Duality and the Kestelman-Borwein-Ditor Theorem

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In memoriam Caspar Goffman (1913 - 2006)
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Abstract

We deduce both the measure and the category cases of the theorem in the title from one category theorem, using the density topology for the first and the usual Euclidean topology for the second. We also obtain a unified proof for the Uniform Convergence Theorem for slowly varying functions.

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1 Introduction

The theme of this paper is measure-category duality – the close parallel between null sets in measure theory and meagre sets in topology. The aspect of duality that motivated us here is the Uniform Convergence Theorem for slowly varying functions, the basic result in the theory of regular variation. See [BGT], where the (Lebesgue) measure and (Baire) category cases are developed in parallel. Now the measure case has traditionally been regarded as the primary case, and the category case as the secondary case that one obtains as a corollary, on substituting ‘Baire’ for ‘measurable’ and ‘meagre’ for ‘null’. This is partly for historical reasons – the measure case was developed first (by Korevaar et al., [KvAEdB] in 1949, as against the Baire case of Matuszewska [Mat] in 1965, building on earlier work by Piccard and Pettis, [P] [Pet1] [Pet2]), partly because, in the representation theorem for regularly varying functions, one has to integrate, which needs measurability. One of the main themes of this paper is that, on the contrary, it is the category case that is primary and the measure case that is secondary. The difficulty about integration in the representation theorem is only apparent, and is dealt with in, e.g., [BGT] Section 1.3.2, [dB], or [BOst4]. The main point is that one can handle both cases together by category arguments, using the ordinary Euclidean topology in the Baire case and the density topology in the measurable case. Thus the real setting is topological – or more properly, bitopological in the sense of Kelly [Kel].

In Section 2 we state and prove our results, a topological Embedding Theorem (on sequences of homeomorphisms on a Baire space) and Proposition 1 (which shows that, under our main condition, this sequence converges to the identity quasi-everywhere). In Section 3, as a first application, we show that this is a generalization of the Kestelman-Borwein-Ditor Theorem of the title (KBD for short); however, unlike the KBD Theorem and its previously known generalizations, Miller’s Theorem of [MilH], which relies on a differentiable homotopy, and the Generic Reflection Theorem of [BOst9], which again relies on some smoothness assumptions, the current generalization not only goes beyond the Euclidean spaces, but also relies merely on continuity (homeomorphisms) – so this is non-smooth analysis. We obtain the measure and the category versions of the Kestelman-Borwein-Ditor Theorem ([Kes], [BoDi], [Trau]; see [BOst9]), by using the density topology for the first and the Euclidean topology for the second. The underlying argument here relies on linearity in such a way that the linearizing tools of ordinary calculus
permit passage, by an appropriate simple modification, from the linear KBD
Theorem to its ‘smooth’ generalizations. In Section 4 we turn, as a second
application, to the Uniform Convergence Theorem for slowly varying func-
tions, obtaining an integrated proof of the measure and category cases, as
had long been sought, from the KBD Theorem. We close with some remarks
in Section 5.

For background on Baire sets (i.e., sets with the Baire property) we refer
to Kechris ([Kech]; see section 8.F p. 47) and on Baire category and Baire
spaces, we refer to Engelking ([Eng]; see especially p.198 Section 3.9 and
Exercises 3.9.J), although we prefer ‘meagre’ to ‘of first category’; we write
as usual ‘almost everywhere’ (a.e.) for ‘off a null set’, and ‘quasi-everywhere’
(q.e.) for ‘off a meagre set’; similarly for ‘for almost all’, or ‘for quasi all’
points. Measure and category are explored at textbook length in [Oxt]; see
Ch. 19 for duality (including the Sierpiński-Erdős Duality Principle under
the Continuum Hypothesis), Ch. 17 (in ergodic theory, duality extends to
some but not all forms of the Poincaré recurrence theorem) and Ch. 21
(in probability theory, duality extends as far as the zero-one law but not to
the strong law of large numbers). Duality also fails to extend to the theory
of random series [Kah]. For further limitations of duality, see [DoF], [Bart],
[BGJS]. For Wilczyński’s theory of a.e.-convergence associated with σ-ideals,
see [PWW]. For a set-theoretic explanation of the duality in regular variation
in terms of forcing see [BOst1] Section 5, [Mill] Section 6. For applications
of regular variation see [Kor] Ch. IV.

