Homotopy and the Kestelman-Borwein-Ditor Theorem

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To David Borwein on his 85th birthday

Abstract. The Kestelman-Borwein-Ditor Theorem, on embedding a null sequence by translation in (measure/category) 'large' sets, has two generalizations. Miller [**MilH**] replaces the translated sequence by a 'sequence homotopic to the identity'. The authors, in [**BOst9**], replace points by functions: a uniform functional null sequence replaces the null sequence and translation receives a functional form. We give a unified approach to results of this kind. In particular, we show that (i) Miller's homotopy version follows from the functional version, and (ii) the pointwise instance of the functional version follows from Miller's homotopy version.

We begin by recalling the following result, due in this form in the measure case to Borwein and Ditor [**BoDi**], but already known much earlier albeit in somewhat weaker form by Kestelman ([**Kes**] Th. 3), and rediscovered by Trautner [**Trau**] (see [**BGT**] p. xix and footnote p. 10). Below, for P a set (or property) of reals that is measurable/Baire, we say that 'P holds for generically all t' to mean that $\{t : t \notin P\}$ is null/meagre.

Theorem 1. ***The Kestelman-Borwein-Ditor Theorem* (*KBD Theorem*).** *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null/Baire and non-meagre, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Furthermore, for any density point u of T, there is $t \in T$ arbitrarily close to u for which the above holds.

We are concerned in this paper with what we loosely term 'smooth generalizations' of the KBD Theorem, in that some form of differentiability is present in the assumptions concerning mappings on the pairs (t, z). In a companion paper [**BOst11**] we derive a common non-smooth generalization in which only continuity is assumed (the mappings are homeomorphisms).

As to the limits of the theorem, these are best seen through terminology motivated by Kestelman's work in [**Kes**]. Let us say that T is *universal,* or respectively

2000 *Mathematics Subject Classification.* 26A03.

Key words and phrases. measure, category, measure-category duality, differentiable homotopy.

subuniversal, for null sequences if any null sequence z_n has a translation $t+z_n$ which is *almost contained* in the set T (i.e. all but a finite number of its terms lie in the set), or respectively has a subsequence in T. The theorem asserts subuniversality. Universality occupied combinatorialists for its limitations. Thus Borwein and Ditor [**BoDi**] constructed a measurable T of positive measure and a null sequence z_n such that no shift of the sequence is almost contained in T. Under Martin's Axiom (MA) Komjáth [**Komj-1**] constructs a measure zero, first category set T such that T is universal, and in fact contains a translated copy of every set of cardinality less than continuum; in [**Komj-2**], generalizing [**BoDi**], he constructs a measurable set T of positive measure and a null sequence z_n such that T fails to contain almost all of any translate of any scalar multiple λz_n. (See [**Mil**] for the associated literature and for 'forcing' connections with genericity.)

We are concerned in this paper with what we loosely term 'smooth generalizations' of the KBD Theorem, in that some form of differentiability is present in the assumptions concerning mappings on the pairs (t, z). In a companion paper [**?**] we derive a common non-smooth generalization in which only continuity is assumed (the mappings are homeomorphisms).

Our point of view is dictated by the central role that subuniversality has in the fundamental theorems of regular variation.

We are also concerned by a further aspect – the 'pointwise' nature of theorem, because of the sequence of *points* z_n which is in the datum. The KBD Theorem was first generalized by Harry Miller [**MilH**], as below, by replacing $t+z$ with a more general function $H(t, z)$ (originally defined on $\mathbb{R}\times\mathbb{R}$). We need a definition (the terminology is ours).

Definition 1. (***Miller homotopy,*** *cf.* [**MilH**]). *Let U be open and let I be a non-degenerate interval (possibly infinite, or semi-infinite). We call a function $H : U \times I \to \mathbb{R}$ a Miller homotopy acting on U with distinguished point z_0 if:*

(i) $H(u, z_0) \equiv u$, for all $u \in U$,
(ii) H has continuous first-order partial derivatives H_1 and H_2, and
(iii) $H_2(u, z_0) > 0$, for all $u \in U$.

Remark 1. *As the function H is differentiable, and hence jointly continuous, it is natural to regard it as establishing a homotopy to the identity (albeit utilizing a distinguished point z_0 other than* 0*, and some interval about z_0 instead of the customary unit interval). Condition (iii) is only a non-stationarity requirement (map $z \to -z$, $z_0 \to -z_0$, if $H_2(u, z_0) < 0$).*

Convention. We will refer to the distinguished point z_0 as the 'null point' and any sequence $z_n \to z_0$ converging to the null point as a 'null sequence'. Thus in the case $H(u, z) = u + z$ with $z_0 = 0$, the sequence $z_n \to z_0$ is a null sequence in the customary sense.

