Normed groups: dichotomy and duality

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Abstract

The key vehicle of the recent development of a topological theory of regular variation based on topological dynamics [BOst13], and embracing its classical univariate counterpart (cf. [BGT]) as well as fragmentary multivariate (mostly Euclidean) theories (eg [MeSh], [Res], [Ya]), are groups with a right-invariant metric carrying flows. Following the vector paradigm, they are best seen as *normed groups*. That concept only occasionally appears explicitly in the literature despite its frequent disguised presence, and despite a respectable lineage traceable back to the Pettis closed-graph theorem, to the Birkhoff-Kakutani metrization theorem and further back still to Banach's *Théorie des opérations linéaires*. We collect together known salient features and develop their theory including Steinhaus theory unified by the Category Embedding Theorem [BOst11], the associated themes of subadditivity and convexity, and a topological duality inherent to topological dynamics. We study the latter both for its independent interest and as a foundation for topological regular variation.

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Contents

1 Introduction

Group norms, which behave like the usual vector norms except that scaling is restricted to the basic scalars of group theory (the units ± 1 in an abelian context and the exponents ± 1 in the non-commutative context), have played a part in the early development of topological group theory. Although ubiquitous, they lack a clear and unified exposition. This lack is our motivation here, since they offer the right context for the recent theory of *topological regular variation.* This extends the classical theory (for which see, e.g. [BGT]) from the real line to metrizable topological groups. Normed groups are just groups carrying a right-invariant metric. The basic metrization theorem for groups, the Birkhoff-Kakutani Theorem of 1936 ([Bir], [Kak], see [Kel], Ch.6 Problems N-R, [Klee], [Bour] Part 2, Section 4.1, and [ArMa], compare also [Eng] Exercise 8.1.G and Th. 8.1.21), is usually stated as asserting that a first-countable Hausdorff group has a right-invariant metric. It is properly speaking a 'normability' theorem in the style of Kolmogorov's Theorem ([Kol], or [Ru-FA2], Th. 1.39; in this connection see also [Jam], where strong forms of connectedness are used in an abelian setting to generate norms), as we shall see below. Indeed the metric construction in [Kak] is reminiscent of the more familiar construction of a Minkowski functional (for which see [Ru-FA2] Sect. 1.33), but is implicitly a supremum norm – as defined below; in Rudin's derivation of the metric (for a topological vector space setting, [Ru-FA2] Th. 1.24) this norm is explicit. Early use by A. D. Michal and his collaborators was in providing a canonical setting for differential calculus (see the review [Mich] and as instance [JMW]) and included the noteworthy generalization of the implicit function theorem by Bartle [Bart] (see Section 6). In name the group norm makes an explicit appearance in 1950 in [Pet1] in the course of his classic closed-graph theorem (in connection with Banach's closed-graph theorem and the Banach-Kuratowski category dichotomy for groups). It reappears in the group context in 1963 under the name 'length function', motivated by word length, in the work of R. C. Lyndon [Lyn2] (cf. [LynSch]) on Nielsen's Subgroup Theorem, that a subgroup of a free group is a free group. (Earlier related usage for function spaces is in [EH].) The latter name is conventional in geometric group theory despite the parallel usage in algebra (cf. [Far]) and the recent work on norm extension (from a normal subgroup) of Bökamp [Bo].

When a group is topologically complete and also abelian, then it admits a metric which is *bi-invariant*, i.e. is both right- and left-invariant, as [Klee] showed in solving a problem of Banach. This context is of significance for the calculus of regular variation (in the study of products of regularly varying functions with range a normed group) – see [BOst15].

Fresh interest in metric groups dates back to the seminal work of Milnor [Mil] in 1968 on the metric properties of the fundamental group of a manifold and is key to the global study of manifolds initiated by Gromov [Gr1], [Gr2] in the 1980s (and we will see quasi-isometries in the duality theory of normed groups), for which see [BH] and also [Far] for an early account; [PeSp] contains a variety of generalizations and their uses in interpolation theory (but the context is abelian groups).