2 Results

The topological theorem below demonstrates why there is measure-category
duality in the fundamental ‘Uniform Convergence Theorem – UCT’ of regular
variation (see below, or [BGT] Th. 1.2.1). It is a topological version of the
KBD Theorem, so is an embedding theorem (Trautner uses the term covering
principle in [Trau]).

**Category Embedding Theorem.** Let $X$ be a Baire space. Suppose
given homeomorphisms $h_n : X \to X$ for which the following weak ‘category
convergence condition’ is met:

For any non-empty open set $U$, there is a non-empty open set $V \subseteq U$
such that, for each \( k \in \omega \),
\[
\bigcap_{n \geq k} V \setminus h_n^{-1}(V) \text{ is meagre,}
\]
equivalently, such that, for each \( k \in \omega \), there is a meagre set \( M \) such that, for \( t \notin M \),
\[
t \in V \implies (\exists n \geq k) \ h_n(t) \in V.
\]

Then, for any non-meagre Baire set \( T \), for locally quasi all \( t \in T \), there is an infinite set \( \mathbb{M}_t \) such that
\[
\{h_m(t) : m \in \mathbb{M}_t \} \subseteq T.
\]

**Proof.** Suppose \( T \) is Baire and non-meagre. We may assume that \( T = U \setminus M \) with \( U \) non-empty and \( M \) meagre. Let \( V \subseteq U \) satisfy (1).

Since the functions \( h_n \) are homeomorphisms, the set
\[
M' := M \cup \bigcup_n h_n^{-1}(M)
\]
is meagre. Put
\[
W = h(V) := \bigcap_{k \in \omega} \bigcup_{n \geq k} V \cap h_n^{-1}(V) \subseteq V \subseteq U.
\]

Then \( V \cap W \) is co-meagre in \( V \). Indeed
\[
V \setminus W = \bigcup_{k \in \omega} \bigcap_{n \geq k} V \setminus h_n^{-1}(V),
\]
which by assumption is meagre.

Let \( t \in V \cap W \setminus M' \) so that \( t \in T \). Now there exists an infinite set \( \mathbb{M}_t \) such that, for \( m \in \mathbb{M}_t \), there are points \( v_m \in V \) with \( t = h_m^{-1}(v_m) \). Since \( h_m^{-1}(v_m) = t \notin h_m^{-1}(M) \), we have \( v_m \notin M \), and hence \( v_m \in T \). Thus \( \{h_m(t) : m \in \mathbb{M}_t \} \subseteq T \) for \( t \) in a co-meagre set, as asserted. \( \square \)

Clearly the result relativizes to any open subset of \( T \) on which \( T \) is non-meagre. The following lemma sheds some light on the significance of the category convergence condition (1). The result is capable of improvement, for instance by replacing the countable family generating the coarser topology.
by a $\sigma$-discrete family (which is then metrizable by Bing’s Theorem, given
regularity assumptions – see [Eng] Th. 4.4.8).

**Proposition 1 (Convergence to the identity).** Assume that the
homeomorphisms $h_n : X \to X$ satisfy the weak category convergence con-
dition (1) and that $X$ is a Baire space. Suppose there is a countable family
$B$ of open subsets of $X$ which generates a (coarser) Hausdorff topology on
$X$. Then, for quasi all (under the original topology) $t$, there is an infinite $\mathbb{N}_t$
such that
\[
\lim_{m \in \mathbb{N}_t} h_m(t) = t.
\]

**Proof.** For $U$ in the countable base $B$ of the coarser topology and for
$k \in \omega$ select open $V_k(U)$ so that $M_k(U) := \bigcap_{n \geq k} V_k(U) \setminus h_n^{-1}(V_k(U))$ is meagre.
Thus
\[
M := \bigcup_{k \in \omega} \bigcup_{U \in B} M_k(U)
\]
is meagre. Now $B_t = \{U \in B : t \in U\}$ is a basis for the neighbourhoods of $t$.
But, for $t \in V_k(U) \setminus M$, we have $t \in h_m^{-1}(V_k(U))$ for some $m = m_k(t) \geq k$, i.e.
$h_m(t) \in V_k(U) \subseteq U$. Thus $h_{m_k(t)}(t) \to t$, for all $t \notin M$. \qed

