2}}) + x_0e^{-T\sqrt{2}}$$

$$A = -\frac{x_0 e^{-T\sqrt{2}}}{e^{T\sqrt{2}} - e^{-T\sqrt{2}}}$$

$$B = x_0\left\{1 + \frac{e^{-T\sqrt{2}}}{e^{T\sqrt{2}} - e^{-T\sqrt{2}}}\right\} = x_0\left\{\frac{e^{T\sqrt{2}}}{e^{T\sqrt{2}} - e^{-T\sqrt{2}}}\right\}$$

$$x(t) = \frac{x_0}{e^{T\sqrt{2}} - e^{-T\sqrt{2}}}\left\{e^{T\sqrt{2}}e^{-t\sqrt{2}} - e^{-T\sqrt{2}}e^{t\sqrt{2}}\right\} = \frac{e^{(T-t)\sqrt{2}} - e^{t-T)\sqrt{2}}}{e^{T\sqrt{2}} - e^{-T\sqrt{2}}}x_0$$

or

$$x(t) = x_0\frac{sh(T-t)\sqrt{2}}{sh\sqrt{2}T}, \qquad 0 \le t \le T.$$

Hence

$$u = \dot{x} + x = \frac{x_0}{sh\sqrt{2}T}\left\{-\sqrt{2}ch(T-t)\sqrt{2} + sh\sqrt{2}(T-t)\right\}.$$

4.3 Discontinuous Lagrange multiplier: an elucidation

We are entitled to assume $\alpha(T) = 0$ as in the notes. However, what need not be true is that $\alpha(t)$ is continuous (a fact we are used to in Riemann-Stieltjes integration). In fact:

Theorem *The Lagrange multiplier, α, in the control problem of 4.1 is continuous in $(0, T)$. However:*

(a) if $x(T)$ specified, α is discontinuous at $t = T$, whereas

(b) if $x(T)$ is not specified, then α is continuous at $t = T$.

It follows from this and the Euler-Lagrange equations that $\alpha(t)$ is differentiable in $(0, 1)$. However, the condition $\alpha(T) = 0$ is only meaningful in case (b). In case (a) the specification of $x(T)$ will determine the extremal curve and a limiting value of $\alpha(t)$ as $t \to T$ which will be different from 0.

The above result explains why certain exercise problems may at first appear to be over-specified when $\alpha(T) = 0$ is used to determine constants in the solution of the adjoint equation. In fact the end-point condition can only be applied if the solution of the adjoint equation is continuous at that end-point. Thus in fact the condition $\alpha(T) = 0$ does not contradict the determination of constants of integration from the initial and terminal conditions of the state function(s). A partial explanation of the Theorem is offered in a following exercise set.

4.4 Several variables

In general the state of a dynamical system is described by a vector of state variables $(x_1(t), \ldots, x_n(t))$ and the control problem takes the form

$$\min \int_o^T f_0(\mathbf{x}, u)dt$$

$$s.t. \qquad \left.\dot{x}_1 = f_1(\mathbf{x}, u)\dot{x}_2 = f_2(\mathbf{x}, u) \qquad \ldots \dot{x}_n = f_n(\mathbf{x}, u)\right\} \quad \text{i.e. } \dot{\mathbf{x}} = f(\mathbf{x}, u)$$

with $\mathbf{x}(0)$ and $\mathbf{x}(T)$ specified.

For that matter the controls available on the dynamical system might well form a vector $(u_1(t), \ldots, u_m(t))$ but we shall
refrain from this generalization for now.

Example Minimize $\int_0^T u^2 dt$ subject to $\ddot{x} = u$ with $x(0) = x_0$, $x(T) = x_1$ and $\dot{x}(0)$ also specified.

Remark This is a very natural example - the state equation takes the form of Newton's Law of Motion - acceleration equals force.

Reformulation We can turn the single equation of second order $\ddot{x} = u$ into two first order equations by letting $x_1 = x$ and $x_2 = \dot{x}$ so that $\ddot{x} = \dot{x}_2$. We thus have the equivalence

$$\ddot{x} = u \quad \Leftrightarrow \quad \left.\dot{x}_1 = x_2 \dot{x}_2 = u\right\}$$

which is in the form considered in the first chapter.

