# MA409. Solutions to 2009 Examination

## 0.1 Question 1

(a) Lemma that if a continuous function $g$ has the property that

$$\int_0^T g\dot{h}dt = 0$$

for all continuously differentiable function $h$ with $h(0) = h(T) = 0$, then for some constant $c$ we have

$$g(t) = c.$$

(b) If $x$ is an extremal trajectory consider any continuously differentiable function $h$ with $h(0) = h(T) = 0$. Thus for scalar $s$ the the functions $x + sh$ satisfy the end-point constraint.

Now the function

$$\psi(s) = F(x + sh) = \int_0^T f(x + sh, \dot{x} + s\dot{h}, t)dt$$

has an extremum at $s = 0$. Thus $\psi'(0) = 0$. We compute that

$$\begin{aligned}\psi'(0) &= \int_0^T (f_x h + f_{\dot{x}}\dot{h})dt \\ &= \int_0^T (f_{\dot{x}} - F)\dot{h}dt + [Fh]_0^T \\ &= \int_0^T (f_{\dot{x}} - F)\dot{h}dt = 0,\end{aligned}$$

where

$$F(t) = \int_0^t f_x(x(u), \dot{x}(u), u)du.$$

We apply the lemma that if a continuous function $g$ has the property that

$$\int_0^T g\dot{h}dt = 0$$

for all continuously differentiable function $h$ with $h(0) = h(T) = 0$, then for some constant $c$ we have

$$g(t) = c.$$

Here we conclude that

$$f_{\dot{x}} - F = c$$

i.e.

$$f_{\dot{x}}(x(t), \dot{x}(t), t) = \int_0^t f_x(x(u), \dot{x}(u), u)du + c.$$

Since the right-hand side is differentiable with respect to $t$ we conclude the equation

$$\frac{d}{dt}\left(f_{\dot{x}}(x(t), \dot{x}(t), t)\right) = f_x(x(t), \dot{x}(t), t)$$

holds along the optimal trajectory $x(t)$assuming such a trajectory exists.

(c) Formula:

$$f_{\dot{x}}|_T = 0.$$

If $x(T)$ is unrestricted we consider $h$ with $h(0) = 0$ as the only restriction. As the set of admissible functions $h$ is now larger, the E-L equation holds. Now we have

$$\begin{aligned}\psi'(0) &= \int_0^T (f_x h + f_{\dot{x}} \dot{h}) dt \\ &= \int_0^T (\frac{d}{dt} f_{\dot{x}} - f_x) dt + [f_{\dot{x}} h]_0^T \\ &= h(T)\ f_{\dot{x}}|_T\,.\end{aligned}$$

Take $h(T) = 1$. Since $\psi'(0) = 0$ we have

$$f_{\dot{x}}|_T = 0.$$

(d) We are to minimize the functional

$$F(x) = \int_0^T \left(x(t)^4 + x(t)^2 \dot{x}(t)^2\right)\ dt$$

subject to $x(0) = 1$. Here the natural boundary condition is

$$0 = f_{\dot{x}}|_T = 2x(T)^2 \dot{x}(T), \text{ i.e. } \dot{x}(T) = 0 \text{ or } x(T) = 0.$$

The integrated form is:

$$\begin{aligned} f - \dot{x} f_{\dot{x}} &= c \\ x^4 + x^2\dot{x}^2 - \dot{x}(x^2 2\dot{x}) &= c \\ x^4 - x^2 \dot{x}^2 &= c \\ \dot{x}^2 &= \frac{x^4 - c}{x^2}, \text{ if } x \neq 0. \\ \frac{dx}{dt} &= \pm \frac{\sqrt{x^4 - c}}{x} \\ \frac{2x dx}{\sqrt{x^4 - c}} &= \pm 2 dt \text{ (put } z = x^2) \\ \int \frac{dz}{\sqrt{z^2 - c}} &= \pm 2t + b. \end{aligned}$$

(i) If $c = 0$, then since $x(0) = 1$, we must have

$$\begin{aligned} \dot{x}^2 &= x^2, \\ \dot{x} &= x \text{ or } \dot{x} = -x. \end{aligned}$$

So we obtain

$$x(t) = e^t \text{ or } x(t) = e^{-t},$$

which fails the natural boundary condition.

