Chapter 1 Additional material

1.x Remark It may be shown that in general the Euler-Lagrange Equation can be re-written in the form:

$$\frac{d}{dt}\left(f - \dot{x} f_{\dot{x}}\right) = f_t,$$

where f_t denotes the partial derivative of $f(x, \dot{x}, t)$ with respect to t. See Hestenes [], p. 59.

1.xx Further Exercises on the Euler-Lagrange Equation

1. Referring to the lectures, write down a formula for $\psi''(s)$ when $F(x)$ is in the conventional form and compute it when the integrand is $f(x, \dot{x}, t) = x^2 p(t) + \dot{x}^2 q(t)$, where $p(t), q(t)$ are functions of t only. Show by reference to $\psi''(0)$ that any solution $x(t)$ of:

$$\frac{d}{dt}(\dot{x} q(t)) = x p(t)$$

is in fact a *minimum* of $F(x)$ if $p(t), q(t)$ are positive in $0 \leq t \leq 1$.

2. Referring to the derivation of the Euler-Lagrange equation, show that if $x(1)$ is not specified then

$$f_{\dot{x}}(x(1), \dot{x}(1),\ 1) = 0.$$

Hint: Note that $h(1)$ need no longer be zero; in fact any value may be assigned to it including zero.

3. Show that if $x(0)$ and $x(1)$ are not specified then for $\tau = 0$ and $\tau = 1$ we have:

$$f_{\dot{x}}(x(\tau), \dot{x}(\tau),\ \tau) = 0.$$

Use these conditions to find the particular function $x(t)$ which minimises the integral:

$$\int_0^1 \{x + \dot{x} + x \cdot \dot{x} + \frac{1}{2}\dot{x}^2\}\ dt$$

when the end-point values of $x(t)$ are not specified.

4. Use the Lagrange Multiplier Method to solve:

$$\min \quad F(x) = \int_0^1 \dot{x}(t)^2\ dt$$

subject to

$$\int_0^1 x(t)^2 dt = 2$$

and $x(0) = x(1) = 0$.

5. Use the Lagrange Multiplier Method to solve:

$$\min \quad F(x) = \int_0^\infty \{x^2 + \dot{x}(t)^2\}\ dt$$

subject to

$$\int_0^\infty x(t)e^{-2t}\ dt = 1$$

and $x(0) = 1$ and $\lim_{t\to\infty} x(t) = 0$. [Hint: Take the upper limit to be τ, solve, and then take τ to the limit. See also the poscript section below.]

6. Maximise over $x(t)$ the expression $\int_0^{\pi/2} \sin t \cdot x(t)\ dt$ subject to:

$$\int_0^{\pi/2} x(t)\ dt = 0, \quad \int_0^{\pi/2} t \cdot x(t)\ dt = 0, \quad \int_0^{\pi/2} x(t)^2\ dt = 1.$$

Comment: This problem may be treated geometrically; it asks for a vector x of length 1 (in the appropriate norm $||.||_2$) perpendicular to the vectors u, v: the constant function function $u \equiv 1$ and the function $v \equiv t$ and with maximal projection onto the vector $w \equiv \sin t$. A geometric tool is provided by the Gram-Schmidt process, from where it is clear that $x(t)$ needs to be an appropriate linear combination of u, v, w. The same conclusion is drawn from the Euler-Lagrange equations, the appropriate scalars being Lagrange multipliers.

7. Maximise the integral $\int_{-\infty}^{+\infty} x(t) \log x(t)\, dt$ subject to:

$$\int_{-\infty}^{+\infty} x(t)\, dt = 1, \qquad \int_{-\infty}^{+\infty} t^2 \cdot x(t)\, dt = \sigma^2.$$

This is a problem similar to the last, but with connections with probability; it asks for a probability distribution $x(t)$ of a random variable t, with the first constraint being about its expectation and the second about its variance. [Hint: Refer to the definition of the gamma function.]

8. A country's debt D_t obeys the equation:

$$D_t' = rD_t + M_t - S_t$$

where r is the bank rate, M_t is the rate at which raw materials are imported and S_t is the rate at which goods are exported. In order to clear an initial debt of D_0 by time T precisely the government proposes to maximize production over the time range $[0, T]$ on the assumption that the output is always sold. Assuming a production function $S_t = a\sqrt{M_t}$ with $a > 0$ and writing $x_t = M_t - S_t = S_t^2/a^2 - S_t$ show that the government's problem amounts to maximising:

$$\int_0^T \sqrt{a^2 + 4x_t}\, dt$$

subject to

$$\int_0^T e^{-rt} x_t\, dt = -D_0,$$

where it is assumed that $x_0 = 0$. Find the extremal curve for the problem and show that the plan is feasible if and only if:

$$D_0 = \frac{a^2}{2r} \cdot (1 - \cosh rT).$$

[Hint: Find S_t in terms of x_t and integrate the debt equation.]