**Notes.**
1. Interest in the strong assumption of second countability (or metrizabil-
ity) is motivated in the first place by the paradigm of the density topology on
$\mathbb{R}$ (for definition see below). Formally, in the sense of [Kel] (cf. [LMZ]), that
is a bi-topological context: $\mathbb{R}$ carries two inter-related topologies, both of
interest. The finer one is not even first-countable, whereas the coarser, usual
one is second-countable. They are inter-related in several delicate ways be-
yond mere refinement, for instance:

(i) any open set in the finer density-topology is, modulo a meagre set
of the finer topology, a $G_\delta$-set in the coarser usual topology – that is, the
density-open sets are $G_\delta$-supported, to borrow and extend a term due to
Solecki [Sol];

(ii) likewise singletons (points) are $G_\delta$-sets in the coarser usual topology.
It is not true, for $z_n \to 0$ a fixed null sequence (converging to zero,
in the usual sense), that the homeomorphisms $h_n(x) := x + z_n \to x$ in
the sense of the finer density-topology; but the same assertion is true in
the coarser usual topology. The Proposition above can thus be invoked in
contexts where there is a natural metrizable coarser topology, with the two
topologies inter-related, say as in (i) and (ii). For a bitopological analysis see [LMZ] – the density topology taken in tandem with the usual topology being one of two paradigm examples of abstract fine topology, the other being the fine topology of potential theory, for which see Doob [Doo] Ch. 1.XI, [LMZ].

2. In the second countable case (1) implies a form of $\mathcal{I}$-a.e. convergence to the identity, with $\mathcal{I}$ the $\sigma$-ideal of meagre sets. See [PWW] for further information on this concept. Note also the possibility of a stronger convergence condition, noted in Remark 3 after Proposition 3, akin to convergence down a sub-subsequence of a given subsequence, as occurs in the notion of convergence with respect to $\mathcal{I}$ [PWW] – in this connection see [Mil2] regarding circumstances (and their dependence on the Souslin Hypothesis) when the two modes of convergence relative to $\mathcal{I}$ are equivalent.

3. The result just cited is the abstract form of the relationship between convergence in probability and almost-sure convergence. The latter implies the former, but not conversely in general, although a sequence converging in probability has a subsequence converging almost surely. The former is metric, the latter not even topological in general (see [Dud] Section 9.2 Problem 2). But the latter reduces to the former in exceptional circumstances (when the measure space is purely atomic, when there are no non-trivial null sets); again, see [Mil2] and [WW].

We now deduce the category and measure cases of the Kestelman-Borwein-Ditor Theorem (stated below) as two corollaries of the above theorem by applying it first to the usual and then to the density topology on the reals, $\mathbb{R}$.

For our first application we take $X = \mathbb{R}$ with the usual topology, a Baire space. We let $z_n \to 0$ be a null sequence. It is convenient to take

$$h_n(x) = x - z_n, \text{ so that } h_n^{-1}(x) = x + z_n.$$ 

This is a homeomorphism. We verify (1). If $U$ is non-empty and open, let $V = (a, b)$ be an interval contained in $U$. For later use we identify our result thus.

**Proposition 2.** Let $V$ be an open interval. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,

$$H_k = \bigcap_{n \geq k} V \setminus (V + z_n) \text{ is empty.}$$
**Proof.** Assume first that the null sequence is positive. Then, for all $n$ so large that $a + z_n < b$, we have

$$V \cap h_n^{-1}(V) = (a, a + z_n),$$

and so, for any $k \in \omega$,

$$\bigcap_{n \geq k} V \setminus h_n^{-1}(V)$$

is empty.

The same argument applies if the null sequence is negative, but with the end-points exchanged. □

**Note.** It is interesting to observe that, in any normed vector space, and with $V$ an open ball, the same conclusion follows. We make use of this in a companion paper [BOst12].

For our second application we enrich the topology of $\mathbb{R}$, retaining the functions $h_n$. We consider instead the density topology (introduced in [HauPau], [GoWa] and studied also in [GNN] – see also [CLO], or [BOst9] for a short account and the bibliography therein, and for a textbook treatment [Kech]). Recall that for $T$ measurable, $t$ is a (metric) density point of $T$ if $\lim_{\delta \to 0} |T \cap I_\delta(t)|/\delta = 1$, where $I_\delta(t) = (t - \delta/2, t + \delta/2)$; by the Lebesgue Density Theorem almost all points of $T$ are density points ([Hal] Section 61, [Oxt] Th. 3.20, or [Goff]). A set $U$ is $d$-open (open in the density topology) if each of its points is a density point of $U$. Thus the density topology is translation-invariant and so each $h_n$ continues to be a homeomorphism. The intervals remain open. A set is $d$-nowhere dense iff it is measurable and null.