Theorem 2. ***Miller's Homotopy Theorem.*** *Let H be a Miller homotopy acting on an open set U with distinguished point z_0. Let $z_n \to z_0$ be a null sequence* *and let $T \subseteq U$ be measurable, non-null/Baire, non-meagre. Then, for generically all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{H(t, z_m) : m \in \mathbb{M}_t\} \subseteq T.$$

Stated thus, this too is a 'pointwise' theorem, but it is noteworthy that the substitutions,

$$\mathbf{z}_n(t) := H(t, z_n) - t \text{ and } \mathbf{u}_n(t) = t + \mathbf{z}_n(t), \tag{0.1}$$

allow a functional reinterpretation of the theorem (we have used bold type to distinguish functions from points). We may regard the sequence of functions $\{\mathbf{z}_n(t)\}$, which converge to zero (see below), as the datum and now the conclusion of Miller's theorem reads: $\{t + \mathbf{z}_m(t) : m \in \mathbb{M}_t\} \subseteq T$, or, in short,

$$\{\mathbf{u}_m(t) : m \in \mathbb{M}_t\} \subseteq T. \tag{0.2}$$

Thus Miller's Theorem becomes simply a functional version of the KBD Theorem. We now quote one of the functional generalizations which goes beyond the KBD setting. This involves a continuously differentiable function $f(.)$; see [**BOst9**] for the proof. It will be clear from its statement that the case case $f(u) = u$ yields the Miller Theorem in the form (0.2). We will need several definitions.

Definition 2. (***Uniformity - pointwise***)**.** *We say that the null sequence $\{z_n\} \to z_0$ is a uniformly null sequence, or that $z_n \to z_0$ uniformly, if for some positive constant K,*

$$|z_n - z_0| \leq K2^{-n}, \text{ for all } n \in \omega.$$

Definition 3. (***Uniformity - functionwise***)**.** *We say that the sequence of functions $\{\mathbf{z}_n(.)\}$ is a uniformly null function sequence on U, or that $\mathbf{z}_n(.) \to z_0$ uniformly on U, if each $\mathbf{z}_n(.)$ is measurable/Baire and, for some positive constant K,*

$$\max\{|\mathbf{z}_n(u)|\} \leq K \cdot 2^{-n}, \text{ for all } n \in \omega \text{ and all } u \in U.$$

Definition 4. (***Bi-Lipschitz***)**.** *We call a uniformly null sequence $\{\mathbf{z}_n(.)\}$ bi-Lipschitz if the mappings $t \to \mathbf{u}_n(t)$ are bi-Lipschitz uniformly in n, i.e. for some α, β and all n we have*

$$0 < \alpha \leq 1 + \frac{\mathbf{z}_n(u) - \mathbf{z}_n(v)}{u - v} \leq \beta, \text{ for } u \neq v.$$

In particular $\mathbf{z}_n'$ is bounded away from -1, except perhaps at countably many points.

The following theorem is proved in [**BOst9**] (where further improvements, motivated by convex analysis, are given); it is manifestly a 'functionwise' theorem.

Theorem 3. (***Generic Reflection Theorem***)***.*** *Let T be measurable/Baire. Let $f(.)$ be continuously differentiable and non-stationary at generically all points. Let $\{\mathbf{z}_n(.)\} \to 0$ be a uniformly null sequence that is bi-Lipschitz with*

$$1 + f'(t)z_n'(t) > 0, \text{ for all } n, \tag{0.3}$$

for generically all $t \in T$. Then, for generically all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that

$$t + f(\mathbf{u}_n) - f(t) \in T, \text{ for all } n \in \mathbb{M}_t. \tag{0.4}$$

In particular, if f is linear and $f(t) = \alpha t$ with $\alpha \neq 0$, then, for generically all $u \in T$, there is an infinite set $\mathbb{M}_u$ such that

$$\alpha \mathbf{u}_n(u) + (1 - \alpha)u \in T \text{ for all } n \in \mathbb{M}_u. \tag{0.5}$$

Setting $\alpha = 1$ in (0.5) thus yields (0.2). We will see that the apparently stronger form – the Homotopic Reflection Theorem – is equivalent to this.