Evidently we can always re-write the constraints in the form

$$G(\mathbf{x}, u) \equiv \left(\dot{x}_1(t) - x_1(0) - \int_0^T f_1(\mathbf{x}, u)dt \dot{x}_2(t) - x_2(0) - \int_0^T f_2(\mathbf{x}, u)dt \vdots \right) = \left(0 0 \vdots \right)$$

so that G maps into $\mathcal{Y} = \mathcal{X} \times \mathcal{X} \times \ldots \times \mathcal{X}$ (n times) and say $\mathcal{X} = \mathcal{C}[0, T]$, as earlier. We need to determine the dual of $\mathcal{Y}$.

Lemma $(\mathcal{X} \times \mathcal{Y})^* = \mathcal{X}^* \oplus \mathcal{Y}^*$ *for any normed vector spaces* $\mathcal{X}$ *and* $\mathcal{Y}$.

Sketch Proof In the first place we need to remark that $\mathcal{X} \times \mathcal{Y}$ is the vector space of pairs (x, y) where

$$\alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

and $\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}.$

Now if $H : \mathcal{X} \times \mathcal{Y} \to R$ is a continuous linear functional then

$$H(x, y) = H((x, 0) + (0, y)) = H(x, 0) + H(0, y).$$

Evidently $f(x) \equiv H(x, 0)$ is a member of $\mathcal{X}^*$ and $g(y) \equiv H(0, y)$ is a member of $\mathcal{Y}^*$, hence the result.

Returning to our control problem the Lagrange Multiplier appropriate to $G(\mathbf{x}, u)$ is thus of the form

$$\lambda^*(\mathbf{y}) = \lambda_1^*(y_1) + \lambda_2^*(y_2) + \ldots + \lambda_n^*(y_n)$$

where each λ_i^* is in $\mathcal{X}^*$. It is now easy to check that the argument of section 4.2 may be expanded to yield a Hamiltonian

$$H = f_0 + \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n$$

corresponding to the Lagrangian $L = \int_0^T l dt$ where

$$l = \mathcal{H} - \lambda_1 \dot{x}_1 - \lambda_2 \dot{x}_2 - \ldots - \lambda_n \dot{x}_n.$$

We thus obtain from the Euler-Lagrange equations the following co-state equations

$$\dot{\lambda}_i = -\frac{\partial \mathcal{H}}{\partial x_i} \qquad (i = 1, 2, ..., n),$$

as well as the original state equations

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i},$$

and the stationarity condition

$$0 = \frac{\partial H}{\partial u}.$$

We now address the example in two variables given at the beginning of this section.

Example Extremize $\int_0^1 u^2 dt$ with u unconstrained subject to $\ddot{x} = u$ with $x(0) = \dot{x}(0) = 1$; $x(1), \dot{x}(1)$ not specified.

Solution We put $x_1 = x, x_2 = \dot{x}$. Thus

$$\left.\dot{x}_2 (= \ddot{x}) = u, \dot{x}_1 = x_2.\right\}$$

The Hamiltonian is $\mathcal{H} = u^2 + \lambda_1 x_2 + \lambda_2 u$ and in the Hamiltonian formulation the Euler-Lagrange equations are:

$$\dot{\lambda}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = 0, \dot{\lambda}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} = -\lambda_1.$$

Thus $\lambda_1 =$ constant and $\lambda_2 = -\lambda_1 t + c$.
To minimize $\mathcal{H}$ we set

$$\frac{\partial \mathcal{H}}{\partial u} = 0.$$

Thus $2u + \lambda_2 = 0$ i.e. $u = -\frac{1}{2}\lambda_2$ giving

$$u = -\frac{1}{2}(-\lambda_1 t + c) = \frac{1}{2}(\lambda_1 t - c)$$

i.e. a general linear expression.
We may now integrate up the equation

$$\ddot{x} = \frac{1}{2}(\lambda_1 t - c)$$

yielding a general cubic

$$x(t) = At^3 + Bt^2 + Ct + D.$$

In particular $t = 0$ gives $D = 1$, $\dot{x}(0) = C = 1$. Hence

$$1 = x(1) = A + B + 2 \quad \text{i.e.} \quad A + B = -1. \tag{1}$$

Since $\dot{x}(1) = x_2(1)$ is not specified, we apply the continuity condition to λ_2 at $t = 1$ to obtain $0 = -\lambda_1 + c$, i.e. $\ddot{x}(T) = 0$, so

$$0 = [A3.2t + B2.1.t]_{t=1} \quad \text{i.e. } 2B + 6B = 0. \tag{2}$$

Finally

$$B = \frac{1}{2} \quad A = -\frac{3}{2}.$$