First we verify that the space is Baire on any non-empty $d$-open set $U$. Suppose that $A_n$ is a sequence of sets $d$-nowhere dense in $U$. Since this means that each set $A_n$ has measure zero, their union has measure zero and so the complement in $U$ is non-empty, since $U$ has positive measure.

To verify (1) consider $U$ non-empty and $d$-open, and choose a finite interval $(a, b)$ so that $V = U \cap (a, b)$ has positive measure. It now suffices to prove the following Proposition, which is of independent interest. (The reader unfamiliar with the density topology may find it helpful to think, in the style of Littlewood’s First Principle, of basic opens sets as being intervals less some measurable set. See [Lit] Ch. 4, [Roy] Section 3.6 p.72.)
Proposition 3. Let $V$ be measurable and non-null. For any null sequence \( \{z_n\} \to 0 \) and each $k \in \omega$,

\[
H_k = \bigcap_{n \geq k} V \setminus (V + z_n)
\]
is of measure zero, so meagre in the $d$-topology.

**Proof.** Suppose otherwise.

Then for some $k$, $|H_k| > 0$. Write $H$ for $H_k$. Since $H \subseteq V$, we have, for $n \geq k$, that $\emptyset = H \cap h_n^{-1}(V) = H \cap (V + z_n)$ and so a fortiori $\emptyset = H \cap (H + z_n)$.

Let $u$ be a density point of $H$. Thus for some interval $I_\delta(u) = (u - \delta/2, u + \delta/2)$ we have

\[
|H \cap I_\delta(u)| > \frac{3}{4} \delta.
\]

Let $E = H \cap I_\delta(u)$. For any $z_n$, we have $|(E + z_n) \cap (I_\delta(u) + z_n)| = |E| > \frac{3}{4} \delta$. For $0 < z_n < \delta/4$, we have $|(E + z_n) \setminus I_\delta(u)| \leq |(u + \delta/2, u + 3\delta/4)| \leq \delta/4$.

Put $F = (E + z_n) \cap I_\delta(u)$; then $|F| > \delta/2$.

But $\delta > |E \cup F| = |E| + |F| - |E \cap F| \geq \frac{3}{4} \delta + \frac{1}{2} \delta - |E \cap F|$. So

\[
|H \cap (H + z_n)| \geq |E \cap F| \geq \frac{1}{4} \delta,
\]

contradicting $\emptyset = H \cap (H + z_n)$. This establishes the claim. \(\square\)

**Remarks.**
1. Although being introduced much later, the density topology is exactly the topology suited to Denjoy’s concept of *approximate continuity* of 1915 [Den]: a function is approximately continuous iff it is continuous under the density topology. See [LMZ] p.1, 149.

2. In fact the proof above requires only that, for any interval $I$ and any $\varepsilon > 0$, there is an infinite set $M$ such that

\[
|I \triangle h_n^{-1}(I)| < \varepsilon, \text{ for } m \in M.
\]

3. For $z_n$ a positive sequence with $z_n \to z_0 = 1$, one may likewise consider the autohomeomorphisms $h_n(t) = tz_n$. In the density topology case the weak convergence condition (1) this time requires that

\[
H_k = \bigcap_{n \geq k} V \setminus (z_n \cdot V) \text{ is of measure zero.}
\]
Since (2) holds, a similar proof (computing potential overlaps) establishes (3). This in turn implies the multiplicative analogue of the Kestelman-Borwein-Ditor Theorem, due to Harry Miller – see the next section.

4. In each of the applications above, the following (stronger) convergence condition is met (compare the study of convergence in category [PWW]).

For each infinite \( M \), \( \bigcap_{m \in M} V \setminus h_m^{-1}(V) \) is meagre.

3 The Kestelman-Borwein-Ditor Theorem and its variants

An immediate first corollary of our theorem is the following result, due in this form in the measure case to Borwein and Ditor [BoDi], but already known much earlier albeit in somewhat weaker form by Kestelman ([Kes] Th. 3), and rediscovered by Trautner [Trau] (see [BGT] p. xix and footnote p. 10).