The very recent [CSC] (see Sect. 2.1.1, Embedding quasi-normed groups into Banach spaces) employs norms in considering Ulam's problem (see [Ul]) on the *global* approximation of nearly additive functions by additive functions. This is a topic related to regular variation, where the weaker concept of *asymptotic* additivity is the key. Recall the classical definition of a regularly varying function, namely a function $h : \mathbb{R} \to \mathbb{R}$ for which the limit

$$\partial_{\mathbb{R}} h(t) := \lim_{x\to\infty} h(tx)h(x)^{-1} \tag{1}$$

exists everywhere; for f Baire, the limit function is a continuous homomorphism (i.e. a multiplicative function). Following the pioneering study of [BajKar] launching a general (i.e., topological) theory of regular variation, [BOst13] has re-interpreted (1), by replacing $|x| \to \infty$ with $||x|| \to \infty$, for functions $h : X \to H$, with tx being the image of x under a T-flow on X (defined in Section 4), and with X, T, H all groups with right-invariant metric (right because of the division on the right) – i.e. normed groups (making ∂h_X a differential at infinity, in Michal's sense [Mi]). In concrete applications the groups may be the familiar Banach groups of functional analyis, the associated flows either the ubiquitous domain translations of Fourier transform theory or convolutions from the related contexts of abstract harmonic analysis (e.g. Wiener's Tauberian theory so relevant to classical regular variation – see e.g. [BGT, Ch. 4]). In all of these one is guaranteed right-invariant metrics. Likewise in the foundations of regular variation the first tool is the group $\mathcal{H}(X)$ of bounded self-homeomorphisms of the group X under a supremum metric (and acting transitively on X); the metric is again right-invariant and hence a group norm. It is thus natural, in view of the applications and the Birkhoff-Kakutani Theorem, to demand right-invariance.

We show in Section 4 and 6 that normed groups offer a natural setting for subadditivity and for (mid-point) convexity.

2 Metric versus normed groups

This section is devoted to group-norms and their associated metrics. We collect here some pertinent information (some of which is scattered in the literature). A central tool for applications is the construction of the subgroup of bounded homeomorphisms of a given group $\mathcal{G}$ of self-homeomorphisms of a topological group X; the subgroup possesses a guaranteed right-invariant metric. This is the archetypal example of the symbiosis of norms and metrics, and it bears repetition that, in applications just as here, it is helpful to work simultaneously with a right-invariant metric and its associated group norm.

We say that the group X is *normed* if it has a group-norm as defined below (cf. [DDD]).

Definition. We say that $||\cdot|| : X \to \mathbb{R}_+$ is a *group-norm* if the following properties hold:

(i) Subadditivity (Triangle inequality): $||xy|| \leq ||x|| + ||y||$;

(ii) Positivity: $||x|| > 0$ for $x \neq e$;

(iii) Inversion (Symmetry): $||x^{-1}|| = ||x||$.

If (i) holds we speak of a group *semi-norm*; if (i) and (iii) and $||e|| = 0$ holds one speaks of a *pseudo-norm* (cf. [Pet1]); if (i) and (ii) holds we speak of a group *pre-norm* (see [Low] for a full vocabulary).

We say that a group pre-norm, and so also a group-norm, is *abelian*, or more precisely *cyclically permutable,* if

(iv) Abelian norm (cyclic permutation): $||xy|| = ||yx||$ for all x, y.

Other properties we wish to refer to are :

$(\mathrm{i})_K$ for all $x, y : ||xy|| \leq K(||x|| + ||y||)$,

$(\mathrm{i})_{\mathrm{ult}}$ for all $x, y : ||xy|| \leq \max\{||x||, ||y||\}$.

Remarks 1

1. Mutatis mutandis this is just the usual vector norm, but with scaling restricted to the units ± 1. The notation and language thus mimick the vector space counterparts, with subgroups playing the role of subspaces; for example, for a symmetric, subbadditive $p : X \to \mathbb{R}_+$, the set $\{x : p(x) = 0\}$ is a subgroup. Indeed the analysis of Baire subadditive functions (see Section 4) is naturally connected with norms, via regular variation. That is why normed groups occur naturally in regular variation theory.