(ii) If $c = a^2 > 0$ with $a > 0$, put $z = a \cosh u$ so that $dz = a \sinh u du$.

$$\int \frac{a \sinh u du}{\sqrt{a^2 \cosh^2 u - a^2}} = u = 2t + b,$$

so

$$\begin{aligned} x^2 &= z = a \cosh(2t + b) = \frac{a}{2}(e^{2t+b} + e^{-2t-b}), \\ 1 &= \frac{a}{2}(e^b + e^{-b}) \end{aligned}$$

so

$$2x(T)\dot{x}(T) = a(e^{2T+b} - e^{-2T-b}).$$

Hence

$$0 = e^{2T+b} - e^{-2T-b} \text{ or } 2T + b = -2T - b, \text{ i.e. } b = -2T$$

and so

$$1 = a\cosh(2T).$$

(iii) If $c = -a^2$ with $a > 0$, put $z = a\sinh u$ so that $dz = a\cosh u du$.

$$\int \frac{dz}{\sqrt{z^2+a^2}} = \int \frac{a\cosh u du}{a\sqrt{\sinh^2 u + 1}} = u = \pm 2t + b.$$

So

$$\begin{aligned} x^2 &= z = a\sinh(2t+b) = \frac{a}{2}(e^{2t+b} - e^{-2t-b}), \\ 1 &= \frac{a}{2}(e^b - e^{-b}). \end{aligned}$$

Here

$$2x(T)\dot{x}(T) = a(e^{2T+b} + e^{-2T-b})$$

and this is never zero unless $a = 0$, in which case $x(0) = 1$ is impossible.

**Remark.** The alternative approach (not encouraged) via the E-L equation runs as follows:

$$\frac{d}{dt}\left(2x^2\dot{x}\right) = 4x^3 + 2x\dot{x}^2,$$

or

$$2x\dot{x}^2 + x^2\ddot{x} = 2x^3 + x\dot{x}^2,$$

i.e. either $x(t) = 0$ or

$$\dot{x}^2 + x\ddot{x} = 2x^2.$$

This equation may be solved by putting $y = x^2$. Then $\dot{y} = 2x\dot{x}$ and $\ddot{y} = 2\dot{x}^2 + 2x\ddot{x}$ and so

$$\ddot{y} = 4y.$$

The auxiliary equation is $m^2 - 4 = 0$. Thus

$$x^2 = y = Ae^{2t} + Be^{-2t}.$$

Since $x(T)\dot{x}(T) = 0$ we have $\dot{y}(T) = 0$. Hence

$$\begin{aligned} A + B &= 1, \\ 2Ae^{2T} - 2Be^{-2T} &= 0. \end{aligned}$$

So

$$B = Ae^{4T}$$

and

$$A = \frac{1}{1+e^{4T}} = \frac{e^{-2T}}{e^{2T}+e^{-2T}}, \; B = \frac{e^{2T}}{e^{2T}+e^{-2T}}.$$

## 0.2 Question 2

(a) We assume the Relative Stationarity Condition viz.

$$DG(x)h = 0 \Rightarrow DF(x)h = 0$$

where $F : X \to R$, $G : X \to Y$ and we read this to mean that

$$DF(x) \perp h$$

for each $h$ in $N(DG(x))$, i.e. that

$$DF(x) \in N(DG(x))^{\perp}.$$

But by the Duality Theorem

$$N(DG(x))^{\perp} = R(DG(x)^{*})$$

so for some $\mu^* \in Y^*$ we have for all $h$

$$DF(x)h = D(G(x)^* \mu^*(h),$$

or, by the definition of the adjoint ( $A^* y^*(x) = y^*(Ax)$), we have

$$DF(x)h = \mu^*(DG(x)h).$$

But since $\mu^*$ is linear

$$DF(x)h = D(\mu^* G(x))h$$

Thus we obtain that

$$D\{F(x) - \mu^* G(x)\} = 0$$

i.e the Lagrangian

$$F(x) - \mu^* G(x) = 0$$

is stationary for some $\mu^*$ in $Y^*$.

(b) We apply the method of Lagrange multipliers to solve for $(x(t), y(t))$ the constrained problem of maximizing

$$\int_0^1 e^{-rt}(x(t)^{1/2} + 2y(t)^{1/2})\, dt,$$

where $r > 0$ is a constant, subject to

$$\int_0^1 (x(t) + y(t))\, dt = 1,$$

with $x(t) \geq 0, y(t) \geq 0$. For $\mathcal{X}$ take the continuously differentiable functions with values in $\mathbb{R}^2$ and $\mathcal{Y} = \mathbb{R}$.

