Foundations of Regular Variation
by

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In memoriam Paul Erdős, 1913-1996

Abstract. The theory of regular variation is largely complete in one dimension, but is developed under regularity or smoothness assumptions. For functions of a real variable, Lebesgue measurability suffices, and so does having the property of Baire. We find here that the preceding two properties have two kinds of common generalization, both of a combinatorial nature; one is exemplified by 'containment up to translation of subsequences', the other, drawn from descriptive set theory, requires non-emptiness of a Souslin $\mathbf{\Delta}^1_2$-set. All of our generalizations are equivalent to the uniform convergence property.

1 *Introduction and results*

1.1 Preamble

The theory of regular variation, or of regularly varying functions, is a chapter in the classical theory of functions of a real variable, dating from the work of Karamata in 1930. It has found extensive use in probability theory, analysis (particularly Tauberian theory and complex analysis), number theory and other areas; see [BGT] for a monograph treatment, and [Kor] Chapter IV. Henceforth we identify our numerous references to [BGT] by BGT. The theory explores the consequences of a relationship of the form

$$f(\lambda x)/f(x) \to g(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0, \qquad (RV)$$

for functions defined on $\mathbb{R}_+$. The limit function g must satisfy the Cauchy functional equation

$$g(\lambda\mu) = g(\lambda)g(\mu) \qquad \forall \lambda, \mu > 0. \qquad (CFE)$$

Subject to a mild regularity condition, (CFE) forces g to be a power:

$$g(\lambda) = \lambda^\rho \qquad \forall \rho > 0. \qquad (\rho)$$

Then f is said to be *regularly varying* with *index* ρ, written $f \in R_\rho$.

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The case $\rho = 0$ is basic. A function $f \in R_0$ is called *slowly varying*; slowly varying functions are often written ℓ (for *lente*, or *langsam*). The basic theorem of the subject is the Uniform Convergence Theorem (UCT), which states that if

$$\ell(\lambda x)/\ell(x) \to 1 \qquad (x \to \infty) \qquad \forall \lambda > 0, \tag{SV}$$

then the convergence is *uniform* on compact λ-sets in $(0, \infty)$.

The basic facts are:
(i) if ℓ is (Lebesgue) measurable, then the UCT holds;
(ii) if ℓ has the Baire property (for which see e.g. Kuratowski [Kur], Oxtoby [Oxt]), then the UCT holds;
(iii) in general, the UCT need not hold.
Similarly, if f is measurable or has the Baire property, (CFE) implies (ρ), but not in general.

See BGT §§1.1, 1.2; for background on the Cauchy functional equation, see [Kucz],[AD]. Although in this context measure and category are interchangeable, we will warn the reader in Section 5 that interchangeability is not guaranteed.

The UCT extends easily to regularly as well as slowly varying functions; see BGT Th. 1.5.2. The basic case is $\rho = 0$, so we lose nothing by restricting attention to it here.

The basic foundational question in the subject, which we address here, concerns the search for natural conditions for the above to hold, and in particular for a substantial common generalization of measurability and the Baire property. We find such a common generalization, indeed two kinds of generalization, which are actually both necessary and sufficient. The paper thus answers an old problem noted in BGT p. 11 Section 1.2.5.

While regular variation is usually used in the multiplicative formulation above, for proofs in the subject it is usually more convenient to use an additive formulation. Writing $h(x) := \log f(e^x)$ (or $\log \ell(e^x)$ as the case may be), $k(u) := \log g(e^u)$, the relations above become

$$h(x+u) - h(x) \to k(u) \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}, \tag{RV_+}$$

$$h(x+u) - h(x) \to 0 \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}, \tag{SV_+}$$

$$k(u+v) = k(u) + k(v) \qquad \forall u, v \in \mathbb{R}. \tag{CFE_+}$$

Here the functions are defined on $\mathbb{R}$, whereas in the multiplicative notation functions are defined on $\mathbb{R}_+$.

In BGT, conditions are imposed on *functions* f. It is more helpful here to identify a function with its graph – so that $y = f(x)$ means $(x, y) \in f$, etc. Conditions are thus imposed on sets, and we are able to use the language of descriptive set theory, for which see e.g. [Mos], as in §2 below. See the comment in Section 5 on classification by reference to pre-images (which needs the material of Section 2).

It is convenient to describe the context of the Uniform Convergence Theorem (UCT) by writing

$$h_x(u) = h(u + x) - h(x)$$

and regarding $h_x(u)$, with x as parameter, as though it were an 'approximately-additive' function of u (a term defined explicitly in [Kucz] p. 424). Then, granted assumptions on the function h, (UCT) asserts that pointwise convergence of the family $\{h_x\}$ implies uniform convergence over compact sets of u.

The entire analysis rests on two key definitions and one purely set-theoretic *combinatorial principle* that can address practicalities within 'naive set theory' (without any need for formal axiomatics). In Section 5 we state, but do not prove, a new theorem in the style of Heiberg's Theorem aimed at testing for UCT by employing those of our results that rely only on 'naive set theory'. For a proof of this and of more powerful variants see the companion paper [BOst2].

The relevant combinatorial principle and its variants are defined in Section 1.2 and used to establish a number of interesting nececessary and sufficient conditions equivalent to UCT.

In Section 1.3 we find a further set of necessary and sufficient conditions equivalent to UCT which require some knowledge of 'definability' as formalized by Descriptive Set Theory. The point of view adopted there leads to connections with infinitary games (an alternative 'canon' for combinatorial principles, calling for further research into UCT). That subsection has been written so that it can be read with a minimal exposure to technicalities and so we recommend that analysts who prefer to depend only on 'naive set theory' should not flinch from reading this. Indeed, to place them in the right frame of mind, we recommend that they read the delightful article [Kan] tracing the development of modern set theory.

In Section 2 we address ourselves more carefully to the connections between Regular Variation Theory and Descriptive Set Theory but from the point of view of an analyst without a background in Descriptive Set Theory. Indeed, in Section 2 we first assemble the machinery that we shall need from Descriptive Set Theory (for which see [Rog], [Mos]). We then make the case that the family of sets Δ^1_2 plays the natural role in the theory of regular variation. The basis of the argument centres on the role of the lim sup operation and its variants. Our purpose in this section is to show how the theory of regular variation comes to depend on the axiomatic assumptions of set theory.

In Section 3 we clarify our approach to the sequence trapping at the heart of our *combinatorial principle* by reference to such standard notions as Luzin and Sierpiński sets, Hamel bases, and automatic continuity. That section requires only a background in 'naive' set theory. In Section 4 we collect proofs of all our results. In Section 5 we note some interesting open problems and offer further comments.

1.2 Uniform convergence and combinatorial principles

The combinatorial principles which are at the heart of UCT do not directly require additional axiomatic justification, so we will name them (with appropriate prefixes) **NoTrumps** (NT) by analogy with the set-theoretic combinatorial principles of various strengths. The latter were named after the card suits starting with Jensen's Diamond $\diamondsuit$ appearing in [Je] and ranging upwards and downwards to the weaker $\clubsuit$ introduced in [Ost1] and studied by Devlin [Dev3], where he refers to it as Ostaszewski's Principle, (OP). All of these follow from additional set-theoretic axioms. See for instance [Dev1], [Dev2] , [Kun], [DW] Chapter 7, [FSS] for details.

The concepts we need for our analysis are embodied in the following definitions. They have been extracted from a close reading of the standard treatment of UCT in BGT, but whilst only implicit there, here they are now identified as quintessential.

Definitions.

(i) The **ε-level set** (of h_x) is defined to be the set

$$H^\varepsilon(x) = \{t : |h(t+x) - h(x)| < \varepsilon\}.$$

(ii) For $\mathbf{x} = \{x_n : n \in \omega\}$ an arbitrary sequence tending to infinity, the **x-stabilized ε-level set** (of h) is defined to be the set

$$T_k^\varepsilon(\mathbf{x}) = \bigcap\nolimits_{n=k}^{\infty} H^\varepsilon(x_n) \text{ for } k \in \omega.$$

Here ω denotes the set of natural numbers $0, 1, 2, \ldots$. Note that

$$T_0^\varepsilon(\mathbf{x}) \subseteq T_1^\varepsilon(\mathbf{x}) \subseteq T_2^\varepsilon(\mathbf{x}) \subseteq \ldots \text{ and } T_k^\varepsilon(\mathbf{x}) \subseteq T_k^\eta(\mathbf{x}) \text{ whenever } \varepsilon < \eta. \qquad (1)$$

If h is slowly varying, then $\mathbb{R} = \bigcup_{k\in\omega} T_k^\varepsilon(\mathbf{x})$.

(iii) The basic **No Trumps** combinatorial principle (there are several), denoted $\mathbf{NT}(\{T_k : k \in \omega\})$, refers to a family of subsets of reals $\{T_k : k \in \omega\}$ and means the following.

For every bounded sequence of reals $\{u_m : m \in \omega\}$ there are $k \in \omega$, $t \in \mathbb{R}$ and an infinite set $\mathbb{M} \subseteq \omega$ such that

$$u_m + t \in T_k \text{ for all } m \text{ in } \mathbb{M}.$$

In words: the translate of some *subsequence* of $\{u_m\}$ is contained in some T_k. We will also say that $\{T_k : k \in \omega\}$ **traps sequences by translation.**

Since any bounded set has a convergent subsequence, if the (**NT**) applies to convergent sequences, then it applies also to bounded sequences. There is thus no need to make any distinction. The existing literature has apparently concentrated not on trapping but on avoiding images of *entire* convergent sequences (affine images, including translates); see for example [Kom] in regard to sets of positive measure avoiding translates of a given convergent sequence (see [Mil1] for additional references). Our rather different approach is motivated by proof structure, so in Section 3 we clarify this weaker concept in its present context of measure and category by reference to the notions of Lusin set (or, to use the modern transliteration, Luzin set), Sierpiński set, Hamel basis, and automatic continuity. We begin by noting the following strong property of the stabilized ε-level sets.

Proposition (Sequence containment). *Suppose the UCT holds for a function h. Let $\mathbf{u}$ be any bounded sequence, and let $\varepsilon > 0$. Then, for every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level set $T_k^\varepsilon(\mathbf{x})$ for some k contains the sequence $\mathbf{u}$. In particular, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ trap bounded sequences by translation.*

For a proof see Section 4. Our main result is the following 'converse'.