2. When $(\mathrm{i})_K$, for some constant K, replaces (i), one speaks of quasi-norms (see [CSC], cf. 'distance spaces' [Rach] for a metric analogue). When $(\mathrm{i})_{\mathrm{ult}}$ replaces (i) one speaks of an ultra-norm, or non-Archimedean norm.

3. Note that (i) implies joint continuity of multiplication, while (iii) implies continuity of inversion, but in each case only at the identity, e_X, a matter we return to in Section 3. (Montgomery [Mon1] shows that joint continuity is implied by separate continuity when the group is locally complete.) See below for the stronger notion of uniform continuity invoked in the Uniformity Theorem of Conjugacy.

4. Abelian groups with ordered norms may also be considered, cf. [JMW].

Remarks 2

Subadditivity implies that $||e|| \geq 0$ and this together with symmetry implies that $||x|| \geq 0$, since $||e|| = ||xx^{-1}|| \leq 2||x||$; thus a group norm cannot take negative values. Subadditivity also implies that $||x^n|| \leq n||x||$, for natural n. The norm is said to be 2-*homogeneous* if $||x^2|| = 2||x||$; see [CSC] Prop. 4.12 (Ch. IV.3 p.38) for a proof that if a normed group is amenable or *weakly commutative* (defined in [CSC] to mean that, for given x, y, there is m of the form 2^n, for some natural number n, with $(xy)^m = x^m y^m$), then it is embeddable as a subgroup of a Banach space. In the case of an abelian group 2-homogeneity corresponds to sublinearity, and here Berz's Theorem characterizes the norm (see [Berz] and [BOst5]). The abelian property implies only that $||xyz|| = ||zxy|| = ||yzx||$, hence the alternative name of 'cyclically permutable'. Harding [H], in the context of quantum logics, uses this condition to guarantee that the group operations are jointly continuous (cf. Theorem 2 below) and calls this a *strong norm.* See [Kel] Ch. 6 Problem O (which notes that a locally compact group with abelian norm has a bi-invariant Haar measure). We note that when X is a locally compact group continuity of the inverse follows from the continuity of multiplication (see [Ell]). The literature concerning when joint continuity of $(x, y) \to xy$ follows from separate continuity reaches back to Namioka [Nam] (see e.g. [Bou], [HT], [CaMo]).

Convention. For a variety of purposes and for the sake of clarity, when we deal with a metrizable group X if we assume a metric d^X on X is right/left invariant we will write d_R^X or d_L^X, omitting the superscript and perhaps the subscript if context permits.

Remarks 3

For X a metrizable group with right-invariant metric d^X and identity e_X, the canonical example of a group-norm is identified in Proposition 2.3 below as

$$||x|| := d^X(x, e_X).$$

Remarks 4

If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, subadditive with $f(0) = 0$, then

$$|||x||| := f(||x||)$$

is also a group-norm. See [BOst5] for recent work on Baire (i.e., having the Baire property) subadditive functions. These will appear in Section 3.

We begin with two key definitions.

Definition and notation. For X a metric space with metric d^X and $\pi : X \to X$ a bijection the π-*permutation* metric is defined by

$$d^X_\pi(x, y) := d^X(\pi(x), \pi(y)).$$

When X is a group we will also say that d^X_π is the π-*conjugate* of d^X. We write

$$||x||_\pi := d^X(\pi(x), \pi(e)),$$

and for d any metric on X

$$B^d_r(x) := \{y : d(x, y) < r\},$$

suppressing the superscript for $d = d^X$; however, for $d = d^X_\pi$ we adopt the briefer notation

$$B^\pi_r(x) := \{y : d^X_\pi(x, y) < r\}.$$

Following [BePe] $Auth(X)$ denotes the group of auto-homeomorphisms of X under composition, but without a topological structure. We denote by id_X the identity map $id_X(x) = x$ on X.

Examples A. Let X be a group with metric d^X. The following permutation metrics arise naturally in this study.