We take with $\lambda \in \mathbb{R}$

$$\begin{aligned} L &= \int_0^1 e^{-rt}(x(t)^{1/2} + 2y(t)^{1/2})\, dt - \lambda \left[ \int_0^1 (x(t) + y(t))\, dt - 1 \right] \\ &= \int_0^1 e^{-rt}(x(t)^{1/2} + 2y(t)^{1/2} - \lambda(x(t) + y(t)))\, dt - \lambda. \end{aligned}$$

Applying the E-L eqns in $x$ and $y$ we obtain

$$\begin{aligned} 0 &= \frac{d}{dt}(0) = e^{-rt}\frac{1}{2}x(t)^{-1/2} - \lambda, \\ 0 &= \frac{d}{dt}(0) = e^{-rt}\frac{1}{2}2y(t)^{-1/2} - \lambda \end{aligned}$$

Hence for all $t$

$$\begin{aligned} e^{-rt}\frac{1}{2}x(t)^{-1/2} &= \lambda = e^{-rt}\frac{1}{2}2y(t)^{-1/2}, \\ x(t)^{-1/2} &= 2y(t)^{-1/2} \\ x(t) &= \frac{1}{4}y(t). \end{aligned}$$

Thus $\lambda$ is non-zero, and so we have

$$x(t) = \frac{1}{4\lambda^2}e^{-2rt}$$

and so

$$\begin{aligned} \int_0^1 5x(t)\,dt &= 1, \\ \frac{5}{4\lambda^2}\int_0^1 e^{-2rt}\,dt &= 1, \\ \frac{5}{8r\lambda^2}(1-e^{-2r}) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} x(t) &= \frac{2r}{5}e^{-2rt}(1-e^{-2r})^{-1}, \\ y(t) &= \frac{8r}{5}e^{-2rt}(1-e^{-2r})^{-1}. \end{aligned}$$

Hence the maximum is

$$\begin{aligned} & \int_0^1 e^{-rt}(x(t)^{1/2}+2y(t)^{1/2})\,dt \\ = \;& K\int_0^1 e^{-2rt}dt = \frac{K}{2r}(1-e^{-2r}), \end{aligned}$$

where

$$\begin{aligned} K &= \left(\frac{2r}{5}\right)^{1/2}(1-e^{-2r})^{-1/2}(1+4) \\ &= \sqrt{10r}(1-e^{-2r})^{-1/2}. \end{aligned}$$

So the value is

$$\sqrt{\frac{10}{4r}(1-e^{-2r})}.$$

## 0.3 Question 3

Before stating the Pontryagin Principle, we need to identify various equations.

The *state equation* is

$$\ddot{x} = 4\dot{x} - 3x + u, \qquad |u| \le 1.$$

Now the first-order formulation with $x_1 = x, x_2 = x'$ is

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -3x_1 + 4x_2 + u, \end{aligned}$$

with $|u| \le 1$. Thus the *Hamiltonian* is

$$H = 1 + \lambda_1 x_2 + \lambda_2(-3x_1 + 4x_2 + u).$$

Let $u^*(t)$ be the optimal control function and let the corresponding optimal state trajectory be $x = (x_1, x_2)$ and let the co-state variables be defined by the *co-state equations*:

$$-\dot{\lambda}_1 = \frac{\partial H}{\partial x_1}, -\dot{\lambda}_2 = \frac{\partial H}{\partial x_2}.$$

The Pontryagin Principle asserts that the Hamiltonian, as a function of $u$ only, is minimized by setting $u = u^*(t)$ when $x$ and $\lambda$ are the state and co-state trajectories.

(i) In the current situation the principle asserts that

$$\begin{aligned} u &= +1, \text{ for } \lambda_2 < 0, \\ &= -1, \text{ for } \lambda_2 > 0. \end{aligned}$$

To determine the behaviour of the co-state variable explicitly, we have in this case

$$-\dot{\lambda}_1 = -3\lambda_2, -\dot{\lambda}_2 = \lambda_1 + 4\lambda_2,$$

so

$$-\lambda_2'' = \lambda_1' + 4\lambda_2' = 3\lambda_2 + 4\lambda_2'$$
$$\lambda_2'' + 4\lambda_2' + 3\lambda_2 = 0.$$

The auxiliary equation

$$m^2 + 4m + 3 = 0$$

or

$$(m+3)(m+1) = 0$$

tell us the associated eigenvalues are $-1, -2$ . Now

$$\begin{aligned} \lambda_2 &= Ae^{-t} + Be^{-3t} \\ &= e^{-t}\{A + Be^{-2t}\} \end{aligned}$$

so $\lambda_2$ changes sign at most once.