Main Theorem (UCT). *For h slowly varying, the following are equivalent.*
(i) *The UCT holds for h.*
(ii) *The principle $\mathbf{1}$-$\mathbf{NT}_h$ holds: for every $\varepsilon > 0$ and every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ of h trap bounded sequences by translation.*

In loose notation: $(\forall\varepsilon > 0)(\forall\mathbf{x})\ \mathbf{NT}(\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\})$.
(iii) *For every $\varepsilon > 0$ and for every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ of h contain all the bounded sequences.*

That this is indeed the sought-for generalization of the UCT in BGT is shown by the special case of the following general result. We term the latter the No Trumps Theorem, as it justifies the combinatorial framework of No Trumps.

Theorem (No Trumps Theorem). *Let T be an interval. Suppose that $T = \bigcup_{k\in\omega} T_k$, where the sets T_k are measurable/Baire. Then the sets $\{T_k : k \in \omega\}$ trap bounded sequences, i.e. $\mathbf{NT}(\{T_k : k \in \omega\})$.*

Theorem (Existence Theorem for trapping families). *Suppose the slowly varying function h is measurable, or has the property of Baire. Let $\mathbf{x} = \{x_n\}$ be any sequence tending to infinity. Then, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ trap bounded sequences, i.e. $\mathbf{NT}(\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\})$.*

The proof of the Existence Theorem is implicitly given, albeit bound up with its context, as the 'fourth proof of UCT' in BGT, p. 9, due to Csiszár and Erdős, see [CsEr]. We repeat the short (disentangled) proof for convenience in Section 4. In fact, much more is true (see [BOst6]); we restrict attention here to the simplest case, which suffices for our present purposes.

We note the strength of the sequence trapping property in the following.

Theorem (Bounded Equivalence Principle). *For h a slowly varying function the following are equivalent.*
(i) *The principle $\mathbf{1}$-$\mathbf{NT}_h$ holds: the family $\{T_n^\varepsilon(\mathbf{x}) : n \in \omega\}$ traps bounded sequences for any **real** sequence $\mathbf{x}$ tending to infinity, and any positive ε.*

In loose notation: $(\forall\varepsilon > 0)(\forall \text{ real } \mathbf{x})\ \mathbf{NT}(\{T_k^{\varepsilon}(\mathbf{x}) : k \in \omega\})$.

(ii) *Whenever* $\{u_n\}$ *is a bounded sequence, and* $\{x_n\}$ *tends to infinity*

$$\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0. \tag{2}$$

(ii)* *For any sequence* $\mathbf{x}$ *tending to infinity, and any positive* ε*, the family* $\{T_n^{\varepsilon}(\mathbf{x}) : n \in \omega\}$ ***ultimately contains*** *almost all of any bounded sequence* $\mathbf{u}$*.*

That is, for any bounded sequence $\mathbf{u}=\{u_n\}$ *there is* k *such that*

$$\{u_m : m > k\} \subseteq T_n^{\varepsilon}(\mathbf{x}) \text{ for all } n > k. \tag{3}$$

(iii) *Whenever* $\{u_n\}$ *is a bounded sequence, and* $\mathbf{m}=\{m_n\}$ *is an* ***integer*** *sequence tending to infinity*

$$\lim_{n\to\infty} |h(u_n + m_n) - h(m_n)| = 0. \tag{4}$$

(iv) **2-NT**$_h$ *holds: the family* $\{T_n^{\varepsilon}(\mathbf{m}) : n \in \omega\}$ *traps bounded sequences for any* ***integer*** *sequence* $\mathbf{m}$ *tending to infinity, and any positive* ε*.*

In loose notation: $(\forall\varepsilon > 0)(\forall \text{ integer } \mathbf{m})\ \mathbf{NT}(\{T_k^{\varepsilon}(\mathbf{m}) : k \in \omega\})$.

(v) **3-NT**$_h$ *holds:for all* $\varepsilon > 0$*, the family* $\{T_n^{\varepsilon}(\mathbf{m}) : n \in \omega\}$ *traps bounded sequences with* $\mathbf{m}$ *restricted to just the one sequence* **id** *defined by* $m_n = n$*.*

In loose notation: $(\forall\varepsilon > 0)\ \mathbf{NT}(\{T_k^{\varepsilon}(\mathbf{id}) : k \in \omega\})$.

(vi) *The UCT holds for* h*.*

In particular, for h *slowly varying, the three combinatorial principles* **1-NT**$_h$, **2-NT**$_h$, **3-NT**$_h$ *involving trapping of subsequences by translation are all equivalent.*

The assertion (ii)* , which is actually a transcription of (ii), clearly alludes to some further variations on the **i-NT**$_h$ theme. The sequence $\{T_k^{\varepsilon}(\mathbf{y}) : k \in \omega\}$ may have one of three 'inclusion properties' in relation to a bounded sequence $\mathbf{u}$. For some k, $T_k^{\varepsilon}(\mathbf{y})$ could:

(F) include all of $\mathbf{u}$, i.e. fully include $\mathbf{u}$, or,

(A) include almost all terms of $\mathbf{u}$, or,

(ST) include a subsequence of $\mathbf{u}$ *by translation*, i.e. precisely **NT** itself.

We refer to these various strengthenings of trapping as $F/A/ST$ analogues of trapping. Furthermore the inclusion property might be applied to:

($\mathbf{x}$) $\mathbf{y}$ ranging over real sequences $\mathbf{x}$,

($\mathbf{m}$) $\mathbf{y}$ ranging over integer sequences $\mathbf{m} = \{m_n\}$,

(**id**) $\mathbf{y}$ restricted to just the one integer sequence **id** defined by $m_n = n$.

The implications can be summarized in a 'contingency table', shown below in the style of the *Cichoń diagram*, for which see [F2]. The minimal one is thus **NT**$_h$:= **3-NT**$_h$. (referring to the sequence **id**).

When restricted to a slowly varying function h all these properties are equivalent.

$$
\begin{array}{ccccccc}
 & ST(\mathbf{x}) & \Longrightarrow & ST(\mathbf{m}) & \Longrightarrow & ST(\mathbf{id}) & \Longrightarrow UCT_h \\
 & \Uparrow & & \Uparrow & & \Uparrow & \\
 & A(\mathbf{x}) & \Longrightarrow & A(\mathbf{m}) & \Longrightarrow & A(\mathbf{id}) & \\
 & \Uparrow & & \Uparrow & & \Uparrow & \\
UCT_h \Longrightarrow & F(\mathbf{x}) & \Longrightarrow & F(\mathbf{m}) & \Longrightarrow & F(\mathbf{id}) &
\end{array}
$$

Here

$$P(\cdot) = F/A/ST \text{ analogue of the property } \forall\varepsilon\forall(\cdot)\mathbf{NT}(\{T_k^\varepsilon(\cdot) : k \in \omega\}),$$

and

$$
\begin{aligned}
F &= \text{Full inclusion,} \\
A &= \text{Almost inclusion,} \\
ST &= \text{Subsequence inclusion by translation.}
\end{aligned}
$$

Of course in combination with the existence theorem, the bounded equivalence principle contributes a 'sixth' proof of UCT complementing the five given in BGT, Chapter 1.

As a consequence of the bounded equivalence principle, in the general setting of a regularly varying function h, one may relax the definition of the associated limit function to

$$k(u) = \lim_{n\to\infty} |h(u+n) - h(n)|.$$

It is of course possible to limit attention everywhere to 'effectively definable' sequences $\mathbf{x}$ (see [Mos] Chapter 3).

1.3 A Souslin representation approach

We now offer an alternative point of view. This requires some notation and further definitions. Let ω^ω denote the space of sequences of natural numbers. We identify a natural number n with its set of predecessors, that is, $n = \{0, 1, 2, ..., n-1\}$. If $n \in \omega$ and $\alpha \in \omega^\omega$ we denote by $\alpha \upharpoonright n$ the restriction of α to the set n and identify this with with the n-term sequence $(\alpha(0), ..., \alpha(n-1))$. This notation is used in [Rog] p. 404, alongside the alternate notation of p. 9-10 for sequences σ in $\mathbb{N}^\mathbb{N}$ where $\sigma|n = (\sigma_1, ..., \sigma_n)$.

Definitions. (i) For $\mathcal{H}$ a family of sets, the set S is said to be Souslin-$\mathcal{H}$, if

$$S = \bigcup_{\alpha\in\omega^\omega} \cap_{n=1}^\infty H(\alpha \upharpoonright n),$$

provided that the sets $H(\alpha \upharpoonright n)$ are in the family $\mathcal{H}$.

(ii) Let

$$A = \{\alpha \in \omega^\omega : (\forall n)(\exists m > n)\alpha(m) \geq n\}.$$

Thus α is in A iff the range of α is unbounded.

(iii) Given $\varepsilon > 0$, two positive sequences $\mathbf{x} = \{x_n\}$ and $\mathbf{u} = \{u_n\}$, let

$$Z^\varepsilon(\mathbf{x}, \mathbf{u}, k) = \bigcup_{\alpha \in A} \bigcap_{n=1}^{\infty} Z_{\alpha(n),k}, \qquad \text{with } Z_{m,k} = T_k^\varepsilon(\mathbf{x}) - u_m,$$

so that $Z_{m,k}$ are the translates of the stabilized ε-level set $T_k^\varepsilon(\mathbf{x})$.

Comment. The set A is an $\mathcal{F}_{\sigma\delta}$ set, and hence a set of the form

$$S = \bigcup_{\alpha \in A} \cap_{n=1}^{\infty} H(\alpha \restriction n)$$

is Souslin-$\mathcal{H}$ provided that the sets $H(\alpha \restriction n)$ are in the family $\mathcal{H}$.

Proposition (Trapping Representation). *$T_k^\varepsilon(\mathbf{x})$ traps the sequence $\mathbf{u}$ by translation by z if, and only if, $z \in Z^\varepsilon(\mathbf{x}, \mathbf{u}, k)$.*

The next observation refers to the class of sets $\mathcal{H} = \Delta_2^1$ (definition to follow in Section 2).