1. With $\pi(x) = x^{-1}$ we refer to the π-permutation metric as the involution-conjugate, or just the conjugate, metric and write

$$\tilde{d}^X(x, y) = d^X_\pi(x, y) = d^X(x^{-1}, y^{-1}), \qquad \text{so that} \qquad ||x||_\pi = ||x|| = ||x^{-1}||.$$

2. With $\pi(x) = \gamma_g(x) := gxg^{-1}$, the inner automorphism, we have (dropping the additional subscript, when context permits):

$$d^X_\gamma(x, y) = d^X(gxg^{-1}, gyg^{-1}), \qquad \text{so that} \qquad ||x||_\gamma = ||gxg^{-1}||.$$

3. With $\pi(x) = \lambda_g(x) := gx$, the left shift by g, we refer to the π-permutation metric as the g-conjugate metric, and we write

$$d^X_g(x, y) = d^X(gx, gy).$$

If d^X is right-invariant, cancellation on the right gives

$$d^X(gxg^{-1}, gyg^{-1}) = d^X(gx, gy), \text{ i.e. } d^X_\gamma(x, y) = d^X_g(x, y) \text{ and } ||x||_g = ||gxg^{-1}||.$$

For d^X right-invariant, $\pi(x) = \rho_g(x) := xg$, the right shift by g, gives nothing new:

$$d_\pi^X(x, y) = d^X(xg, yg) = d^X(x, y).$$

But, for d^X left-invariant, we have

$$||x||_\pi = ||g^{-1}xg||.$$

4 (**Topological permutation**). For $\pi \in Auth(X)$, i.e. a homeomorphism and x fixed, note that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$d_\pi(x, y) = d(\pi(x), \pi(y)) < \varepsilon,$$

provided $d(x, y) < \delta$, i.e

$$B_\delta(x) \subset B_\varepsilon^\pi(x).$$

Take $\xi = \pi(x)$ and write $\eta = \pi(y)$; there is $\mu > 0$ such that

$$d(x, y) = d_{\pi^{-1}}(\xi, \eta) = d(\pi^{-1}(\xi), \pi^{-1}(\eta)) < \varepsilon,$$

provided $d_\pi(x, y) = d(\pi(x), \pi(y)) = d(\xi, \eta) < \mu$, i.e.

$$B_\mu^\pi(x) \subset B_\varepsilon(x).$$

Thus the topology generated by d_π is the same as that generated by d. This observation applies to all the previous examples provided the permutations are homeomorphisms (e.g. if X is a topological group under d^X). Note that for d^X right-invariant

$$||x||_\pi = ||\pi(x)\pi(e)^{-1}||.$$

5. For $g \in Auth(X), h \in X$, the bijection $\pi(x) = g(\rho_h(x)) = g(xh)$ is a homeomorphism provided right-shifts are continuous. We refer to this as the *shifted g-h-permutation metric*

$$d_{g\text{-}h}^X(x, y) = d^X(g(xh), g(yh)),$$

which has the associated g-$h$shifted norm

$$||x||_{g\text{-}h} = d^X(g(xh), g(h)).$$

6 (**Equivalent Bounded norm**). Set $\bar{d}(x, y) = \min\{d^X(x, y), 1\}$. Then $\bar{d}$ is an equivalent metric (cf. [Eng] Th. 4.1.3, p. 250). We refer to

$$||x|\bar{|} := \bar{d}(x, e) = \min\{d^X(x, e), 1\} = \min\{||x||, 1\},$$

as the **equivalent bounded norm**.

7. For $\mathcal{A} = Auth(X)$ the evaluation pseudo-metric at x on $\mathcal{A}$ is given by

$$d_x^{\mathcal{A}}(f,g) = d^X(f(x), g(x)),$$

and so

$$||f||_x = d_x^{\mathcal{A}}(f, id) = d^X(f(x), x)$$

is a pseudo-norm.