Below, for \( P \) a set of reals (or property) that is measurable/Baire, we say that ‘\( P \) holds for generically all \( t \)’ to mean that \( \{ t : t \notin P \} \) is null/meagre.

**Theorem (Kestelman-Borwein-Ditor Theorem).** Let \( \{z_n\} \to 0 \) be a null sequence of reals. If \( T \) is Lebesgue, non-null/Baire, non-meagre, then for generically all \( t \in T \) there is an infinite set \( M_t \) such that

\[ \{ t + z_m : m \in M_t \} \subseteq T. \]

Furthermore, for any density point \( u \) of \( T \), there is \( t \in T \) arbitrarily close to \( u \) for which the above holds.

From here we proceed to illustrate why the Category Embedding Theorem implies also other ‘smooth generalizations’ of the KBD Theorem. We term this a ‘pointwise’ theorem because of the sequence of points \( z_n \). This result was generalized by Harry Miller [MilH], as below, by replacing \( t + z \) with a more general function \( H(t, z) \) (originally defined on \( \mathbb{R} \times \mathbb{R} \)). We need a definition (the terminology is ours).

**Definition (Miller homotopy, cf. [MilH]).** Let \( U \) be open and let \( I \) be a non-degenerate interval (possibly infinite, or semi-infinite). We call a function \( H : U \times I \to \mathbb{R} \) a Miller homotopy acting on \( U \) with distinguished point \( z_0 \) if:
(i) \( H(u, z_0) \equiv u \), for all \( u \in U \),
(ii) \( H \) has continuous first-order partial derivatives \( H_1 \) and \( H_2 \), and
(iii) \( H_2(u, z_0) > 0 \), for all \( u \in U \).

Note. As the function \( H \) is differentiable, and hence jointly continuous, it is natural to regard it as establishing a homotopy to the identity (albeit utilizing a distinguished point \( z_0 \) other than 0, and some interval about \( z_0 \) instead of the customary unit interval). Condition (iii) is only a non-stationarity requirement (map \( z \rightarrow -z, z_0 \rightarrow -z_0 \), if \( H_2(u, z_0) < 0 \)).

Convention. We will refer to the distinguished point \( z_0 \) as the ‘null point’ and any sequence \( z_n \rightarrow z_0 \) converging to the null point a ‘null sequence’. Thus in the case \( H(u, z) = u + z \) with \( z_0 = 0 \), the sequence \( z_n \rightarrow z_0 \) is a null sequence in the customary sense.

Miller’s Homotopy Theorem. Let \( H \) be a Miller homotopy acting on an open set \( U \) with distinguished point \( z_0 \). Let \( z_n \rightarrow z_0 \) be a null sequence and let \( T \subseteq U \) be measurable, non-null/Baire, non-meagre. Then, for generically all \( t \in T \), there is an infinite set \( M_t \) such that

\[
\{ H(t, z_m) : m \in M_t \} \subseteq T.
\]

As stated this is also a ‘pointwise’ theorem. It is noteworthy that the substitution

\[
z_n(t) := H(t, z_n) - t, \quad \text{and} \quad u_n(t) = t + z_n(t)
\]

allows a functional reinterpretation of the theorem (we have used bold type to distinguish functions from points). We may now regard the sequence of functions \( \{z_n(t)\} \), which converge to the null point \( z_0 \) (see below), as given and so the conclusion of Miller’s theorem reads:

\[
\{ t + z_m(t) : m \in M_t \} \subseteq T.
\]

Thus Miller’s Theorem becomes simply a functional version of the KBD Theorem. We quote one possible functional generalization which goes beyond the KBD setting. Again we need some definitions.
Definition (uniformity - pointwise). We say that the null sequence \( \{z_n\} \to z_0 \) is a uniformly null sequence, or that \( z_n \to z_0 \) uniformly, if for some positive constant \( K \),
\[
|z_n - z_0| \leq K 2^{-n}, \text{ for all } n \in \omega.
\]

Definition (uniformity - functionwise). We say that the sequence of functions \( \{z_n(.)\} \) is a uniformly null function sequence on \( U \), or that \( z_n(.) \to z_0 \) uniformly on \( U \), if each \( z_n(.) \) is measurable/Baire and, for some positive constant \( K \),
\[
\max\{|z_n(u)|\} \leq K \cdot 2^{-n}, \text{ for all } n \in \omega \text{ and all } u \in U.
\]

Definition (bi-Lipschitz). We call a uniformly null sequence \( \{z_n(.)\} \) bi-Lipschitz if the mappings \( t \to u_n(t) \) are bi-Lipschitz uniformly in \( n \), i.e. for some \( \alpha, \beta \) and all \( n \) we have
\[
0 < \alpha \leq 1 + \frac{z_n(u) - z_n(v)}{u - v} \leq \beta, \text{ for } u \neq v.
\]

In particular \( z_n' \) is bounded away from \(-1\), except perhaps at countably many points.