Proposition 1. (***Canonical Homotopy***)***.*** *Let U be an open set and let H be a Miller homotopy acting on U with distinguished point z_0. Let f be continuously differentiable and increasing on U. Then*

$$F(u, z) := u + f(H(u, z)) - f(u)$$

is a Miller homotopy acting on U with distinguished point z_0. In particular, the canonical homotopy

$$F(u, z) := u + f(u + z) - f(u)$$

is a Miller homotopy acting on U with distinguished point $z_0 = 0$.

Proof. This is clear since $F(u, z_0) = u$, and $F_2(u, z_0) = f'(u)H_2(u, z_0)$. □

We call the particular case canonical for two reasons. In the first place, if $F(u, z) := f(H(u, z)) + g(u)$ is a Miller homotopy, then the substitution $z = z_0$ yields $g(u) = u - f(u)$, making the choice of $g(.)$ unique, and H is then recoverable from F. The second reason is even more fundamental; we defer this to the end of the paper.

Proposition 2. (***Composition Theorem***)***.*** *Let U be an open set and let H and F be Miller homotopies acting on U with distinguished point z_0.Then*

$$G(u, z) := F(H(u, z), z)$$

is a Miller homotopy acting on some open subset of U with distinguished point z_0.

Proof. As $H(u, z_0) = u$, by continuity, for any $u \in U$, there is a neighbourhood $W \times J$ of (u, z_0), so that H maps $W \times J$ into U and $W \subseteq V$. The rest is clear since

$$G_2(u, z_0) = F_1(H(u, z_0), z_0)H_2(u, z_0) + F_2(H(u, z), z_0) = H_2(u, z_0) + F_2(H(u, z), z_0) > 0.$$

□

PROPOSITION 3. *Let H be a Miller homotopy acting on an open set U with distinguished point z_0. Let $z_n \to z_0$ uniformly. Put*

$$\mathbf{z}_n(u) := H(u, z_n) - u.$$

Then

(i) $\{\mathbf{z}_n(u)\} \to 0$,
(ii) $\{\mathbf{z}_n(u)\}$ *is locally uniformly null in U,*
(iii) *for some large enough N, $\{\mathbf{z}_n(u) : n \geq N\}$ is locally bi-Lipschitz in U.*

PROOF. Since $H_1(t, z_0) = 1$, for any t, we may invoke the Mean Value Theorem to write the Taylor expansion for (u, z) near (t, z_0) as

$$H(u, z) = t + (u - t) + H_2(t, z_0)(z - z_0) + o(||(u - t, z - z_0)||). \tag{0.6}$$

Hence,

$$\mathbf{z}_n(u) = H_2(t, z_0)(z_n - z_0) + o(||(u - t, z_n - z_0)||). \tag{0.7}$$

Thus the sequence has limit zero, and uniformity is clear provided u is close enough to t.Again by the Mean Value Theorem, for some $w_n = w_n(u, v)$, we have

$$H(u, z_n) - H(v, z_n) = H_1(w_n, z_n)(u - v),$$

so

$$\mathbf{z}_n(u) - \mathbf{z}_n(v) = (H_1(w_n, z_n) - 1)(u - v).$$

Hence

$$1 + \frac{\mathbf{z}_n(u) - \mathbf{z}_n(v)}{u - v} = H_1(w_n, z_n).$$

But $H_1(t, z_0) = 1$, so near (t, z_0) we can ensure that $\frac{1}{2} \leq H_1(w_n, z_n) \leq 2$. □

REMARK 2. *Formula (0.7) indicates that in practice $\{\mathbf{z}_n(u)\}$ is close to monotonic if $\{z_n\}$ is (see e.g.* [**BGT**] *Section 1.7.6 for slow decrease and related matters).*

PROPOSITION 4. *Let H be a Miller homotopy acting on an open set U with distinguished point z_0. Let $z_n \to z_0$ monotonically. Then the functions*

$$h_n(t) := H(t, z_n)$$

are all homotopic to the identity, and local diffeomorphisms, hence locally 'bi-Lipschitz' (thus preserve null sets both ways); moreover

$$h_n(t) \to t, \text{ ultimately monotonically.}$$

PROOF. Invertibility of h_n follows from the Inverse Function Theorem. As before the Taylor expansion near (t_0, z_0) is given by (0.6). From here we deduce that

$$h_n(t) = t + H_2(t_0, z_0)(z_n - z_0) + o(||(t - t_0, z_n - z_0)||), \text{ as } t \to t_0 \text{ and } n \to \infty.$$