Definition (Refinements). 1 (cf. [GJ] Ch. 15.3 which works with pseudometrics.) Let $\Delta = \{d_i^X : i \in I\}$ be a family of metrics on a group X. The *weak (Tychonov)* Δ*-refinement topology* on X is defined by reference to the local base at x obtained by a finite intersections of ε-balls about x :

$$\bigcap_{i\in F} B_\varepsilon^i(x), \text{ for } F \text{ finite, i.e. } B_\varepsilon^{i_1}(x) \cap ... \cap B_\varepsilon^{i_n}(x), \text{ if } F = \{i_1, ..., i_n\},$$

where

$$B_\varepsilon^i(x) := \{y \in X : d_i^X(x,y) < \varepsilon\}.$$

2. The *strong* Δ*-refinement topology* on X is defined by reference to the local base at x obtained by a full intersections of ε-balls about x :

$$\bigcap_{d\in\Delta} B_\varepsilon^d(x).$$

Clearly

$$\bigcap_{d\in\Delta} B_\varepsilon^d(x) \subset \bigcap_{i\in F} B_\varepsilon^i(x), \text{ for } F \text{ finite,}$$

hence the name. We will usually be concerned with a family Δ of conjugate metrics. We note the following, which is immediate from the definition. (For (ii) see the special case in [dGMc] Lemma 2.1, [Ru-FA2] Ch. I 1.38(c), or [Eng] Th. 4.2.2 p. 259, which uses a sum in place of a supremum, and identify X with the diagonal of $\prod_{d\in\Delta}(X,d)$; see also [GJ] Ch. 15.)

Proposition 2.1. (i) *The strong* Δ*-refinement topology is generated by the metric*

$$d_\Delta^X(x,y) = \sup\{d_i^X(x,y) : i \in I\}.$$

(ii) *The weak Δ-refinement topology for Δ a countable family of metrics indexed by $I = \mathbb{N}$ is generated by the metric*

$$d^X_\Delta(x,y) = \sup_{i \in I} 2^{-i} \frac{d^X_i(x,y)}{1 + d^X_i(x,y)}.$$

Examples B. 1. For X a group we may take $\Delta = \{d^X_z : z \in X\}$ to obtain

$$d^X_\Delta(x,y) = \sup\{d^X(zx, zy) : z \in X\},$$

and if d^X is right-invariant

$$||x||_\Delta = \sup_z ||zxz^{-1}||.$$

2. For X a topological group we may take $\Delta = \{d^X_h : h \in Auth(X)\}$, to obtain

$$d^X_\Delta(x,y) = \sup\{d^X(h(x), h(y)) : h \in Auth(X)\}.$$

3. In the case $\mathcal{A} = Auth(X)$ we may take $\Delta = \{d^{\mathcal{A}}_x : x \in X\}$, the evaluation pseudo-metrics, to obtain

$$\begin{aligned} d^{\mathcal{A}}_\Delta(f,g) &= \sup_x d^{\mathcal{A}}_x(f,g) = \sup_x d^X(f(x), g(x)), \qquad \text{and} \\ ||f||_\Delta &= \sup_x d^{\mathcal{A}}_x(f, id_X) = \sup_x d^X(f(x), x). \end{aligned}$$

In Proposition 2.12 we will show that the strong Δ-refinement topology restricted to the subgroup $\mathcal{H}(X) := \{f \in \mathcal{A} : ||f||_\Delta < \infty\}$ is the topology of uniform convergence. The weak Δ-refinement topology here is just the topology of pointwise convergence.

The following result illustrates the kind of use we will make of refinement.

Proposition 2.2 (Symmetrization refinement) *If $|x|$ is a group pre-norm, then the symmetrization refinement*

$$||x|| := \max\{|x|, |x^{-1}|\}$$

is a group-norm

Proof. Positivity is clear, likewise symmetry. Noting that, for any A, B,

$$a + b \leq \max\{a, A\} + \max\{b, B\},$$

and supposing w.l.o.g. that

$$\max\{|x| + |y|, |y^{-1}| + |x^{-1}|\} = |x| + |y|,$$

we have

$$\begin{aligned} ||xy|| &= \max\{|xy|, |y^{-1}x^{-1}|\} \leq \max\{|x| + |y|, |y^{-1}| + |x^{-1}|\} \\ &= |x| + |y| \leq \max\{|x|, |x^{-1}|\} + \max\{|y|, |y^{-1}|\} \\ &= ||x|| + ||y||. \qquad \square \end{aligned}$$

Remark. One can use summation and take $||x|| := |x| + |x^{-1}|$, as

$$||xy|| = |xy| + |y^{-1}x^{-1}| \leq |x| + |y| + |y^{-1}| + |x^{-1}| = ||x|| + ||y||.$$

However, here and below, we prefer the more general use of a supremum or maximum.