(ii) We must now study the trajectories of constant control $u = u^* = \pm 1$. Now

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -3x_1 + 4x_2 + u, \end{aligned}$$

so the singular points ( $x_1' \equiv 0, x_2' \equiv 0$) are

$$x_2 = 0, x_1 = u^*/3.$$

Letting $y_1 = x_1 - u^*/3, y_2 = x_2$ we have

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -3y_1 + 4y_2 \end{aligned}$$

so that the characteristic equation is

$$0 = \begin{vmatrix} -\mu & 1 \\ -3 & 4-\mu \end{vmatrix} = \mu^2 - 4\mu + 3 = (\mu - 1)(\mu - 3)$$

so that $\mu = 1$ or $\mu = 3$. Alternatively note that

$$y_1'' - 4y_1' + 3y_1 = 0.$$

The auxiliary equation

$$m^2 - 4m + 3 = 0$$

or

$$(m - 3)(m - 1) = 0$$

tell us the associated eigenvalues are $2, 1$ which are of the same sign.Thus

$$y_1 = Ke^{3t} + Le^t$$

and so

$$y_2 = y_1' = 3Ke^{3t} + Le^t.$$

It follows that we obtain two linear trajectories:

$$\frac{y_2}{y_1} = \frac{3Ke^{3t} + Le^t}{Ke^{3t} + Le^t} = 3, \text{ for } L = 0$$

this one having state receding from the singular point as time tends to plus infinity since $|K|e^{2t} \to \infty$ (i.e. the singular state is repelling) and the other

$$\frac{y_2}{y_1} = +1, \text{ for } K = 0$$

and corresponds again to the state leaving the singular point (again a repelling state).

The general trajectory relative to the *canonical co-ordinates* (using eigenspaces as axes), however, takes the form

$$Y_1^3 = (e^t)^3 = (e^{3t}) = Y_2$$

which is thus a cubic.

(iii) See sketch to be attached.

(iv) Choosing for each singular point $(\pm\frac{1}{3}, 0)$ a trajectory passing through the origin in a direction away from the singular point we obtain two curves meeting at the origin which together form the switching curve. Optimal control is available only for initial states in the open finite region indicated in the sketch.

## 0.4 Question 4

**Derivation.** We argue that if $x$ is the minimizer, then with $s = t - \Delta t$ and with $y(s) = x(s + \Delta t)$

$$S(c,T) = \min_{x(0)=c} \int_0^T (x^4 + x^2\dot{x}^2)\, dt.$$

For any $x$ we have

$$\begin{aligned}
J(c,T) &= \int_0^T (x^4 + x^2\dot{x}^2)\, dt \\
&= \int_0^{\Delta t} (x^4 + x^2\dot{x}^2)\, dt + \int_{\Delta t}^T (x^4 + x^2\dot{x}^2)\, dt \\
&= (x^4 + x^2\dot{x}^2)\Delta t + \int_0^{T-\Delta t} (x^4(s+\Delta t)^r + x^2(s+\Delta t)\dot{x}^2(s+\Delta t))ds \\
&= (x^4 + x^2\dot{x}^2)\Delta t + \int_0^{T-\Delta t} (y^4(s)^r + y(s)^2\dot{y}(s)^2)ds \\
&= (x^4(\theta\Delta t) + x^2(\theta\Delta t)\dot{x}^2(\theta\Delta t))\Delta t + J(x(\Delta t), T - \Delta t)
\end{aligned}$$

for some $\theta$ with $0 < \theta < 1$.

Hence by the MeanValue Theorem, assuming $V(x,t)$ is differentiable:Hence by the MeanValue Theorem, assuming $V(x,t)$ is differentiable:

$$\begin{aligned}
0 &= (x^4(\theta\Delta t) + x^2(\theta\Delta t)\dot{x}^2(\theta\Delta t))\Delta t + J(x(\Delta t), T - \Delta t) - J(c,T), \\
&= (x^4(\theta\Delta t) + x^2(\theta\Delta t)\dot{x}^2(\theta\Delta t))\Delta t + [J_c\dot{x}(\phi\Delta t) - J_T]\,\Delta t
\end{aligned}$$

for some $\phi$ with $0 < \phi < 1$.