The following theorem is proved in [BOst9] (where further improvements, motivated by convex analysis, are given); it is manifestly a ‘functionwise’ theorem.

Theorem (Generic Reflection Theorem). Let \( T \) be measurable/Baire. Let \( f(.) \) be continuously differentiable and non-stationary at generically all points. Let \( \{z_n(.)\} \to 0 \) be a uniformly null sequence that is bi-Lipschitz with
\[
1 + f'(t)z_n'(t) > 0, \text{ for all } n, \quad (4)
\]
for generically all \( t \in T \). Then, for generically all \( u \in T \), there is an infinite set \( M_u \) such that
\[
f(u_n) + u - f(u) \in T, \text{ for all } n \in M_u. \quad (5)
\]

In particular, if \( f \) is linear and \( f(t) = \alpha t \) with \( \alpha \neq 0 \), then
\[
\alpha u_n + (1 - \alpha)u \in T \text{ for all } n \in M_u, \text{ where } u_n = u + z_n(u). \quad (6)
\]
The homotopy versions of this result (e.g. replacing \( t+z_n(t) \) by \( H(t, z_n(t)) \), say) are only apparently stronger – see [BOst10]; see also [BOst3] Section 2.1 Note 3, [BOst4] Section 2.1 Note 1. For the original proofs see [Kes] and [BoDi]; for a first unified treatment see [BOst9]. Our proof of the Category Embedding Theorem of Section 2 bears a ‘family resemblance’ to Miller’s measure argument, which employs a condition similar to (1) and (3). So, not surprisingly, the Category Embedding Theorem also implies Miller’s result; we now indicate why.

We note that, for any monotonic sequence \( z_n \to z_0 \), the functions

\[
h_n(t) := H(t, z_n),
\]

which are all homotopic to the identity, are in fact local diffeomorphisms (so are locally bi-Lipschitz, and hence preserve null sets both ways); moreover

\[
h_n(t) \to t, \text{ ultimately monotonically.}
\]

We repeat the proof from [BOst10] as it is short. Invertibility of \( h_n \) follows from the Inverse Function Theorem. As for the latter conclusion, note that since \( H_t(t_0, z_0) = 1 \), for any \( t_0 \), we may invoke the Mean Value Theorem to write the Taylor expansion near \((t_0, z_0)\) as

\[
H(t, z) = t_0 + (t - t_0) + H_2(t_0, z_0)(z - z_0) + o(||(t - t_0, z - z_0)||).
\]

From here we deduce that

\[
h_n(t) = t + H_2(t_0, z_0)(z_n - z_0) + o(||(t - t_0, z_n - z_0)||), \text{ as } t \to t_0 \text{ and } n \to \infty.
\]

Thus \( h_n \) is almost a shift and \( h_n(t) \to t \). Here the ultimate monotonicity, at any \( t \), follows from the continuity and positivity of the partial derivative \( H_2 \) at \((t_0, z_0)\).

By Remark 1 to Proposition 3, the argument of Proposition 3 may be amended to cover this case.

Likewise the Generic Reflection Theorem is a corollary of the Category Embedding Theorem. One way to see this is to recognize that the functions

\[
f_n(t) := f(t + z_n(t)) + t - f(t)
\]

used in [BOst9] are the relevant homeomorphisms \( h_n(t) \) here, and that the proof there amounts to verifying the weak category convergence condition (1). Alternatively, since

\[
f_n(t) := t + f'(\tilde{u}_n(t))z_n(t),
\]

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for some \( \tilde{u}_n(t) \) lying between \( t \) and \( u_n(t) \), we may again invoke (3) for an appropriate modification of the proof of Proposition 3.