Proposition 2.3. *If $|| \cdot ||$ is a group-norm, then $d(x, y) := ||xy^{-1}||$ is a right-invariant metric; equivalently, $\tilde{d}(x, y) := d(x^{-1}, y^{-1}) = ||x^{-1}y||$ is the conjugate left-invariant metric on the group.*

Conversely, if d is a right-invariant metric, then $||x|| := d(e, x) = \tilde{d}(e, x)$ is a group-norm.

Thus the metric d is bi-invariant iff $||xy^{-1}|| = ||x^{-1}y|| = ||y^{-1}x||$, i.e. iff the group-norm is abelian.

Proof. Given a group-norm put $d(x, y) = ||xy^{-1}||$. Then $||xy^{-1}|| = 0$ iff $xy^{-1} = e$, i.e. iff $x = y$. Symmetry follows from inversion as $d(x, y) = ||(xy^{-1})^{-1}|| = ||yx^{-1}|| = d(y, x)$. Finally, d obeys the triangle inequality, since

$$||xy^{-1}|| = ||xz^{-1}zy^{-1}|| \leq ||xz^{-1}|| + ||zy^{-1}||.$$

As for the converse, given a right-invariant metric d, put $||x|| := d(e, x)$. Now $||x|| = d(e, x) = 0$ iff $x = e$. Next, $||x^{-1}|| = d(e, x^{-1}) = d(x, e) = ||x||$, and so

$$d(xy, e) = d(x, y^{-1}) \leq d(x, e) + d(e, y^{-1}) = ||x|| + ||y||.$$

Also $d(xa, ya) = ||xaa^{-1}y^{-1}|| = d(x, y)$.

Finally d is bi-invariant iff $d(e, yx^{-1}) = d(x, y) = d(e, x^{-1}y)$ iff $||yx^{-1}|| = ||x^{-1}y||$. Inverting the first term yields the abelian property of the group-norm. □

The two (inversion) conjugate metrics separately define a left and right uniformity; taken together they define what is known as the ambidextrous uniformity, the only one of the three capable of being complete – see [Hal-ET, p. 63], [Kel] Ch. 6 Problem Q, and [Br-2]. We return to these matters in Section 3.

Definitions. 1. For d_R^X a right-invariant metric on a group X, we are justified by Proposition 2.2 in defining the *g-conjugate norm* from the g-conjugate metric by

$$||x||_g := d_g^X(x, e_X) = d_R^X(gx, g) = d_R^X(gxg^{-1}, e_X) = ||gxg^{-1}||.$$

2. For Δ a family of right-invariant metrics on X we put $\Gamma = \{||.||_d : D \in \Delta\}$, the set of corresponding norms defined by

$$||x||_d := d(x, e_X), \text{ for } d \in \Delta.$$

The *refinement norm* is then, as in Proposition 2.1,

$$||x||_\Gamma := \sup_{d\in\Delta} d(x, e_X) = \sup_{d\in\Gamma} ||x||_d.$$

We will be concerned with special cases of the following definition.

Definition ([Gr1], [Gr2], [BH] Ch. I.8). For constants $\mu \geq 1, \gamma \geq 0$, the metric spaces X and Y are said to be *(μ-γ)-quasi-isometric* under the mapping $\pi : X \to Y$ if

$$\begin{aligned} \frac{1}{\mu} d^X(a, b) - \gamma &\leq d^Y(\pi a, \pi b) \leq \mu d^X(a, b) + \gamma \qquad (a, b \in X), \\ d^Y(y, \pi[X]) &\leq \gamma \qquad (y \in Y). \end{aligned}$$