Dividing by $\Delta t$ and passing to the limit we obtain

$$0 = (x^4(0) + x^2(0)\dot{x}^2(0)) + [J_c\dot{x}(0) - J_T]$$

For arbitrary $x$ we have merely

$$V(c,T) \le (x(\theta\Delta t)^r + \dot{x}(\theta\Delta t)^r)\Delta t + V(x(\Delta t), T - \Delta t).$$

and so

$$0 \le (x^4(0) + x^2(0)\dot{x}^2(0)) + [J_c\dot{x}(0) - J_t]$$

In general thus putting $w = \dot{x}(0)$ we have

$$0 = \min_w\{(c^4 + c^2w^2) + [J_cw - J_T]\}.$$

The substitution $z(t) = x(t)/c$ easily yields

$$S(c,T) = c^4S(1,T).$$

Thus the Bellman equation reads

$$0 = \min[c^4 + c^2v^2 + 4vc^3G - c^4G'].$$

FOC is

$$\begin{aligned}
& 2vc^2 + 4c^3G \\
v &= -2cG
\end{aligned}$$

Substitution back yields:

$$\begin{aligned} c^4 + 4c^2c^2G^2 - 8c^4G^2 - c^4G' &= 0, \\ 1 - 4G^2 - G' &= 0, \\ \frac{dG}{dT} &= 1 - 4G^2 \end{aligned}$$

So

$$\begin{aligned} T &= \int \frac{dG}{1-4G^2} = \frac{1}{2}\int \frac{1}{1-2G} + \frac{1}{1+2G} dG \\ 4T &= \log(1+2G) - \log(1-2G) \\ \frac{1+2G}{1-2G} &= Ae^{4T} \end{aligned}$$

$T = 0$ yields $G = 0$ and so

$$1 = A.$$

Hence

$$\begin{aligned} \frac{1+2G}{2} &= \frac{e^{4T}}{1+e^{4T}}, \\ G &= \frac{1}{2}\frac{e^{4T}-1}{1+e^{4T}} \end{aligned}$$

Hence

$$\lim G(T) = \frac{1}{2}.$$

(ii)

$$v = -2cG = -c.$$

Solving

$$\dot{x} = -x,$$

we obtain

$$x = Ae^{-t} = e^{-t}.$$

## 0.5 Question 5

We have for the optimal $u$ if it exists and under the smoothness assumptions that

$$\begin{aligned}
C(x) &= C(x,0) = E[\int_0^\infty e^{-\rho s}(aX_s^2 + bu_s^2)ds \\
&= E[\int_0^{\Delta t} e^{-\rho s}(aX_s^2 + bu_s^2)ds + E[\int_{\Delta t}^\infty e^{-\rho s}(aX_s^2 + bu_s^2)ds, \text{ put } \theta = s - \Delta t \\
&= E[\int_0^{\Delta t} e^{-\rho s}(aX_s^2 + bu_s^2)ds + E[\int_0^\infty e^{-\rho(\theta+\Delta t)}(aX_{\theta+\Delta t}^2 + bu_{\theta+\Delta t}^2)ds \\
&= E[\int_0^{\Delta t} e^{-\rho s}(aX_s^2 + bu_s^2)ds + e^{-\rho\Delta t}E[\int_0^\infty e^{-\rho\theta}(aY_\theta^2 + bv_\theta^2)ds,
\end{aligned}$$

where $\Delta t > 0$ and

$$Y_\theta = X_{\theta+\Delta t}, v(\theta) = u(\theta + \Delta t).$$

Then conditioning on the information available at time $\Delta t$ , $Y_s$ solves

$$dY_s = uds + \sigma Y_s dz_s,$$

with $Y_0 = X_{\Delta t} = x + \Delta x$.