## 4 The Uniform Convergence Theorem

A second corollary of our theorem is the fundamental theorem of regular variation below. Its brief proof, inspired by one due to Csiszár and Erdős [CsEr] (called the ‘fourth proof’ in [BGT]), appeals only to the Kestelman-Borwein-Ditor Theorem. That was discovered in [BOst9]; we quote it here for completeness – shortened (indeed, “from \( 3\varepsilon \) to \( 2\varepsilon \)”), and abstracted from a wider combinatorial context. For another proof, albeit for the measurable case only, see [Trau], where Trautner employs also a theorem of Egorov (cf. Littlewood’s Third Principle, see [Lit] Ch. 4, [Roy] Section 3.6 and Problem 31, or [Hal] Section 55 p. 243). Recall (see [BGT]) that a function \( h : \mathbb{R} \to \mathbb{R} \) is *slowly varying* (in additive notation) if for every sequence \( \{x_n\} \to \infty \) and each \( u \in \mathbb{R} \)

\[
\lim_{n \to \infty} |h(u + x_n) - h(x_n)| = 0.
\]

**Theorem (Uniform Convergence Theorem).** If \( h \) is slowly varying and measurable, or Baire, then uniformly in \( u \) on compacts:

\[
\lim_{n \to \infty} |h(u + x_n) - h(x_n)| = 0.
\]

**Proof.** Suppose otherwise. Then for some measurable/Baire slowly varying function \( h \) and some \( \varepsilon > 0 \), there is \( \{u_n\} \to u \) and \( \{x_n\} \to \infty \) such that

\[
|h(u_n + x_n) - h(x_n)| \geq 2\varepsilon. \tag{7}
\]

Now, for each point \( y \), we have

\[
\lim_{n} |h(y + x_n) - h(x_n)| = 0,
\]

so there is \( k = k(y) \) such that, for \( n \geq k \),

\[
|h(y + x_n) - h(x_n)| < \varepsilon.
\]

For \( k \in \omega \), define the measurable/Baire set

\[
T_k := \bigcap_{n \geq k} \{y : |h(y + u + x_n) - h(x_n)| < \varepsilon\}.
\]

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Since $\{T_k : k \in \omega\}$ covers $\mathbb{R}$, it follows that, for some $k \in \omega$, the set $T_k$ is non-null/non-meagre. Writing $z_n := u_n - u$, we have, for some $t \in T_k$ and for some infinite $M_t$, that

$$\{t + z_m : m \in M_t\} \subseteq T_k.$$ 

Thus

$$|h(t + u_m + x_m) - h(x_m)| < \varepsilon.$$ 

Since $u_m + x_m \to \infty$, we have, for $m$ large enough and in $M_t$, that

$$|h(t + u_m + x_m) - h(u_m + x_m)| < \varepsilon.$$ 

The last two inequalities together imply, for $m$ large enough and in $M_t$, that

$$|h(u_m + x_m) - h(x_m)| \leq |h(u_m + x_m) - h(t + u_m + x_m)| + |h(t + u_m + x_m) - h(x_m)| < 2\varepsilon,$$

and this contradicts (7). □

5 Complements

1. The measure and category cases of the Uniform Convergence Theorem are developed in parallel in [BGT] Section 1.2. A unified proof covering both together has long been sought. The key insight that such a proof could be constructed using the density topology came from the result that a set is (Lebesgue) measurable if and only if it has the Baire property in the density topology ([Kech], p.119, Ex. (17.47), (iv)).

2. Five proofs of the Uniform Convergence Theorem are given in [BGT]. The first, a direct proof due to Delange, uses quantitative measure theory. The second, also direct, uses only qualitative measure theory (distinguishes only between null and non-null sets), and has an immediate category translation. The third, due to Matuszewska, and fourth, due to Csiszár and Erdős, are indirect, and also have immediate category translations. The fifth proof, due to Elliott, uses Egorov’s theorem but covers the measure case only. A sixth proof, due to Trautner [Trau], also uses Egorov’s theorem so again applies only to the measure case, cf. [Oxt] Chapter 8. (Trautner was unaware both of Kestelman’s work and that of Borwein and Ditor – see [BOst9].)
Our seventh proof here (eighth, if the tally is to include one via the Bounded Equivalence Principle, given in [BOst1]) is based on the fourth, and on the insights gained in our earlier papers, cited below.

It is interesting that – contrary to what one might think from the paragraph above – quantitative rather than qualitative measure theory is needed here (see the proof of Proposition 3). Such aspects are intrinsic to the density topology.

References


