Exercises on Time Optimality

1. Transfer a moving point of the x-axis to the origin in shortest time under the governing equation

$$\ddot{x} + x = u \qquad |u| \leq 1.$$

2. Use the transversality condition to drive a moving point of the x-axis governed by the law $\ddot{x} = u$ with $|u| \leq 1$ to the origin (i.e. without requiring the terminal velocity also to be zero).

3. A dynamical system behaves according to the law:

$$\ddot{x} + 4\dot{x} + x = u$$

where $|u| \leq 1$. It is required to steer the system to the state $x = \dot{x} = 0$ in minimum time. Show that the phase plane trajectories for $u = \pm 1$ tend asymptotically to the singular points. Use the Pontryagin Principle to obtain the switching curve for this problem.

4. State the Pontryagin Principle in a form appropriate to finding the trajectory of the dynamical system:

$$\dot{x} = y,$$

$$\dot{y} = x + u,$$

$$| u | \leq 1,$$

which takes the system in shortest time to rest at the origin $(x = 0, y = 0)$.

(i) Show that the co-state variable corresponding to y is either of constant sign or else changes sign at most once. Deduce that the control variable is piecewise constant and equal to either $+1$ or -1 and changes sign at most once.

(ii) Sketch the trajectories $u = +1$ in the phase space (x, y) indicating the direction of change with time. Repeat this for $u = -1$.

Deduce that if the state of the dynamical system is initially in the strip bounded by $x + y = -1$ and $x + y = +1$ then the shortest time problem has a solution.

5. State the Pontryagin Principle in a form suited to deriving the trajectory of minimum time taking the dynamical system with governing equation:

$$\ddot{x} = -5\dot{x} - 6x + u, \qquad |u| \leq 1$$

from a given initial state to rest at the origin (i.e. $x = 0, \quad \dot{x} = 0$).

Use the Pontryagin Principle to show that the optimal control is of 'bang-bang' type and that at most one switch of control takes place.

Find the singular points in the $(x, \dot{x})$ phase plane corresponding to the two constant controls $u = \pm 1$ and the linear trajectories through them. By considering the eigenvalues of the associated first-order formulation say what shape of trajectories to expect in general. Show that all trajectories for each of the two controls $u = \pm 1$ except one approach a singular point tangentially to just one of the linear trajectories.

Sketch the trajectories $u = \pm 1$ which pass through the origin of the phase plane. Sketch also the switching curve, being careful to indicate how it is used to obtain optimal trajectories from any initial state in the phase plane.

6. Discuss the time optimality problem for the dynamical system:

$$\ddot{x} = -2x - 3\dot{x} + u, \quad |u| \leq 1.$$

7. Discuss the time optimality problem for the dynamical system:

$$\ddot{x} = \frac{1}{6}(x + \dot{x}) + u, \quad |u| \leq 1.$$

8. If $\mathbf{v}_1, \mathbf{v}_2$ are the eigenvectors of A, show that the linear trajectories are in the directions $\mathbf{v}_1, \mathbf{v}_2$. (See hint below.)

9. When the eigenvalues of A are distinct, show that A *is symmetric, if and only if, the eigenvectors of* A *are orthogonal, if and only if, the linear trajectories are orthogonal.*

Hint: With respect to the basis of eigenvectors the dynamical system is:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and $X_1 = 0$ and $X_2 = 0$ are then the linear trajectories, e.g. for $X_1 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X_2 \cdot \mathbf{v}_2 \quad (X_2 \in \mathbf{R})$$

etc. and if $\mathbf{v}_1, \mathbf{v}_2$ are orthonormal

$$A = (\mathbf{v}_1, \mathbf{v}_2) \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (\mathbf{v}_1, \mathbf{v}_2)^T$$

is symmetric.

10. Verify the result of the last exercise, by reference to the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as follows. Working with the singular point as origin, write $x_1 = Be^{\alpha t} + Ce^{\beta t}$, compute appropriate k, l from a, b, c, d such that $x_2 = kBe^{\alpha t} + lCe^{\beta t}$. Observe that the linear trajectories are: $x_2 = kx_1$ (for $C = 0$) and $x_2 = lx_1$ (for $D = 0$). These are orthogonal when $kl = -1$.

11. Assuming the eigenvalues of A are negative show that the trajectories with $u = \pm 1$ are all but one tangential to a linear trajectory passsing through the singular point, the exceptional trajectory being the other linear trajectory.

Other things to do

1. Use Plot and ParametricPlot in Maple (or Mathematica) to derive the shapes of trajectories.

2. Investigate trajectories for the case of repeated eigenvalues. Plot the curves $y = c \log x^x$ and $y = x + k \log |x|$.