Proposition (Souslin Representation). *Let h be a slowly varying function with graph in the family of sets Δ_2^1. Then the following set is Souslin-Δ_2^1 :*

$$Z^\varepsilon(\mathbf{x}, \mathbf{u}, k).$$

The essence of the Csiszár-Erdős proof [CsEr] of the UCT may now be stated as follows. The theorem follows from the Main Theorem (UCT) by an application of the above proposition on trapping representation.

Theorem (The non-empty Souslin set condition). *For h a slowly varying function with Δ_2^1 graph the following are equivalent.*

(i) *The UCT holds for h.*

(ii) *For any $\varepsilon > 0$, any sequence $\mathbf{x}$ tending to infinity and any bounded sequence $\mathbf{u}$, there is some k so that the set $Z^\varepsilon(\mathbf{x}, \mathbf{u}, k)$ (for h) is non-empty.*

(iii) *For any $\varepsilon > 0$, any sequence $\mathbf{x}$ tending to infinity and any bounded sequence $\mathbf{u}$, there is some k so that $0 \in Z^\varepsilon(\mathbf{x}, \mathbf{u}, k)$ (for h).*

So, “It contains 0 or contains nothing” (we are indebted to Anatole Beck for this aphorism.). In [BOst2] we will find this theorem useful in proving the Second Heiberg-Seneta Theorem (see Section 5).

The result may be viewed as game-theoretic, cf. Section 5. Thus the question arises as to when the set $Z^\varepsilon(\mathbf{x}, \mathbf{u}, k)$ may be non-empty. The answer is now seen to depend both on the function h and on the axioms of set theory one assumes. Suppose to fix ideas that h has a graph that is a projective set of type $\boldsymbol{\Delta}_2^1$. Under this circumstance any axiom which guarantees that h is measurable (as in the model of Solovay [So]) or has the property of Baire (as in

Shelah's model described in [She], or in models where the axiom of projective determinacy is restricted to $\boldsymbol{\Delta}^1_2$ sets of reals) thus guarantees the non-emptiness that we seek. Determinacy has had much study since the original paper of Mycielski and Steinhaus [MySt] (see the chapter on 'The playful universe' and the bibliography in [Mos]). That literature has led to an understanding of the implications for determinacy of large cardinal assumptions in set theory, for which see Woodin [Wo]. But it is also possible under Gödel's Axiom of Constructibility to construct a function h with graph a projective set of type $\boldsymbol{\Delta}^1_2$ for which this set is empty. This may be done by the standard construction with reference to a Hamel basis, provided the Hamel basis is taken to be a $\boldsymbol{\Pi}^1_1$ set. See [Mil1] or the textbook [Mil2] for details (Section 18 and also p. 97).

2 *Descriptive character of limits in regular variation theory*

Inevitably, we work in the standard mathematical framework of Zermelo-Fraenkel set theory (ZF). Unless we make explicit use of the Axiom of Choice (AC), we will use ZF rather than ZFC (ZF plus AC). However, the underlying classical theory is principally that of (Lebesgue) measurability Since one cannot exhibit a non-measurable set, or non-measurable function, without (AC), the framework of measurability *itself* may be regarded as tacitly assuming (AC). In this sense, the classical theory works with (ZF) explicitly and with (ZFC) implicitly.

The theme of this section is to argue the case for restricting attention to functions h in certain naturally occurring classes of sets. (Recall that we identify a function with its graph.) These classes refer to 'definability', that is, to the way a function h may come to the attention of an analyst by way of its definition when written out in the usual semi-formal language of mathematics. We present the case for our preferred class of functions (the class $\boldsymbol{\Delta}^1_2$ to be defined below) at the end of this section. Our argument is based on an analysis of two interrelated examples of 'limit taking' which are at the heart of regular variation theory.

Our first and main example refers to a function obtained using the lim sup operation. Consider the definition:

$$h^*(x) := \limsup_{t\to\infty} [h(t+x) - h(t)].$$

We will first establish the following result, which we call the First Character Theorem, and then contrast it with two alternative character theorems. Undefined terms are explained below in the course of the proof. Indeed the purpose behind the proof is to familiarize the analyst with the logical apparatus of descriptive set theory, which we use to examine the character of relevant functions and their graphs. (As in BGT, we reserve the name Characterization Theorem

CT for a result identifying the g of (RV) and (CFE) as a power function, as in (ρ).)

First Character Theorem. (i) *Suppose that h is Borel (has Borel graph). In general the graph of the function $h^*(x)$ is a difference of two analytic sets, hence is measurable and $\mathbf{\Delta}^1_2$. If the graph of h is $\mathcal{F}_\sigma$, then the graph of $h^*(x)$ is Borel.*

(ii) *Suppose that h is analytic (has analytic graph). Then the graph of the function $h^*(x)$ is $\mathbf{\Pi}^1_2$.*

(iii) *Suppose that h is co-analytic (has co-analytic graph). Then the graph of the function $h^*(x)$ is $\mathbf{\Pi}^1_3$.*

Proof. Let us suppose that h is Borel (that is, h has a Borel graph). As a first step consider the graph of the function of two variables: $h(t+x)-h(t)$, namely the set

$$G = \{(x,t,y) : y = h(t+x) - h(t)\}.$$

One expects this to be a Borel set and indeed it is. For a proof, we must refer back to the set h itself, and to do this we must re-write the defining clause appropriately. This re-writing brings out explicitly an implicit use of quantifiers, a common enough occurrence in analysis, often missed by the untrained eye (see Section 5 for another important example). We have:

$$y = h(t+x) - h(t) \Leftrightarrow (\exists u, v, w \in \mathbb{R}) r(x,t,y,u,v,w),$$

where

$$r(x,t,y,u,v,w) = [\ y = u - v \ \&\ w = t + x \ \&\ (w,u) \in h \ \&\ (t,v) \in h]. \qquad (5)$$

From a geometric viewpoint, the set of points

$$\{(x,t,y,u,v,w) : r(x,t,y,u,v,w)\}$$

is Borel in $\mathbb{R}^6$, hence the set $G = \{(x,t,y) : (\exists u,v,w \in \mathbb{R}) r(x,t,y,u,v,w)\}$, being a projection of a Borel set, is an analytic set in $\mathbb{R}^3$, and in general not Borel. (The theory of analytic sets dates from work of Souslin in 1916, Luzin in 1917, Luzin and Sierpiński in 1918. For monograph treatments, see [Lu], [Rog]. The historical origins, in an error of Lebesgue in 1905, are given there - in Lebesgue's preface to [Lu] and in [Rog] Section 1.3.) However, in the particular present context the 'sections'

$$\{(u,v,w) : r(x,y,z,t)\},$$

corresponding to fixed $(x,t,y) \in G$, are single points (since u, v, w are defined uniquely by the values of x and t). In consequence, the projection here is Borel. The reason for this is that any Borel set is a continuous injective image of the irrationals, and so a continuous injective image, as here under projection, of a Borel set is Borel. (So here the hidden quantifiers are 'innocuous' to the character of G.) The current result may also be seen as the simplest instance of a more general result, the Rogers-Kunugui-Arsenin Theorem, which asserts that if the sections of a Borel set are $\mathcal{F}_\sigma$ (that is, countable unions of closed sets), then its projection is Borel ([Rog] p. 147/148).

By abuse of notation, let us put $h(t,x) = h(t+x) - h(t)$ and think of t as parametrizing a family of functions. We continue to assume that the family of functions $h(t,x)$ is Borel. That is, the graph $\{(x,y,t) : y = h(t,x)\}$ is a Borel set. We will weaken this restriction appropriately in later paragraphs.

As a second step, we now consider the formal definition of $h^*(x)$, again written out in a predicate calculus using a semi-formal apparatus. The definition comes naturally as a conjunction of two clauses:

$$y = h^*(x) \Leftrightarrow P(x,y) \ \& \ Q(x,y),$$

where

$$\begin{aligned} P &= (\forall n)(\forall q \in \mathbb{Q}^+)(\exists t \in \mathbb{R})(\exists z \in \mathbb{R})[t > n \ \& \ z = h(t,x) \ \& \ |z-y| < q], \\ Q &= (\forall q \in \mathbb{Q}^+)(\exists m)(\forall t \in \mathbb{R})(\forall z \in \mathbb{R})[t > n \ \& \ (t,x,z) \in h \implies z < y + q]. \end{aligned}$$

The first clause (predicate) asserts that y is a limit point of the set $\{h(t,x) : t \in \mathbb{R}\}$ and this requires an existential quantifier; the second clause asserts that, with finitely many exceptions, no member of the set exceeds y by more than q and this must require a universal quantifier.

From a geometric viewpoint, for fixed $q > 0$ the set of points

$$G_1 = \{(x,y,z,t) : p(x,y,z)\}, \text{ where } p(x,y,z,t) = [(t,x,z) \in h \ \& \ |z-y| < q],$$

is Borel in $\mathbb{R}^4$, hence again the set $\{(x,y) : (\exists z, t \in \mathbb{R})p(x,y,z)\}$, being a projection of a Borel set, is an analytic set in $\mathbb{R}^2$. Again, for fixed (x,y) we look at the section of G_1. Evidently $\{z : |z-y| < q\}$ is an open set so $\mathcal{F}_\sigma$. However, only if we assume that h is $\mathcal{F}_\sigma$ can we deduce that $\{(x,y) : (\exists t \in \mathbb{R})(\exists z \in \mathbb{R})[t > n \ \& \ z = h(t,x) \ \& \ |z-y| < q\}$ is Borel. Otherwise it is merely analytic.

From the viewpoint of mathematical logic, since the quantifiers in $(\exists z \in \mathbb{R})(\exists t \in \mathbb{R})p(x,y,z,t)$ are at the front of the defining formula, that formula is said to be $\mathbf{\Sigma}_1^1$ (read: bold-face sigma-1-1), where Σ refers to the opening quantifier block being existential, the superscript identifies that the quantification is of order 1 (i.e. ranging over reals rather than integers), and the subscript refers to the fact that there is only one (existential) block of quantifiers at the front. (That is, p may be written out without using any further order 1 quantifiers.) See [Rog] for a modern side-by-side exposition of the two viewpoints of mathematical logic and geometry.

Finally, the set $\{(x,y) : P(x,y)\}$ is seen to be obtainable from analytic set (or Borel in the special case) by use of countable union and intersection operations. It is thus an analytic set (or Borel as the case may be).