Corollary 2.4. *For π a homomorphism, the normed groups X, Y are (μ-γ)-quasi-isometric under π for the corresponding metrics iff their norms* *are (μ-γ)-quasi-equivalent, i.e.*

$$\begin{aligned} \frac{1}{\mu}||x||_X - \gamma &\leq ||\pi(x)||_Y \leq \mu||x||_X + \gamma \qquad (a, b \in X), \\ d^Y(y, \pi[X]) &\leq \gamma \qquad (y \in Y). \end{aligned}$$

Proof. This follows from $\pi(e_X) = e_Y$ and $\pi(xy^{-1}) = \pi(x)\pi(y)^{-1}$. $\square$

Remark. Note that $p(x) = ||\pi(x)||_Y$ is subadditive and bounded at $x = e$. It will follow that p is locally bounded at every point when we later prove Lemma 4.3.

The following result (which we use in [BOst14]) clarifies the relationship between the conjugate metrics and the group structure. We define the ε-*swelling* of a set K in a metric space X, for a given (e.g. right-invariant) metric d^X, to be

$$B_\varepsilon(K) := \{z : d^X(z, k) < \varepsilon \text{ for some } k \in K\}$$

and for the conjugate (resp. left-invariant) case we can write similarly

$$\tilde{B}_\varepsilon(K) := \{z : \tilde{d}^X(z, k) < \varepsilon \text{ for some } k \in K\}.$$

We write $B_\varepsilon(x_0)$ for $B_\varepsilon(\{x_0\})$, so that

$$B_\varepsilon(x_0) := \{z : ||zx_0^{-1}|| < \varepsilon\}.$$

When $x_0 = e_X$, the ball $B_\varepsilon(e_X)$ is the same under either of the conjugate metrics, as

$$B_\varepsilon(e_X) := \{z : ||z|| < \varepsilon\}.$$

Proposition 2.5. *(i) In a locally compact group X, for K compact and for $\varepsilon > 0$ small enough so that the closed ε-ball $B_\varepsilon(e_X)$ is compact, the swelling $B_{\varepsilon/2}(K)$ is pre-compact.*

(ii) $B_\varepsilon(K) = \{wk : k \in K, ||w||_X < \varepsilon\} = B_\varepsilon(e_X)K$, where the notation refers to swellings for d^X a right-invariant metric; similarly, for $\tilde{d}^X$, the conjugate metric, $\tilde{B}_\varepsilon(K) = KB_\varepsilon(e)$.

Proof. (i) If $x_n \in B_{\varepsilon/2}(K)$, then we may choose $k_n \in K$ with $d(k_n, x_n) < \varepsilon/2$. W.l.o.g. k_n converges to k. Thus there exists N such that, for $n > N$,

$d(k_n, k) < \varepsilon/2$. For such n, we have $d(x_n, k) < \varepsilon$. Thus the sequence x_n lies in the compact closed ε-ball centred at k and so has a convergent subsequence.

(ii) Let $d^X(x, y)$ be a right-invariant metric, so that $d^X(x, y) = ||xy^{-1}||$. If $||w|| < \varepsilon$, then $d^X(wk, k) = d^X(w, e) = ||w|| < \varepsilon$, so $wk \in B_\varepsilon(K)$. Conversely, if $\varepsilon > d^X(z, k) = d^X(zk^{-1}, e)$, then, putting $w = zk^{-1}$, we have $z = wk \in B_\varepsilon(K)$. □

The significance of the following simple corollary is manifold. It explicitly demonstrates that small either-sided translations λ_x, ρ_y do not much alter the norm. Its main effect is on the analysis of subadditive functions.

Corollary 2.6. *With* $||x|| := d^X(x, e)$, *where* d^X *is a right-invariant metric on* X,

$$|(||x|| - ||y||)| \leq ||xy|| \leq ||x|| + ||y||.$$

Proof: By Proposition 2.2, the triangle inequality and symmetry holds for norms, so $||y|| = ||x^{-1}xy|| \leq ||x^{-1}|| + ||xy|| = ||x|| + ||xy||$. □

We now generalize (1), by letting T, X be subgroups of a normed group G with X invariant under T.