Thus h_n is almost a shift and $h_n(t) \to t$. The ultimate monotonicity, at any t, follows from the continuity and positivity of the partial derivative H_2 at (t, z_0). □

COROLLARY 1. (***Miller's Theorem***) *The functionwise Generic Reflection Theorem implies the pointwise Miller Homotopy Theorem.*

Proof. Indeed, the definition (0.1) and the argument following it are now justified by Proposition 3. So Miller's Theorem follows from the Generic Reflection Theorem by taking $f(u) = u$. □

Now we obtain a pointwise converse: Miller's Homotopy Theorem implies the pointwise Homotopic Generic Reflection Theorem.

Theorem 4. (***Pointwise homotopic generic reflection***)**.** *Let U be an open set and let H be a Miller homotopy acting on U with distinguished point z_0. Let $T \subseteq U$ be measurable, non-null/Baire, non-meagre and let $z_n \to z_0$. Then Miller's theorem implies that, for generically all $u \in T$, there is an infinite $\mathbb{M}_u$ such that*

$$\{u + f(u_n) - f(u) : m \in \mathbb{M}_u\} \equiv \{u + f(H(u, z_m)) - f(u) : m \in \mathbb{M}_u\} \subseteq T.$$

In particular, for $H(t, z) = t + z$ and $z_0 = 0$, we have

$$\{u + f(u + z_m) - f(u) : m \in \mathbb{M}_u\} \subseteq T.$$

Proof. By Proposition 1

$$F(t, z) := t + f(H(t, z)) - f(t)$$

is a Miller homotopy, so we may apply Miller's Theorem to the homotopy $F(t, z)$ to obtain

$$\{F(t, z_m) : m \in \mathbb{M}_t\} \subseteq T.$$

□

A first homotopic generalization of the Generic Reflection theorem may be obtained by taking a function sequence $z_n(u)$ and transforming by a Miller homotopy H. Then,

$$\tilde{z}_n(u) = H(u, z_n(u)) - u$$

is uniformly null and locally bi-Lipschitz. However, a conclusion in the form

$$\{u + f(H(u, z_m(u))) - f(u) : m \in \mathbb{M}_u\} \subseteq T$$

is already available, in the equivalent form

$$\{u + f(u + \tilde{z}_m(u)) - f(u) : m \in \mathbb{M}_u\} \subseteq T.$$

Our final result is obtained by replacing the f construction here by the obvious generalization, suggested by Propositions 1 and 2, a composition Miller homotopy F. We see below that the Generic Reflection Theorem implies such a generalization of itself. We thus have the following result.

Theorem 5. (***Homotopic Generic Reflection***)**.** *Let H and F be Miller homotopies acting on an open set U with distinguished point z_0. Let $T \subseteq U$ be non-null/non-meagre and let $\{\mathbf{z}_n(u)\}$ be a uniformly null sequence that is bi-Lipschitz on U (so converging to z_0). If*

$$1 + [F_2(u, z_0) + H_2(u, z_0)]\mathbf{z}_n'(u) > 0, \text{ for all } n,$$

for generically all $u \in U$, then, for generically all $u \in T$, there is an infinite $\mathbb{M}_u$ such that

$$\{F(H(u, \mathbf{z}_m(u)), \mathbf{z}_m(u)) : m \in \mathbb{M}_u\} \subseteq T.$$

In particular, let f be continuously differentiable and increasing in U, and

$$1+[f'(u)H_2(u,z_0)]\mathbf{z}_n'(u)>0, \text{ for all } n,$$

then, for generically all $u\in T$, there is an infinite $\mathbb{M}_u$ such that

$$\{u+f(\mathbf{u}_n(u))-f(u):m\in\mathbb{M}_u\}=\{u+f(H(u,\mathbf{z}_m(u)))-f(u):m\in\mathbb{M}_u\}\subseteq T.$$

Proof. According to Proposition 2 the equation

$$G(t,z)=F(H(t,z),z)$$

defines a homotopy provided the composition is valid. Let

$$\bar{\mathbf{z}}_n(t):=F(H(t,\mathbf{z}_n(t)),\mathbf{z}_n(t))-t=G(t,\mathbf{z}_n(t))-t.$$