By contrast, the set $\{(x,y) : (\forall z, t \in \mathbb{R})q(x,y,z,t)\}$, where $q(x,y,z,t) = [[z = h(t,x)] \implies z < y + q]$, is said to be co-analytic, since its complement is the analytic set $\{(x,y) : (\exists z, t \in \mathbb{R})[z = h(t,x) \ \& \ z \geq y + q]\}$. Again for given q and for arbitrary fixed (x, y) the sections of $\{(x, y, z, t) : [z = h(t, x)$ & $z \geq y + q]\}$ will be be $\mathcal{F}_\sigma$ if the graph of h is $\mathcal{F}_\sigma$, but is otherwise analytic. Thus $\{(x, y) : Q(x, y)\}$ is seen to be obtainable from co-analytic sets (or at best Borel sets) by use of countable union and intersection operations. It is thus co-analytic (or Borel as the case may be).

On a syntactic, logical analysis the formula $(\forall z \in \mathbb{R})q(x, y, z, t)$ is said to be $\mathbf{\Pi}^1_1$, since the opening quantifier is universal of order 1.

The set $\{(x, y) : Q(x, y)\}$ is seen to be obtainable from co-analytic sets by use of countable union and intersection operations. It is thus co-analytic since such operations preserve this character. Finally, note that the sets which are differences of analytic sets are both in the classes $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$, and so are in the common part of the two classes denoted $\mathbf{\Delta}^1_2$. We have of course neglected the possibility that the lim sup is infinite, but for this case we need only note that

$$\begin{array}{rcl} h^*(x) & = & \infty \Leftrightarrow (\forall n)(\exists t \in \mathbb{R})(\exists z \in \mathbb{R})[t > n \ \& \ z = h(t, x) \ \& \ z > n], \\ h^*(x) & < & \infty \Leftrightarrow (\exists y \in \mathbb{R})(y = h^*(x)), \end{array}$$

so that this case is simultaneously $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

We have thus proved part (i) of the Character Theorem. □(i)

We now work with weaker assumptions than that the function h is Borel (has a Borel graph).

We consider part (ii) of the theorem. First suppose instead that the function h has an analytic graph. It follows from (5) that G, being the projection of an analytic set, is now analytic. That is, we may write

$$y = h(t, x) \Leftrightarrow (\exists w \in \mathbb{R})F(t, x, y, w),$$

where the set $\{(t, x, y, w) : F(t, x, y, w)\}$ is Borel. Then

$$\{(x, y) : (\exists z \in \mathbb{R})(\exists w \in \mathbb{R})[F(t, x, z, w) \ \& \ |z - y| < q]\}$$

is only analytic, since we have no information about special sections; however, the set

$$\{(x, y) : (\forall z \in \mathbb{R})(\exists w \in \mathbb{R})[t > n \ \& \ F(t, x, z, w) \implies z < y + q]\},$$

requires for its definition a quantifier alternation which begins with a universal quantifier, so is said to be $\mathbf{\Pi}^1_2$ (read: bold-face pi-1-2). Since $\mathbf{\Sigma}^1_1$ sets are necessarily a subclass of $\mathbf{\Pi}^1_2$ sets, the graph of $\limsup_t f(t, x)$ in this case is $\mathbf{\Pi}^1_2$. □(ii)

Finally, we consider part (iii) of the theorem. Suppose that the function $h(x)$ has a co-analytic graph. Then by (5) the set G is of class $\mathbf{\Sigma}^1_2$, i.e. the function $h(t, x)$ has a $\mathbf{\Sigma}^1_2$ graph. That is, we now have to write

$$y = h(t, x) \Leftrightarrow (\exists u \in \mathbb{R})(\forall w \in \mathbb{R})F(t, x, y, u, w),$$

where as before the set $\{(t,x,y,w) : F(t,x,y,u,w)\}$ is Borel. Then

$$\{(x,y) : (\exists z, u \in \mathbb{R})(\forall w \in \mathbb{R})[F(t,x,z,u,w) \ \& \ |z-y| < q]\}$$

is now $\mathbf{\Sigma}^1_2$. On the other hand the set

$$\{(x,y) : (\forall z \in \mathbb{R})(\exists u \in \mathbb{R})(\forall w \in \mathbb{R})[F(t,x,z,u,w) \implies z < y+q]\}$$

is $\mathbf{\Pi}^1_3$. Since $\mathbf{\Sigma}^1_1$ sets are necessarily a subclass of $\mathbf{\Pi}^1_3$ sets, the graph of $\limsup_t h(t,x)$ in this case is $\mathbf{\Pi}^1_3$. □(iii)

In our next theorem we assume much more than in the First Character Theorem.

Second Character Theorem. *Suppose $h \in \Delta^1_2$ and the following limit exists.*

$$h^*(x) := \lim_{t\to\infty}[h(t+x) - h(t)].$$

Then the graph of h^* is Δ^1_2.

Proof. Here we have

$$y \quad = \quad h^*(x) \Longleftrightarrow (\forall q \in \mathbb{Q}^+)(\exists n \in \omega)(\forall t > n)(\forall zuvw)P,$$

where

$$P = [[z = u - v \ \& \ w = t + x \ \& \ (t,v) \in h \ \& \ (w,u) \in h \ ] \Longrightarrow \ |z-y| < q]],$$

and

$$y \quad \neq \quad h^*(x) \Longleftrightarrow (\forall q \in \mathbb{Q}^+)(\exists n \in \omega)(\forall t > n)(\forall zuvw)Q,$$

where

$$Q = [[z = u - v \ \& \ w = t + x \ \& \ (v,t) \in h \ \& \ (u,w) \in h \ ] \Longrightarrow \ |z-y| \geq q]].$$

□

The point of the next theorem is that it may be applied under the assumption of Gödel's Axiom ($V = L$), (see [Dev1]) as the axiom implies that Δ^1_2 ultrafilters on ω exist (see for instance [Z], where Ramsey ultrafilters are considered). For information on various types of ultrafilter on ω see [CoNe]. In particular this means that we have a midway position between the results of the First and Second Character Theorem.

Third Character Theorem. *Suppose the following are of class $\mathbf{\Delta}^1_2$: the function h and an ultrafilter $\mathcal{U}$ on ω. Then the following is of class $\mathbf{\Delta}^1_2$:*

$$h^*(t) = \mathcal{U}\text{-}\lim_n[h(t + x(n)) - h(x(n))].$$

Comment 1. In the current circumstances $h^*(t)$ is an additive function, whereas in the circumstances envisaged by the First Character Theorem we would have had only sub-additivity. See BGT p. 62 equation (2.0.3).

Comment 2. For an 'effective' version of the proposition one would need to specify the effective descriptive character of the sequence $\mathbf{x} : \omega \to \omega^\omega$.

Proof. By (5) the function $y = h(t, x)$ is of class $\mathbf{\Sigma}^1_2$. We show that $y = h^*(t)$ is of class $\mathbf{\Sigma}^1_2$. The result will follow since the negation satisfies

$$y \neq h^*(t) \iff \exists z[z \neq y \ \& \ [z = h^*(t) \text{ or } h^*(t) = \pm\infty]],$$

and so is of class $\mathbf{\Sigma}^1_2$. Finally,

$$y = h^*(t) \iff (\forall \varepsilon \in \mathbb{Q}^+)(\exists U)(\forall n \in \omega)(\exists t)P,$$

where

$$P = [U \in \mathcal{U} \ \& \ n \in U \ \& \ (n, t) \in \mathbf{x} \ \& \ |t - y| < \varepsilon],$$

and

$$h^*(t) = \infty \iff (\forall M \in \mathbb{Q}^+)(\exists U \in \mathcal{U})(\forall n \in U)(\exists t)[(n, t) \in \mathbf{x} \ \& \ t > M].$$

□

Comment on the virtues of the class $\mathbf{\Delta}^1_2$. It seems to us that the class $\mathbf{\Delta}^1_2$ offers an attractive class within which to carry out the analysis of regularly varying functions. It admits a pluralist interpretation. Either the members of the class $\mathbf{\Delta}^1_2$ may be taken to be measurable in the highly regular world governed by the Axiom of Projective Determinacy, or else the limit function $h^*(t)$ is guaranteed to exist in a world otherwise filled with Hamel-type pathology governed by Gödel's Axiom.

In summary, regular variation theory has occasion in a natural way to make use of the 'projective sets' of level 2. We suggest that therefore a natural setting for the theory of regular variation is slowly varying functions of class $\mathcal{H}$, where $\mathcal{H}$ may be taken according to need to be one of the classes $\mathbf{\Sigma}^1_2$, or $\mathbf{\Pi}^1_2$, or their intersection $\mathbf{\Delta}^1_2$.

The latter class is the counterpart of the Borel sets thought of as $\mathbf{\Delta}^1_1$, namely the intersection of the classes $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ (according to Souslin's characterization of Borel sets as being simultaneously analytic and co-analytic).

In certain axiom schemes for set theory, the sets in these three classes are all measurable and have the Baire property. The notable case is Zermelo-Fraenkel set theory enriched with the Axiom of Projective Determinacy (PD), which asserts the existence of winning strategies in Banach-Mazur games with projective target sets (see [Tel2] and [MaKe] for surveys); this axiom is a replacement for the Axiom of Choice (AC), some of whose reasonable consequences it upholds, at the same time negating consequences that are sometimes held to be glaringly counter-intuitive (such as the paradoxical decompositions, for which see [Wag]).

Though somewhat inadequate from the point of view of the lim-sup operation, the class $\mathbf{\Delta}^1_2$ is quite rich. In Zermelo-Fraenkel set theory enriched with Gödel's Axiom of Constructibility $V = L$ (a strong form of AC), the class $\mathbf{\Delta}^1_2$ contains a variety of singular sets. In particular, the class $\mathbf{\Delta}^1_2$ is rich enough to contain the well-known Hamel pathologies (see BGT p. 5 and 11), since the axiom furnishes a $\mathbf{\Pi}^1_1$ set of reals which is a Hamel basis. On this latter point see [Mil1], and for a classical treatment of Hamel bases see [Kucz].