Using Ito's formula in the intuitive form:

$$C(x + \Delta x) = C(x) + [C'(x)u + \frac{1}{2}\sigma^2 x^2 C''(x)]\Delta t + C'(x)\sigma x \Delta z_t + o(\Delta t)]$$

we obtain for the optimal $u_0$:

$$\begin{aligned}
C(x) &= E_0 E_{\Delta t}[(ax^2 + bu_0^2)\Delta t + e^{-\rho\Delta t}C(x + \Delta x)] \\
&= E_0 E_{\Delta t}[(ax^2 + bu_0^2)\Delta t + (1 - \rho\Delta t)C(x + \Delta x)] \\
&= E_0 E_{\Delta t}[(ax^2 + bu_0^2)\Delta t + (1 - \rho\Delta t)[C(x) + [C'(x)u + \frac{1}{2}\sigma^2 x^2 C''(x)]\Delta t + C'(x)\sigma x \Delta z_t + o(\Delta t)]] \\
&= E_0 E_{\Delta t}[(ax^2 + bu_0^2)\Delta t + [C(x) + [C'(x)u + \frac{1}{2}\sigma^2 x^2 C''(x)]\Delta t + C'(x)\sigma x \Delta z_t + o(\Delta t)]] \\
&\quad -\rho\Delta t C(x).
\end{aligned}$$

Thus

$$0 = (ax^2 + bu_0^2)\Delta t - \rho\Delta t C(x) + [C'(x)u + \frac{1}{2}\sigma^2 x^2 C''(x)]\Delta t + o(\Delta t).$$

Dividing by $\Delta t$ and passing to the limit we obtain

$$0 = (ax^2 + bu_0^2) - \rho C(x) + [C'(x)u + \frac{1}{2}\sigma^2 x^2 C''(x)].$$

For a general $u$ we obtain only an inequality and so

$$\rho C(x,0) = \min_u [ax^2 + bu^2 + u\frac{\partial C(x,0)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C(x,0)}{\partial x^2}].$$

Minimization over $u$ gives

$$u = -C_x/2b$$

and so

$$\rho C = ax^2 - \frac{1}{4b}{C_x}^2 + \frac{1}{2}\sigma^2 x^2 C_{xx}.$$

We try a solution of the form $C = Ax^\gamma$ and obtain

$$\rho A x^\gamma = ax^2 - \frac{1}{4b}A^2\gamma^2 x^{2\gamma-2} + \frac{1}{2}\sigma^2 x^\gamma A\gamma(\gamma - 1).$$

This will be an identity if $\gamma = 2$ and $A$ satisfies

$$\rho A = a - \frac{1}{b}A^2 + \sigma^2 A,$$

i.e.

$$A^2 + (\rho - \sigma^2)bA - ab = 0.$$

There are two roots $A_{\pm}$, one positive, one negative.

In view of the positive integrand defining $C$, we have $C \geq 0$. Hence $A = A_+$.

**Remark.** Note that since

$$u = -Ax/b$$

and as we seek a minimum, we must take $A = A_+$ in order to reduce $X(t + dt)$ in expectation. Thus

$$u(t) = -A_+ x(t)/b.$$

Note that $C(x, t) = e^{-\rho t} C(x, 0) = e^{-\rho t} A_+ x^2$.

## 0.6 Question 6

(a) (i) The directional derivative is defined by

$$D_hF(x) = \lim_{s\to 0}\frac{F(x+sh)-F(x)}{s}$$

when this exists. This may be rewritten as

$$\lim_{s\to 0}\frac{\psi_h(s)-\psi_h(0)}{s}$$

where reference is made to

$$\psi_h(s) = F(x+sh)$$

and so if it exists it is equal to

$$\psi_h'(0).$$

(ii) The strong derivative if it exists is the (continuous) linear transformation $A$ satisfying: for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|F(x+h) - F(x) - Ah| \le \varepsilon ||h||$$

for any $h$ with $||h|| \le \delta$.

If the strong derivative exists we have for any $h$ that if $s$ is so small that $||sh|| \le \delta$ then

$$||F(x+sh) - F(x) - sAh|| \le \varepsilon ||sh||$$

whence for $s \neq 0$

$$||\frac{F(x+sh)-F(x)}{s} - Ah|| \le \varepsilon ||h||$$

and so $D_hF(x) = A$.