Thus

$$1+\bar{\mathbf{z}}_n'(t)=F_1(H(t,\mathbf{z}_n(t)),\mathbf{z}_n(t))H_1(t,\mathbf{z}_n(t))+[F_1(H(t,\mathbf{z}_n(t)),\mathbf{z}_n(t))H_2(t,\mathbf{z}_n(t))+F_2(H(t,\mathbf{z}_n(t)),\mathbf{z}_n(t))]\mathbf{z}_n'(t).$$

Then, by Proposition 3, this is locally a uniformly null, bi-Lipschitz sequence tending to zero. Hence, the Generic Reflection Theorem (applied with $f(u)=u$) yields the desired conclusion:

$$\{t+\bar{\mathbf{z}}_m(t):m\in\mathbb{M}_u\}\subseteq T,$$

or

$$\{F(H(u,\mathbf{z}_m(u)),\mathbf{z}_m(u)):m\in\mathbb{M}_u\}\subseteq T.$$

□

Remark 3. *The Homotopic Generic Reflection Theorem follows from the special linear case $f(u)=u$ of the Generic Reflection Theorem. In turn the Homotopic Generic Reflection Theorem may be applied to $F(t,z)=t+f(t+z)-f(t)$, for a general $f(.)$ to obtain the conclusion of the Generic Reflection Theorem. Thus the special linear case $f(u)=u$ contains the nub; it is actually equivalent to the general case of the Generic Reflection Theorem. This is ultimately the reason for regarding the homotopy in Proposition 1 as canonical.*

Remark 4. *There is an alternative approach to the Homotopic Reflection Theorem. One can adapt the proof in* [**BOst9**] *of the Generic Reflection Theorem, as follows. Firstly, we need to define the analogue of the f-congugate; the F-conjugate of $\{\mathbf{z}_m(t)\}$ is defined to be*

$$\bar{\mathbf{z}}_m(t):=F(H(t,\mathbf{z}_m(t)),\mathbf{z}_m(t))-F(t,z_0)=F(H(t,\mathbf{z}_m(t)),\mathbf{z}_m(t))-t.$$

Secondly, as may be expected from Proposition 3, we set

$$f_n(t):=F(H(t,\mathbf{z}_n(t)),\mathbf{z}_n(t)).$$

Hence

$$f_n'(t):=F_1(H(t,\mathbf{z}_n(t)),\mathbf{z}_n(t))H_1(t,\mathbf{z}_n(t))+[F_1(H(t,\mathbf{z}_n(t)),\mathbf{z}_n(t))H_2(t,\mathbf{z}_n(t))+F_2(t,\mathbf{z}_n(t))]\mathbf{z}_n'(t),$$

so that $f_n(u)$ is increasing for u near t_0 (with at most countably many exceptions) provided

$$1+[F_2(t,z_0)+H_2(t,z_0)]\mathbf{z}_n'(t)>0,$$

since $H_1(t_0,z_0)=F_1(t_0,z_0)=1$.
Now by (0.6) applied to F we have

$$f_n(t)=H(t,\mathbf{z}_n(t))+(t-t_0)+F_2(t_0,z_0)(\mathbf{z}_n(t)-z_0)+o(||(t-t_0,\mathbf{z}_n(t)-z_0)||),$$

since $H(t_0, z_0) = t_0$. Applying (0.6) again, but now to H, we have

$$f_n(t) = t + [F_2(t_0, z_0) + H_2(t_0, z_0)](\mathbf{z}_n(t) - z_0) + o(||(t - t_0, \mathbf{z}_n(t) - z_0)||).$$

Hence, since H_2 and F_2 are continuous, for u sufficiently close to t and n large enough, we have the critical inequality

$$|f_n(u) - u| \leq M|z_n|,$$

for some constant M. This is all that is needed for the proof in [**BOst9**] *to proceed.*

Remark 5. *The overall conclusion is that all the functional reflection theorems are equivalent. This is because, in the limit, all the null sequences act like first-order infinitesimals added to the identity. Thus, despite its being restricted to the pointwise case, Miller's Theorem falls barely short of the full story. The essence of the KBD Theorem is that it applies to a wide class of sequences homotopic to the identity, as Miller was the first to observe.*

Remark 6. *Our topological generalization of the KBD Theorem, the Category Embedding Theorem of* [**BOst11**] *(involving sequences of homeomorphisms), has recently given rise in* [**BOst12**] *to a KBD Theorem for normed groups (involving left or right translations of sequences). In view of this and in the spirit of McShane* [**McSh**] *it would be interesting to study group analogues of the results here.*

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