3 *Set-theoretic status of sequence trapping*

We recall that a *Luzin* set is one which meets any nowhere dense set in at most a countable set. Similarly a *Sierpiński* set is one which meets any set of measure zero in at most a countable set. See [Kun], [Mil2] p. 32 (where there is a historical attribution to Mahlo, and the two concepts are described as $\mathcal{I}$-Luzin sets for the appropriate σ-ideal $\mathcal{I}$), or [Mil3] for a survey of 'special' subsets of the real line. A Luzin set is measurable and is of measure zero; furthermore, it is of second category, but fails to have the Baire property. See e.g. [Kucz], p. 63 for proofs. Similarly every Sierpiński set is strongly meager, see [Paw].

Proposition. *Assume the Continuum Hypothesis (CH). There exists a Luzin set (resp. Sierpiński set) which contains a Hamel basis and contains all sequences up to translation. Its difference set has empty interior. Assuming Gödel's Axiom $V = L$, there is a $\mathbf{\Delta}^1_2$ such set.*

Remark. Note that if $\{T_n\}$ is a family of sets such that for some n the set T_n contains an interval then the family traps sequences by translation. Indeed, suppose z is in the interior of T_n, and suppose u_m converges to u; then with $y = z - u$ we see that

$$y + u_m = z - (u - u_m)$$

is ultimately in T_n. This observation ties in with the standard textbook approach to UCT where a number of proofs arrange to use measurability and Steinhaus's Theorem (see BGT Theorem 1.1.1 p. 2) to manufacture an interval that traps a translate of a convergent sequence.

One can also relate the sequence trapping property directly to the notion of 'automatic continuity'. Here the natural point of departure from the present perspective is the limit function:

$$k(u) = \lim_{x\to\infty} [h(u + x) - h(x)],$$

which, assuming it exists, is additive. We study in [BOst3] the present combinatorial insights, as they impinge on the Ostrowski and Steinhaus Theorems; there is also the expected connection with the natural classes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ associated with automatic continuity, as defined by Ger and Kuczma (see [Kucz] p. 206 or [GerKucz]).

4 *Proofs*

4.1 Proof that UCT implies sequence containment

Suppose given two sequences $\mathbf{x} = \{x_n\}$ and $\mathbf{u} = \{u_n\}$ with $x_n \to \infty$ and u_n bounded. If the sequence $\{u_m\}$ lies in the compact interval $[a, b]$ then, for any $\varepsilon > 0$, there is k so large that, for any u in $[a, b]$ and any $n \geq k$, we have

$$|h(u + x_n) - h(x_n)| < \varepsilon.$$

This means that any such u is in $T_k^\varepsilon(\mathbf{x})$, so in particular $\{u_m : m \in \omega\} \subset T_k^\varepsilon(\mathbf{x})$. $\square$

4.2 Proof of the Main theorem (UCT)

From the last Proposition we already know that (i) implies (iii) and (iii) implies (ii). It remains to prove that (ii) implies (i).

So suppose that UCT fails for some function h.

Suppose that for the two sequences $\mathbf{x} = \{x_n\}$ and $\mathbf{u} = \{u_n\}$ with $x_n \to \infty$ and u_n bounded there is an $\varepsilon > 0$ such that for $n = 1, 2, ..$ we have

$$|h(x_n + u_n) - h(x_n)| \geq 2\varepsilon. \tag{6}$$

Note that if $y \in T_k^\varepsilon(\mathbf{x})$ then we have, for $n = k, k + 1, ...,$ that

$$|h(x_n + u_n) - h(x_n + y)| \geq \varepsilon. \tag{7}$$

Indeed, otherwise we would have

$$|h(x_n + u_n) - h(x_n + y)| < \varepsilon$$

and

$$|h(x_n + y) - h(x_n)| < \varepsilon,$$

contradicting (6).

Now, by the trapping assumption, for infinitely many m in, say $\mathbb{M}$, we have

$$y_m = u_m + z \in T_k^\varepsilon(\mathbf{x}) \text{ for } m \in \mathbb{M}.$$

Now, for any such $m \in \mathbb{M}$ with $m > k$, by (7) with $y = y_m$, we have that for $n = m$:

$$|h(x_m + u_m) - h(x_m + u_m + z)| \geq \varepsilon.$$

Putting $v_m = x_m + u_m$ this yields that

$$|h(z + v_m) - h(v_m)| \geq \varepsilon,$$

which contradicts that h is slowly varying. Hence the assumption (6) is untenable, and thus after all UCT holds. $\square$

4.3 No Trumps Theorem and the existence of trapping families

This argument is extracted from the Csiszár-Erdős proof [CsEr] of the UCT.

Without loss of generality we take $T = [-1, 1]$. Now let $\mathbf{u} = \{u_n\}$ be a bounded sequence, which we may as well assume is convergent to some u_0. We assume that $|u_n - u_0| \leq 1$. We are to show that for some z, some K, and some infinite $\mathbb{M} \subset \omega$, we have $z + u_m \in T_K$.

By assumption, each T_k is measurable [Baire], so there is K such that T_K has positive measure [is non-meagre]. Let

$$Z_K = \mathbf{u}(T_K) := \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} (T_K - u_n).$$

We now quote almost verbatim from BGT p. 9. 'In the measurable case all the $Z_{n,K}$ have measure $|T_K|$, and as they are subsets of the fixed bounded interval $[u_0 - 2, u_0 + 2]$, Z_K is a subset of the same interval having measure

$$|Z_K| = \lim_{j\to\infty} \left| \bigcup_{n=j}^{\infty} (T_K - u_n) \right| \geq |T_K| > 0.$$

So Z_K is non-empty.

In the Baire case T_K contains some set $I \backslash M$, where I is an open interval of length $\delta > 0$, and M is meagre. So each set $T_K - u_n$ contains $I^n \backslash M^n$, where $I^n = I - u_n$ is an open interval of length δ and $M^n := M_n - u_n$ is meagre. Choosing J so large that $|u_i - u_j| < \delta$ for all $i, j \geq J$, the intervals $I^J, I^{J+1}, ...$ all overlap each other, and so $\bigcup_{n=j}^{\infty} I^n$, for $j = J, J+1, ...$, is a decreasing sequence of intervals, all of length $\geq \delta$ and all contained in the interval $[u_0 - 2, u_0 + 2]$; hence $I^0 = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} I^n$ is an interval of length $\geq \delta$. Since Z_K contains $I^0 \backslash \bigcup_{n=j}^{\infty} M^n$, it follows that Z_K is non-meagre, so non-empty.' Thus in either case, there is a point $z \in Z_K$.

This means that $z \in T_K - u_n$ for infinitely many n. Say that

$$z \in T_K - u_m \text{ for } m \in \mathbb{M}.$$

Without loss of generality, $m \in \mathbb{M}$ implies $m > K$.

Consider $m \in \mathbb{M}$. By definition, for some $y = y_m$, we have $z = y_m - u_m$ with $y_m \in T_K$. But this says that

$$z + u_m \in T_K \text{ for } m \in \mathbb{M},$$

as required. □

Corollary. *The existence theorem holds.*

Proof. Let h be measurable or Baire slowly varying. Let $\mathbf{x} = \{x_n\}$ be a fixed sequence tending to infinity and let $\varepsilon > 0$ be fixed.

By assumption of slow variation, we have

$$[-1,1] = \bigcup_k I_k, \text{ where } I_k = [-1,1] \cap \bigcup_k T_k^{\varepsilon}(\mathbf{x})$$

and

$$T_k^{\varepsilon}(\mathbf{x}) = \bigcap_{n=k}^{\infty} \{y : |h(y + x_n) - h(x_n)| < \varepsilon\}.$$

The corollary is now immediate, as the sets $T_k := T_k^{\varepsilon}(\mathbf{x})$ are, by assumption, measurable [Baire]. □

Comment. A forcing argument due to A. Miller (quoted in Section 5) shows why there is duality present here between measure and category; his proof tells us that the amount by which the subsequence needs to be translated is 'generic' in nature.

4.4 Proof of the bounded equivalence principle

In what follows if we assert that a combinatorial principle holds, then it is to be understood implicitly that it holds for all $\varepsilon > 0$.

(a) The equivalence of (i) and (vi) is the substance of our Main Theorem UCT.

(b) We prove that (i) implies (ii). This is the hardest part of the proof. All the other steps are either simple, or in just one case a nearly verbatim repetition of the current step with $\mathbf{x}$ replaced by $\mathbf{m}$.

Suppose that (2) fails. Then for some $\eta > 0$

$$|h(u_n + x_n) - h(x_n)| \geq \eta, \tag{8}$$

for a subsequence $\mathbb{M}^0 \subset \mathbb{N}$ of $n's$. As $u = \{u_n\}$ is a bounded sequence, by passing to a subsequence $\mathbb{M} \subset \mathbb{M}^0$, we may suppose that $\{u_m\}$ converges for $m \in \mathbb{M}$, to u say.