(b) When

$$S(x)(t) = \dot{x}(t)^3$$

we have

$$(\dot{x}(t)+\dot{h}(t))^3 - \dot{x}(t)^3 - 3\dot{x}(t)^2\dot{h}(t) = 3\dot{x}(t)\dot{h}(t)^2 + \dot{h}(t)^3$$

so

$$\begin{aligned} ||(\dot{x}(t)+\dot{h}(t))^3 - \dot{x}(t)^3 - 3\dot{x}(t)^2\dot{h}(t)|| &= ||3\dot{x}(t)\dot{h}(t)^2 + \dot{h}(t)^3|| \\ &\le ||3\dot{x}(t)\dot{h}(t)^2|| + ||\dot{h}(t)^3|| \\ &\le 3||\dot{h}(t)^2|| + ||\dot{h}(t)^3|| \\ &\le 4\varepsilon||\dot{h}(t)|| \le 4\varepsilon||h||_1. \end{aligned}$$

Put

$$A(h) = 3\dot{x}^2\dot{h}.$$

Then $A$ is linear in $h$ and

$$||S(x+h) - S(x) - Ah||_\infty = 4\varepsilon||h||_1.$$

Thus given $\varepsilon > 0$ with $\varepsilon < 1$ we may take $\delta = \varepsilon$ and then for $h$ with $||h||_1 \le \delta$ we have $||\dot{h}||_\infty \le \delta < 1$ and so

$$||S(x+h) - S(x) - Ah||_\infty \le 4\varepsilon||\dot{h}||_\infty \le 4\varepsilon||h||_1.$$

(c) **9 marks**
(i)

$$G(x+sh, \tau+s\sigma) = ((x+sh)(0), (x+sh)(\tau+s\sigma) - q(\tau+s\sigma))$$

Differentiate wrt $s$ yields

$$(h(0), \sigma\dot{x}(\tau + s\sigma) + h(\tau + s\sigma) + \sigma s\dot{h}(\tau + s\sigma) - \sigma\dot{q}(\tau + s\sigma))$$

and followed by setting $s = 0$ yields

$$G_u(x, \tau) = (h(0), \sigma\dot{x}(\tau) + h(\tau) - \sigma\dot{q}(\tau)).$$

(ii)

$$F(x + sh, \tau + s\sigma) = \int_0^{\tau+s\sigma} f(x + sh, \dot{x} + s\dot{h}, t)dt$$

and so differentiation wrt $s$ yields

$$\begin{aligned} & F(x + sh, \tau + s\sigma) \\ = \; & \sigma f(x + sh, \dot{x} + s\dot{h}, \tau + s\sigma) + \int_0^{\tau+s\sigma} [hf_x(x + sh, \dot{x} + s\dot{h}, t) + \dot{h}f_{\dot{x}}(x + sh, \dot{x} + s\dot{h}, t)]dt \end{aligned}$$

Hence

$$F_u(x, \tau) = \sigma f(x(\tau), \dot{x}(\tau), \tau) + \int_0^{\tau} [hf_x(x, \dot{x}, t) + \dot{h}f_{\dot{x}}(x, \dot{x}, t)]dt$$

(iii) The condition $G_u = 0$ yields $h(0) = 0$ and

$$h(\tau) = \sigma(\dot{q}(\tau) - \dot{x}(\tau)).$$

Under these conditions we have

$$0 = \sigma f(x(\tau), \dot{x}(\tau), \tau) + \int_0^{\tau} [hf_x(x, \dot{x}, t) + \dot{h}f_{\dot{x}}(x, \dot{x}, t)]dt$$

Taking first $\sigma = 0$ yields $h(\tau) = 0$ and

$$0 = \int_0^{\tau} [hf_x(x, \dot{x}, t) + \dot{h}f_{\dot{x}}(x, \dot{x}, t)]dt.$$

This implies the Euler-Lagrange equation holds, and we are instructed to assume it.

For $\sigma = 1$, say, we have

$$\begin{aligned} 0 &= f(x(\tau), \dot{x}(\tau), \tau) + \int_0^{\tau} [hf_x(x, \dot{x}, t) + \dot{h}f_{\dot{x}}(x, \dot{x}, t)]dt \\ &= f(x(\tau), \dot{x}(\tau), \tau) + [hf_{\dot{x}}(x, \dot{x}, t)]_0^{\tau} + \int_0^{\tau} h[f_x(x, \dot{x}, t) - \frac{d}{dt}f_{\dot{x}}(x, \dot{x}, t)]dt \\ &= f(x(\tau), \dot{x}(\tau), \tau) + (\dot{q}(\tau) - \dot{x}(\tau))f_{\dot{x}}(x, \dot{x}, \tau), \end{aligned}$$

as required.