9. Maximise $\int_0^1 f(t)\sqrt{x(t)}\;dt$ subject to $\int_0^1 x(t)\;dt = c$.

10. Maximise $\int_0^1 f(t)x(t)^\beta\;dt$ subject to $\int_0^1 x(t)\;dt = c$ where $1 \leq \beta$.

11. For the problem:

$$\min \int_0^T (x^2 + \dot{x}^2)\;dt$$

subject to

$$x(0) = c, \quad x(T) = 0,$$

the trajectory is known to be of the form

$$x_T(t) = Ae^t + Be^{-t},$$

show that

$$\lim_{T\to\infty} A_T = 0.$$

Show also that, pointwise $\lim_{T\to\infty} x_T(t) = ce^{-t}$. Interpreting $x_T(t) = x_T(T)$ for $t > T$, verify that the limit is uniform.

12. For the problem:

$$\min \frac{1}{2}\int_0^\infty (x^2 + \dot{x}^2)\;dt$$

subject to $x(0) = 1, \quad x(\infty) = 0$ and

$$\int_0^\infty xe^{-2t}\;dt = 1$$

show that the optimal curve is $x = 9e^{-t} - 8e^{-2t}$.

13. For the problem:

$$\min \int_0^{2\pi} \{\phi \cdot r^2 + \phi^2 \dot{r}^2\}^{1/2}\;d\theta$$

where $\phi = (1-\gamma/r)^{-1}$ and γ is a constant, while $\dot{r}$ here denotes $dr/d\theta$, show that the Euler-Lagrange Equation reduces to the (Ricatti) equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{3}{2}\gamma u^2,$$

where $u = 1/r$.

14. Solve the problem:

$$\min \int_0^1 (t^2 + x^2 + \dot{x}^2)\, dt$$

subject to $x(0) = 0, \quad x(1) = 1.$

1.xxx Postscript: Dealing with infinite horizon problems

Recall problem 5 in the last section; the trajectory is found to be of the form:

$$x(t) = Ae^{\alpha t} + Be^{\beta t} + Ce^{-\alpha t} + De^{-\beta t},$$

where $\alpha > \beta > 0$ and $\alpha \neq \beta$. The problem specified $x(0), \dot{x}(0)$ as given and required $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} \dot{x}(t) = 0$. It is natural to set $A = B = 0$ and continue with the problem. In this note we justify the procedure. It will be clear that the method extends beyond just two exponentially growing terms.

We take our terminal conditions in the form:

$$x(T) = \dot{x}(T) = 0.$$

This yields the following matrix equation of boundary conditions:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & -\alpha & -\beta \\ e^{\alpha T} & e^{\beta T} & e^{-\alpha T} & e^{-\beta T} \\ \alpha e^{\alpha T} & \beta e^{\beta T} & -\alpha e^{-\alpha T} & -\beta e^{-\beta T} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} x(0) \\ \dot{x}(0) \\ 0 \\ 0 \end{bmatrix}.$$

By Cramer's rule we obtain:

$$A_T = \frac{\text{linear combinations of:}(e^{(\beta-\alpha)T}, e^{-(\beta+\alpha)T}, 1)}{-1(\alpha-\beta)^2 e^{(\alpha+\beta)T} + \dots},$$

The numerator is obvious from the expansion by the first column; the critical term of the denominator comes from the bottom row, corner and adjacent term both of which create the appropriate term by multiplication with the top right subdeterminant of size 2. It now follows that

$$\lim_{T\to\infty} A_T = 0.$$

Similarly,

$$B_T = \frac{\text{linear combinations of:}(e^{(\alpha-\beta)T}, e^{-(\beta+\alpha)T}, 1)}{-1(\alpha-\beta)^2 e^{(\alpha+\beta)T} + \ldots},$$

and again

$$\lim_{T\to\infty} B_T = 0.$$

We consider a similar problem in the Exercises (11?): **Exercise:** For the problem:

$$\min \int_0^T (x^2 + \dot{x}^2)\, dt$$

subject to

$$\int_0^T xe^{-2t}\, dt = 1 \quad \text{and } x(0) = 1, \quad x(T) = 0,$$

the trajectory is known to be of the form

$$x(t) = Ae^t + Be^{-t} + Ce^{-2t},$$

show that

$$\lim_{T\to\infty} A_T = 0.$$

Solution: The two boundary conditions and the constraint equation lead to the matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ e^T & e^{-T} & e^{-2T} \\ 1-e^{-T} & \frac{1}{3}(1-e^{-3T}) & \frac{1}{4}(1-e^{-4T}) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The matrix has determinant with dominant term $-e^T\left(\frac{1}{4}-\frac{1}{3}\right)=\frac{1}{12}e^T$, so

$$A = A_T = \frac{\text{linear combinations of:}(e^{-T}, e^{-2T}, e^{-\beta T}, e^{-3T}, e^{-4T}, e^{-5T}, 1)}{\frac{1}{12}e^T + ...} \to 0.$$

A similar calculation yields

$$\begin{aligned} B_T &= -e^T(\frac{1}{4}-1)\cdot\frac{\text{linear combinations of:}(e^{-T}, e^{-2T}, e^{-\beta T}, e^{-3T}, e^{-4T}, e^{-5T}, 1)}{\frac{1}{12}e^T + ...} \\ &\to \frac{3}{4}\cdot 12 = 9, \end{aligned}$$

but it is easier to set $A = 0$, now that that has beeen justified, and to use the initial condition and the constraint equation (to ∞).