We begin by establishing that, for the subsequence of $\{u_m\}$ convergent to u, we have

$$\lim_{m \in \mathbb{M}} |h(u + x_m) - h(u_m + x_m)| = 0,$$

where the limit is taken down the subsequence $\mathbb{M}$. More precisely, we show that, with $\varepsilon = \eta/3 > 0$, there is $N = N(u)$ such that if $n > N$ and $n \in \mathbb{M}$, then

$$|h(u + x_n) - h(u_n + x_n)| < 2\varepsilon.$$

Define

$$y_n = u + x_n,$$

which tends to infinity. By the sequence trapping hypothesis, there are t, n and $\mathbb{M}_1 \subset \mathbb{M}$ such that

$$u_m - u + t \in T_n^{\varepsilon}(\mathbf{y}),$$

provided $m \in \mathbb{M}_1$. Let $M_1 = \min \mathbb{M}_1$. Since h is slowly varying, we have

$$\lim_{n\to\infty} |h(t + y_n) - h(y_n)| = 0.$$

That is, transcribing the result, there is M_2 such that, for $n \geq M_2$, we have

$$|h(t + u_m + x_n) - h(u + x_n)| < \varepsilon. \tag{9}$$

Finally, since h is slowly varying, we also have

$$\lim_{n\to\infty} |h(u + x_n) - h(x_n)| = 0,$$

so there is M_3 such that, for $n \geq M_3$, we have

$$|h(u + x_n) - h(x_n)| < \varepsilon. \tag{10}$$

Consider now any $k > N(u) = \max\{M_1, M_2, M_3, n\}$ with $k \in \mathbb{M}_1$. We have, since $k > n$, that

$$u_k - u + t \in T_n^\varepsilon(\mathbf{y}) \subseteq H_k^\varepsilon(\mathbf{y}).$$

Put $v = u_k - u + t$. Then

$$|h(v + y_k) - h(y_k)| < \varepsilon.$$

Substituting in this last inequality for v and for y_k, we obtain

$$|h((u_k - u + t) + (u + x_k)) - h(u + x_k)| < \varepsilon,$$

i.e.

$$|h(t + u_k + x_k) - h(u + x_k)| < \varepsilon. \tag{11}$$

Combining (9) and (11) we obtain

$$\begin{aligned} |h(u + x_k) - h(u_k + x_k)| &\leq |h(t + u_k + x_k) - h(u_k + x_k)| + |h(t + u_k + x_k) - h(u + x_k)| \\ &< 2\varepsilon. \end{aligned}$$

Finally, referring to (10), we obtain

$$\begin{aligned} |h(x_k) - h(u_k + x_k)| &\leq |h(u + x_k) - h(u_k + x_k)| + |h(u + x_k) - h(x_k)| \\ &< 2\varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

This contradicts (8). □ (b)

(c) The assertion (ii)* is a restatement of (ii). Indeed, (2) implies that, for every $\varepsilon > 0$, there is k such that $u_n \in H(x_n)$, for every $n > k$; hence $\{u_m : m > k\} \subseteq T_k^\varepsilon(\mathbf{x})$ from the definition of $T_k^\varepsilon(\mathbf{x})$. So (3) follows from (1). For the reverse direction note that (3) implies that $u_n \in H(x_n)$, for every $n > k$. □ (c)

(d) Since (ii)* asserts that $\mathbf{u}$ is trapped without any need for translation, we have a fortiori (i). □ (d)

(e) We show that (ii) and (iii) are equivalent. Clearly (ii) implies (iii). To see that (iii) implies (ii) write $x_n = m_n + v_n$, where $m_n \in \omega$ and $0 < v_n < 1$ and $w_n = u_n + v_n$, then we have

$$\begin{aligned} h(x_n+u_n)-h(x_n) &= [h(m_n+u_n+v_n)-h(m_n)]-[h(m_n+v_n)-h(m_n)] \\ &= [h(m_n+w_n)-h(m_n)]-[h(m_n+v_n)-h(m_n)] \\ &\to 0-0=0, \end{aligned}$$

in view of (iii). □ (e)

(f) We now proceed by analogy and prove that (iii) is equivalent to (iv). Indeed (b) with $\mathbf{x}$ replaced by $\mathbf{m}$ proves that (iv) implies (iii). Now (iii) is equivalent to the following (just as (ii) and (ii)* were):

(iii)* *For any integer sequence* $\mathbf{m}$ *tending to infinity, and any positive* ε*, the family* $\{T_n^\varepsilon(\mathbf{m}) : n \in \omega\}$ ***ultimately contains*** *almost all of any bounded sequence* $\{u_n\}$*.*

That is, for any bounded sequence $\{u_n\}$, there is k such that

$$\{u_m : m > k\} \subseteq T_n^\varepsilon(\mathbf{m}), \text{ for all } n > k,$$

so a fortiori $\mathbf{2\text{-}NT}_h(\{T_k^\varepsilon(\mathbf{m}) : k \in \omega\})$ holds for all $\mathbf{m}$. □ (f)

(g) Clearly if $\mathbf{2\text{-}NT}_h(\{T_k^\varepsilon(\mathbf{m}) : k \in \omega\})$ holds for all $\mathbf{m}$, then in particular $\mathbf{3\text{-}NT}_h(\{T_k^\varepsilon(\mathbf{id}) : k \in \omega\})$ holds. Noting that

$$\bigcap\nolimits_{n=m_k}^{\infty} H^\varepsilon(n) \subseteq \bigcap\nolimits_{n=k}^{\infty} H^\varepsilon(m_n),$$

we see that if $\mathbf{3\text{-}NT}_h(\{T_k^\varepsilon(\mathbf{id}) : k \in \omega\})$ holds, then $\mathbf{2\text{-}NT}_h(\{T_k^\varepsilon(\mathbf{m}) : k \in \omega\})$ holds for all $\mathbf{m}$. □ (g)

Comment. If (2) holds for $\{u_n\}$ any bounded sequence, and $\{x_n\}$ any real sequence tending to infinity, then one can prove directly that UCT holds for h by repeating the proof step given in BGT p. 8. Clearly the property (2) follows from UCT.

4.5 Proof of the trapping representation

Suppose that

$$z \in Z^\varepsilon(\mathbf{x}, \mathbf{u}, k) = \bigcup_{\alpha \in A} \bigcap_{n=1}^{\infty} Z_{\alpha(n),k}, \text{ where } Z_{m,k} = T_k^\varepsilon(\mathbf{x}) - u_m.$$

Thus, $z \in Z^\varepsilon(\mathbf{x}, \mathbf{u}, k)$ if and only if for infinitely many m we have that

$$z \in T_k^\varepsilon(\mathbf{x}) - u_m.$$

If this is the case, then for such m we may write $z = y_m - u_m$, where $y_m \in T_k^\varepsilon(\mathbf{x})$, or

$$y_m = u_m + z \in T_k^\varepsilon(\mathbf{x}). \tag{12}$$

This means that $T_k^\varepsilon(\mathbf{x})$ traps the sequence $\{u_m\}$ by translation by z.

Conversely, if (12) holds for the set $\mathbb{M} = \{\alpha(n) : n \in \omega\}$ with $\alpha \in A$, then $z \in Z_{\alpha(n),k}$, for all n, and so $z \in Z^\varepsilon(\mathbf{x}, \mathbf{u}, k)$. □

4.6 Proof of the Souslin representation

The set H defined by

$$H(x) = \{y : |h(x+y) - h(x)| \leq \varepsilon\}$$

is in Δ_2^1, since

$$\begin{aligned} y &\in H(x) \Leftrightarrow (\exists ztu)[u = x + y \ \&\ (u,z) \in h \ \&\ (x,t) \in h \ \&\ |z-t| \leq \varepsilon] \\ &\Leftrightarrow (\forall ztu)[[u = x + y \ \&\ (u,z) \in h \ \&\ (x,t) \in h] \implies |z-t| \leq \varepsilon]. \end{aligned}$$

The set T_k is in Δ_2^1, since

$$\begin{aligned} y &\in T_k \Leftrightarrow (\forall n \geq k)(\exists t)[t = x(n) \ \&\ y \in H(t)] \\ &\Leftrightarrow (\forall t)[(\forall n \geq k)[t = x(n)] \implies y \in H(t)]. \end{aligned}$$

Notice that

$$(\cap_{n=1}^\infty H_n) - u = \cap_{n=1}^\infty (H_n - u).$$

Next, put

$$Z_{m,n}^k(\mathbf{x}, \mathbf{u}) = H_{n+k} - u_m.$$

Thus

$$\begin{aligned} y &\in Z_{m,n}^k(\mathbf{x}, \mathbf{u}) \\ &\Leftrightarrow \\ (\exists s,t,v)(\exists j)[y &= t - s \ \&\ s = u(m) \ \&\ v = x(j) \ \&\ j = n + k \ \&\ t \in H(v)] \\ &\Leftrightarrow \\ (\forall s,t,v)(\exists j)[y &= t - s \ \&\ s = u(m) \ \&\ v = x(j) \ \&\ j = n + k \implies t \in H(v)], \end{aligned}$$

so that $Z_{m,n}^k(\mathbf{x}, \mathbf{u})$ is $\boldsymbol{\Delta}_2^1$.

Finally, put

$$Z(\alpha \restriction n, \mathbf{x}, \mathbf{u}, k) = \bigcap_{i,j<n} Z_{\alpha(i),j}^k(\mathbf{x}, \mathbf{u}).$$

Then

$$Z^\varepsilon(\mathbf{x}, \mathbf{u}, k) = \bigcup_{\alpha \in A} \cap_{n=1}^\infty Z(\alpha \restriction n, \mathbf{x}, \mathbf{u}, k).$$

□

4.7 Proof of the Luzin set proposition

In the Luzin [resp. Sierpiński] case, let $\{N_\alpha : \alpha < \omega_1\}$ list all closed nowhere dense sets in $\mathbb{R}$ [all the $\mathcal{G}_\delta$-sets of measure zero] and let $\{\{u_\alpha^n\} : \alpha < \omega_1\}$ list all sequences. We construct, by transfinite induction, points t_α for $\alpha < \omega_1$ so that the sets $T_\alpha = \{t_\beta : \beta \leq \alpha\}$ avoid certain forbidden sets. The forbidden sets will have union a first category set [be a set of measure zero] and so it will be possible to select the next point in the transfinite induction.

We will neglect the Hamel basis property to begin with, and later show how to modify the construction to accommodate this additional property.

To secure the Luzin [Sierpiński] property, we aim to have

$$T_{\omega_1} \cap \bigcup_{\delta<\beta} N_\delta \subset T_\beta,$$

for $\beta < \omega_1$, as then $T = T_{\omega_1}$ meets any N_δ in at most a countable set. This can be arranged in the induction by ensuring that for $\alpha < \omega_1$we have for all $\beta < \alpha$ that

$$T_\alpha \cap \bigcup_{\delta<\beta} N_\delta \subset T_\beta. \tag{13}$$

We also require that the difference set of each T_α avoids $\mathbb{Q}$. Thus $T = T_{\omega_1}$ is the required Luzin set and $T - T$ avoids $\mathbb{Q}$, which implies that $T - T$ has empty interior.

Actually, it is more convenient to carry out the induction over limit ordinals. Suppose that T_α has been defined with α a limit ordinal, so that (13) holds, and

$$T_\alpha - T_\alpha \cap \mathbb{Q} = \emptyset.$$

We intend to select t so that the translates $t + u_\alpha^n$ shall all be included in $T_{\alpha+\omega}$, that is, so that $T_{\alpha+\omega} = T_\alpha \cup \{t + u_\alpha^n : n \in \omega\}$.

Consider our requirements. For the Luzin [Sierpiński] property at $\alpha + \omega$ in place of α in (13), we require:

$$t + u_\alpha^n \notin \bigcup_{\delta<\alpha} N_\delta \text{ i.e. } t \notin \bigcup_{\delta<\alpha} (N_\delta - u_\alpha^n).$$

For the forbidden differences to occur we require that for $\beta < \alpha$ we have

$$\pm(t + u_\alpha^n - t_\beta) \notin \mathbb{Q} \text{ i.e. } t \notin (\mathbb{Q} + t_\beta - u_\alpha^n).$$

Thus t must be selected to avoid the first category set [the measure zero set]

$$C = \bigcup_{\beta<\alpha} \bigcup_{n\in\omega} \left[\bigcup_{\delta<\alpha} (N_\delta - u_\alpha^n) \cup (\mathbb{Q} + t_\beta - u_\alpha^n) \right].$$

Note that it is not possible to arrange that the vectors in $T_\alpha \cup \{\, t + u_\alpha^n : n \in \omega\}$ do not introduce linear dependencies over $\mathbb{Q}$. For instance if the sequence $\mathbf{u}_\alpha = \{u^n\}$ is such that

$$u^{n+1} \in \mathrm{conv}_{\mathbb{Q}}\{u^1, .., u^n\},$$

then for any t we have

$$t+u^{n+1}\in \mathrm{conv}_{\mathbb{Q}}\{t+u^1,..,t+u^n\}$$

and we introduce linear dependencies (over $\mathbb{Q}$). The best that we can achieve is to include a Hamel basis in our Luzin [Sierpiński] set.

Accordingly, we now go on to show how to modify the construction so as to ensure that the set T contains a Hamel basis. We mimic an idea due to Erdős (see [Kucz] p. 267). Let $\{x_\alpha : \alpha<\omega_1\}$ list all real numbers. We assume, as before, that T_α has been defined inductively with the properties identified before and in addition the property that: for $\delta<\alpha$ the points x_δ are represented as rational convex combinations of members of T_α.

We suppose at stage α that x_α is not a rational convex combination of members of T_α. We need to include in the construction of $T_{\alpha+\omega}\backslash T_\alpha$ two real numbers u,v such that x_α will be represented as

$$x_\alpha=u+v.$$

We thus require that

$$\begin{aligned}\{u,v\} &\notin \bigcup_{\delta<\alpha} N_\delta,\text{ i.e. } u\notin \bigcup_{\delta<\alpha} N_\delta \text{ and } u\notin \bigcup_{\delta<\alpha} x_\alpha-N_\delta,\\ \pm(u-v) &\notin \mathbb{Q},\text{ i.e. } 2u\notin\mathbb{Q}+x_\alpha,\text{ and also } 2u\notin\mathbb{Q}-x_\alpha,\\ \pm(u-t_\beta) &\notin \mathbb{Q},\text{ i.e. } u\notin\mathbb{Q}+t_\beta,\text{ and also } u\notin\mathbb{Q}-t_\beta,\\ \pm(v-t_\beta) &\notin \mathbb{Q},\text{ i.e. } u\notin x_\alpha-t_\beta+\mathbb{Q},\text{ and also } u\notin\mathbb{Q}+t_\beta-x_\alpha.\end{aligned}$$

Again such a choice of u is clearly possible. We put $t_\alpha=u, t_{\alpha+1}=x_\alpha-u, t_{\alpha+n+2}=t+u_\alpha^n$ with t selected as earlier but with $T_{\alpha+2}$ replacing T_α. Evidently, this ensures that x_α is represented, that $T-T$ contains no intervals, and T meets every nowhere dense set in at most a countable set. □

Comment. In the absence of the assumption of (CH) the argument may be modified to give a set of reals of power continuum such that the set
(i) contains no non-empty perfect subset (so has inner measure zero),
(ii) has difference set with empty interior,
(iii) contains all sequences up to translation, and
(iv) contains a Hamel basis.

5 *Complements*

This section is devoted to some open problems, thoughts on directions of generalization, and comments to the main material which would have been out of place elsewhere.

Beyond regularity conditions. Having unchained the theory of regular variations from the gold-standard assumptions guaranteeing UCT (namely measure/category), the most pressing question is: to give practical criteria for verifying that the UCT holds. One immediate answer is provided by Heiberg's

Theorem [Hei] (or BGT, Theorem 3.2.5, p. 141) which refers to a limit quotient condition. The natural next answer is to put to use the combinatorial principles of Section 1.2. We offer an example of an alternative to Heiberg's Theorem. It is still formulated in the same spirit, but relaxes conditions imposed on the quotient function, and removes all reference to limits. In the intuitive language of h_x of Section 1.1 the condition we name the Heiberg-Seneta condition 'factorizes out of h_x its dependence on x' locally, whereas the First Heiberg-Seneta Theorem (for which see below) factorizes out 'dependence on x at infinity', studying in essence an appropriate application of L'Hospital's Rule.

The theorem below is inspired by [AER] and our Main Theorem UCT. For proof and variants, see the companion paper [BOst2].

Definition. We say that h satisfies the **Heiberg-Lipschitz condition** if there are two positive functions φ, g defined on $\mathbb{R}_+$ such that:
(i) $g(x)$ is decreasing to 0 as $x \to \infty$;
(ii) $\varphi(t) \to \infty$ as $t \to \infty$;
(iii) for all $x, t > 0$, there is $x(t)$ between x and $x + t$ such that

$$|h(t + x) - h(x)| = \varphi(t)g(x(t)). \tag{14}$$

The final condition is modelled after the mean-value theorem. The assumptions imply that for all $x, t > 0$

$$\varphi(t)g(x + t) \leq |h(t + x) - h(x)| \leq \varphi(t)g(x),$$

and the right-hand inequality mimics the inequality appearing in BGT p. 11 which was our initial motivation.

Observation. *If h satisfies the Heiberg-Lipschitz condition, then h is slowly varying.*

For,

$$\lim_{x\to\infty} |h(t + x) - h(x)| \leq \lim_{x\to\infty} \varphi(t)g(x) = 0.$$

In [BOst2] we prove inter alia the result below. This is a new result, complementing such results as [Hei], [Sen1], [Sen2]; see BGT Theorem 1.4.3 p. 18-19. For details of the *First Heiberg-Seneta Theorem* see below. Since (**NT**) acts here to extend our understanding of regular variation, we are motivated to ask, later, whether (**NT**) might open the door to new results in automatic continuity in algebra.

Theorem (Second Heiberg-Seneta Theorem). *For h satisfying the Heiberg -Lipschitz condition the following are equivalent.*

(i) *UCT holds for h.*

(ii) *The family $\{\varphi^{-1}((0, n)) : n \in \omega\}$ traps sequences by translation.*

(iii) *The family $\{\varphi^{-1}((0, n)) : n \in \omega\}$ contains almost all terms of every bounded sequence.*

(iv) *The family* $\{\varphi^{-1}((0,n)) : n \in \omega\}$ *contains every bounded sequence.*

De Haan theory. The study of functional relations of the form (RV), or (RV_+), is *Karamata theory*, in the terminology of BGT Ch. 1,2. Related is the study of *de Haan theory* – that of relations of the form

$$\frac{f(\lambda x) - f(x)}{g(x)} \to h(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0 \qquad (deH)$$

(BGT, Ch. 3). See BGT §3.0 for the inter-relationships between the two (de Haan theory both contains Karamata theory, and refines it by filling in 'gaps'). Our approach here to Karamata theory extends to de Haan theory along similar lines.

In de Haan theory, the relevant limit function in (deH) is

$$h(\lambda) = \begin{cases} \frac{\lambda^\rho - 1}{\rho}, & \rho \neq 0, \\ \log \lambda, & \rho = 0. \end{cases}$$

The Ash-Erdös-Rubel results [AER] and Heiberg-Lipschitz condition have something of a de Haan rather than a Karamata character. See e.g. BGT Th. 3.1.10a,c for illustrations of this.

Weakening quantifiers. It is both interesting and useful to see to what extent the quantifier $\forall$ in (RV), (deH) may be weakened to 'for some', plus some side-condition. The prototypical result here is (BGT Th. 1.4.3 in the Karamata case – cf. Th. 3.2.5 in the de Haan case) the following result.

Theorem (First Heiberg-Seneta Theorem). *Write*

$$g^*(\lambda) := \limsup_{x \to \infty} f(\lambda x)/f(x),$$

and assume that

$$\limsup_{\lambda \downarrow 1} g^*(\lambda) \leq 1.$$

Then for a positive function f, the following are equivalent:
(i)(RV) and (ρ) hold for some ρ.
(ii) The limit $g(\lambda)$ in (RV) exists for all λ in a set of positive measure, or a non-meagre Baire set.
(iii) The limit $g(\lambda)$ in (RV) exists, finite, for all λ in a dense subset of $(0,\infty)$.
(iv) The limit $g(\lambda)$ in (RV) exists, finite, for $\lambda = \lambda_1$, λ_2 with $(\log \lambda_1)/(\log \lambda_2)$ *finite and irrational.*

This question of weakening of quantifiers is treated in detail in [BG1] (where the above is Th. 5.7). The original motivation was the study of Frullani integrals; see [BG2] §6, BGT §1.6.4, Berndt [Ber], p. 466-467.

Beyond the real line. The theory as presented here is, to quote the preface of BGT, 'essentially a chapter in real variable theory'. We mention here the availability of a well-developed theory going beyond the real line, for which see [DW]. We raise the possibility of extending the theory of regular variation in this direction.

Conditions on graphs rather than preimages. In the current context, there is a subtlety in play in regard to classifying functions according to the character of graphs rather than according to the character of their preimages. Recall that if $\{I_n^k : n \in \omega\}$ is for each k a family of disjoint intervals of diameter $1/k$ covering $\mathbb{R}$, then

$$f = \bigcap_{k=1}^{\infty} \bigcup_{n\in\omega} f^{-1}(I_n^k) \times I_n^k \text{ and } f^{-1}(I_n^k) = \text{proj}(f \cap \mathbb{R} \times I_n^k).$$

Thus f has Borel (analytic) graph iff the preimages $f^{-1}(I_n^k)$ are Borel (analytic). However, if the preimages $f^{-1}(I_n^k)$ are all co-analytic, then the complementary sets $\mathbb{R}\backslash f^{-1}(I_n^k) = f^{-1}(\mathbb{R}\backslash I_n^k) = \bigcup_{m\neq n} f^{-1}(I_m^k)$ are also co-analytic. Thus $f^{-1}(I_n^k)$ is both co-analytic and analytic and hence Borel by Souslin's Theorem. This implies that each $f^{-1}(I_n^k)$ is Borel.

Automatic continuity. The power solutions (ρ) to the Cauchy functional equation (CFE) (or their additive analogues (CFE_+), (ρ_+)) are homomorphisms. As in BGT §1.1.4, *continuous* solutions must be of this form (the others being pathological – the 'Hamel pathology'). The role of assumptions of measurability or the Baire property, or their common generalizations considered here, may thus be considered the elimination of these Hamel pathologies. This relates to the subject of *automatic continuity*, in which algebraic conditions suffice for this purpose; see e.g. [Dal1], [Dal2], [DW]. A sample result ([Dal1], Prop. 4.2): if A is a commutative, semi-simple Banach algebra, then every homomorphism from a Banach algebra to A is continuous. Here, the reals as a vector space over the rationals forms an algebra, but this fails to be Banach as the operations are discontinuous.

We offer in the companion paper [BOst3] some new contributions to automatic continuity, among them a simple result that '*if the sets* $\{T_m : m \in \omega\}$ *trap bounded sequences, then* $T = \bigcup T_m \in \mathfrak{B}$', where $\mathfrak{B}$ is the Ger-Kuczma class alluded to in Section 3. We have seen above that (**NT**) may be used to offer new theorems in the theory of regular variation (the Second Heiberg-Seneta Theorem). We ask whether such a use on (**NT**) may be extended to questions of automatic continuity in the Banach algebra context; we have identified in [BOst3] some immediate applications in the Euclidean context.

Effective versions of the trapping property. Are there 'effective' versions (see [Mos] Chapter 3) of the Existence Theorem (for trapping families, cf. Section 1)? Here we refer to the light-face versions of the bold-face projective classes introduced in Section 2, so that the hyper-arithmetic sets are effective versions of the Borel sets. For example, what may be said about a Σ^1_1 set trapping by translation a hyperarithmetic sequence?

The $\mathbf{NT}_{\mathbf{\Gamma}}$ *property.* Let $\mathbf{NT}_{\mathbf{\Gamma}}$ be the statement that **3-NT**$_h$ holds, i.e. $(\forall\varepsilon > 0)\mathbf{NT}(\{T^{\varepsilon}_k(\mathbf{id}) : k \in \omega\})$, for all functions h of a class $\mathbf{\Gamma}$. The statement holds in the models of Solovay [So] and of Shelah [She] for any $\mathbf{\Gamma}$. We know that the class of models of (PD) with $\mathbf{\Gamma} = \mathbf{\Delta}^1_2$ satisfies $\mathbf{NT}_\Gamma$. What other classes of models of (ZF) and classes $\mathbf{\Gamma}$ have this property?

Similar sequences: generic arguments. One can see that a non-meagre set A with the Baire property traps sequences by an amendment of a forcing argument given by Miller in [Mil1]. Let $\{u_n\}$ be a convergent sequence with limit u. Specifically, suppose that A is co-meagre in the interval (a, b). Choose $\varepsilon > 0$ and a rational q so that $a + \varepsilon < q < b - \varepsilon$. Thus for some N we have that $a + \varepsilon < q + (u_n - u) < b - \varepsilon$ for all $n > N$. Let $x \in (-\varepsilon, \varepsilon)$ be a Cohen real. Then for every $n \in \omega$, the number $q + (u_n - u) + x$ is a Cohen real. Since $a < q + (u_n - u) + x < b$ we deduce that for $n > N$ we have $q + x - u + u_n \in A$. Thus a translate of almost all of the sequence $\{u_n\}$ is in A. A similar argument may be given replacing 'Cohen real' by (Solovay) 'random real' to show that a translate of almost all of any sequence $\{u_n\}$ is contained in a measurable set A of positive measure. This pin-points the 'generic' nature of the arguments in Section 4.3.

Non-duality between measure and category. We have been lucky in the Existence Theorem (for trapping families) in that the measure/category analogy holds. See [DoF], [Bart], [BGJS] for its limitations.

Analogy with topological games. The non-empty Souslin set condition is game-theoretic in content along the lines of the topological 'cutting and choosing' games of Telgársky [Tel1], or the 'point and cover' games of Galvin [G] (dually:'point and omit'). Two players, cutter and shifter, take turns selecting respectively x_n (say with $x_n > n$) and u_n as well as $\alpha(n)$. (As to cutting, recall that $\mathbb{R} = \bigcup_{x>n} H(x)$.) The shifter wins if $\cap^{\infty}_{n=1} Z(\alpha \upharpoonright n)$ is non-empty. Then UCT holds if the shifter has a winning strategy. See also [OT], [Tel2], [Sc], and [ScSz] For background on game-theoretic aspects of analytic sets, see [MaKe].

Intersection properties. When the sets $Z(\alpha \upharpoonright n, \mathbf{x}, \mathbf{u}, k)$ are Souslin-Δ^1_2 we can re-write the set $Z^{\varepsilon}(\mathbf{x}, \mathbf{u}, k)$ as

$$Z^{\varepsilon}(\mathbf{x}, \mathbf{u}, k) = \bigcup_{\alpha \in A} \cap^{\infty}_{n=1} H(\alpha \upharpoonright n, \mathbf{x}, \mathbf{u}, k),$$

where the sets $H(\alpha \upharpoonright n, \mathbf{x}, \mathbf{u}, k)$ are Δ^1_2. The Souslin scheme can be refined so that each of the sets

$$Z(\alpha) = \cap^{\infty}_{n=1} Z(\alpha \upharpoonright n, \mathbf{x}, \mathbf{u}, k)$$

comprises at most one point. Hence if we put

$$\mathcal{N} = \{\alpha : Z(\alpha) \neq \varnothing\},$$

then, since $Z^{\varepsilon}(\mathbf{x},\mathbf{u},k)$ is itself Σ^1_2, we see that $Z : \mathcal{N} \to Z^{\varepsilon}(\mathbf{x},\mathbf{u},k)$ is continuous and $\mathcal{N}$ is Σ^1_2. This last observation is reminiscent of the arguments in [Ost2] (Section 3) where some special Souslin schemes are considered.

The similarity may be somewhat enhanced by the following argument. Suppose that $\mathbf{u} = \{u_m : m \in \omega\}$ is not trapped by $T^{\varepsilon}_k(\mathbf{x})$. Then for all t the set $\{t + u_m : m \in \omega\} \cap T^{\varepsilon}_k(\mathbf{x})$ is finite. That is, for each t there is $N(t)$ such that $\{t + u_m : m > N(t)\} \cap T^{\varepsilon}_k(\mathbf{x})$ is empty, or equivalently, for $m > N(t)$ $t \notin T^{\varepsilon}_k(\mathbf{x}) - u_m = \cap^{\infty}_{n=1} H(x_{n+k}) - u_m$. It follows that for each point t there is $N(t)$ such that for all n we have

$$t \notin H(x_{k+n}) - u_m = Z^k_{n,m} \text{ for } m > N(t).$$

Thus

$$t \in Z^k_{n,m} \text{ for } m \leq N(t).$$

Now recall that

$$Z(\alpha \upharpoonright n, \mathbf{x}, \mathbf{u}, k) = \bigcap_{i,j<n} Z^k_{\alpha(i),j}(\mathbf{x},\mathbf{u}).$$

Hence the Souslin representation has the defining 'point-finite property' making it 'meta-Souslin' in the sense of the just cited [Ost2]. Here there is also a link to a multiple separation theorem of Novikov type on the lines of [DJOR] but at the next projective level.

Character complexity induced by hidden quantifiers. We offer the promised example of a 'far from innocuous' hidden occurrence of quantifiers. The vector sum of two sets S, T is formally defined by

$$S + T = \{r : (\exists s, t)[s \in S \ \& \ t \in T \ \& \ r = s - t]\}.$$

It is the occurrence of the quantifier here that is responsible for altering the complexity of the sum well beyond the complexity of the summands. Thus if the summands are co-analytic sets the vector sum need not be measurable. A specific example may be constructed by appeal to Gödel's Axiom $V = L$ and taking for the summands co-analytic Hamel bases; see [Kucz] p. 256. For further details of the vector sum see [NSW].

Continuum Hypothesis. In elucidating the sequence trapping property we restricted ourselves to the simplest context, that of assuming CH. We draw the reader's attention to two alternative hypotheses: Martin's Axiom (see [F1]) and the Covering Property Axiom CPA (see [CP]).

Multi-dimensional regular variation. As mentioned earlier, the theory in BGT deals with regular variation in one dimension. In recent years, much effort has been devoted to extensions of this theory to many dimensions, including infinitely many dimensions. Since the motivation is mainly probabilistic, we give the probabilistic formulation:

$$nP(\mathbb{X}/a_n \in \cdot) \to \mu(\cdot),$$

where $\mathbb{X}$ is a random vector (possibly infinite dimensional), a_n is a sequence and μ is a measure. For background here, see e.g. [HLMS].

Postscript.

The thinking of Paul Erdös permeates this paper, and so it is a pleasure to dedicate it to his memory.

This paper is, for the first author, a return to the foundational first sections of BGT with the benefit of twenty years' worth of hindsight – or, in the case of [BG1], [BG2], twenty-five. It may be regarded as 'the missing zeroth chapter' of BGT providing sections on 'foundational preliminaries on descriptive set theory' and 'infinitary combinatorics'. We hope that the mathematical logic community will find in this field a new playground. For a similar return to the motivating last chapter of BGT, on probability theory, see [Bin].

To close, we quote Weil [We] p. 234 on foundations: 'We know that mathematicians are seldom influenced in their work by philosophical considerations, even when they profess to take them seriously; we know that they have their own way of dealing with foundational matters by an alternation between possibly reckless disregard and the most painful critical attention'. We have done our best here to steer a middle course between Scylla and Charybdis.

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