Definition. We say that a function $h : X \to H$ is *slowly varying on* X *over* T if $\partial_X h(t) = e_H$, that is, for each t in T

$$h(tx)h(x)^{-1} \to e_H, \text{ as } ||x|| \to \infty \text{ for } x \in X.$$

We omit mention of X and T when context permits. In practice G will be an internal direct product of two normal subgroups $G = TX$. We may verify the property of h just defined by comparison with a slowly varying function.

Theorem 2.7 (Comparison criterion). $h : X \to H$ *is slowly varying iff for some slowly varying function* $g : X \to H$ *and some* $\mu \in H$,

$$\lim_{||x|| \to \infty} h(x)g(x)^{-1} = \mu.$$

Proof. If this holds for some slowly varying g and some μ,

$$h(tx)h(x)^{-1} = h(tx)g(tx)^{-1}g(tx)g(x)^{-1}g(x)h(x)^{-1} \to \mu e_H \mu^{-1} = e_H,$$

so h is slowly varying; the converse is trivial. □

Theorem 2.8. *For d^X a right-invariant metric on X, the norm $||x|| := d^X(x,e)$, as a function from X to the multiplicative positive reals $\mathbb{R}_+^*$, is slowly varying in the multiplicative sense, i.e., for any $t \in X$,*

$$\lim_{||x|| \to \infty} \frac{||tx||}{||x||} = 1.$$

Hence also

$$\lim_{||x|| \to \infty} \frac{||gxg^{-1}||}{||x||} = 1.$$

More generally, for T a one-parameter subgroup of X, any sub-additive Baire function $p : X \to \mathbb{R}_+^*$ with

$$||p||_T := \lim_{x \in T,\ ||x|| \to \infty} \frac{p(x)}{||x||} > 0$$

is multiplicatively slowly varying. (The limit exists by the First Limit Theorem for Baire subadditive functions, see [BOst5]*.)*

Proof: By Corollary 2.6, for $x \neq e$,

$$1 - \frac{||t||}{||x||} \leq \frac{||tx||}{||x||} \leq 1 + \frac{||t||}{||x||},$$

which implies slow variation. We regard p as mapping to $\mathbb{R}_+^*$, the strictly positive reals (since $p(x) = 0$ iff $x = e_X$). If $||p||_T > 0$, we may then take $\mu = ||p||_T$ and the assertion follows from the Comparison criterion above. Explicitly, for $x \neq e$,

$$\frac{p(xy)}{p(x)} = \frac{p(xy)}{||xy||} \cdot \frac{||xy||}{||x||} \cdot \frac{||x||}{p(x)} \to ||p||_T \cdot 1 \cdot \frac{1}{||p||_T} = 1. \qquad \square$$

Corollary 2.9. *If $\pi : X \to Y$ is a group homomorphism and $|| \cdot ||_Y$ is $(1$-$\gamma)$-quasi-isometric to $|| \cdot ||_X$ under the mapping π, then the subadditive function $p(x) = ||\pi(x)||_Y$ is slowly varying. For general $(\mu$-$\gamma)$-quasi-isometry the function p satisfies*

$$\mu^{-2} \leq p_*(z) \leq p^*(z) \leq \mu^2,$$

where

$$p^*(z) = \lim \sup_{||x|| \to \infty} p(zx)p(x)^{-1} \qquad p_*(z) = \lim \inf_{||x|| \to \infty} p(zx)p(x)^{-1}.$$

Proof. Subadditivity of p follows from homomorphism, since $p(xy) = ||\pi(xy)||_Y = ||\pi(x)\pi(y)||_Y \leq ||\pi(x)||_Y + ||\pi(y)||_Y$. Assuming that, for $\mu = 1$ and $\gamma > 0$, the norm $||\cdot||_Y$ is (μ-γ)-quasi-isometric to $||\cdot||_X$, we have, for $x \neq e$,

$$1 - \frac{\gamma}{||x||_X} \leq \frac{p(x)}{||x||_X} \leq 1 - \frac{\gamma}{||x||_X}.$$

So

$$\lim_{||x|| \to \infty} \frac{p(x)}{||x||} = 1 \neq 0,$$

and the result follows from the Comparison criterion (Th. 2.7) and Theorem 2.5.

If, for general $\mu \geq 1$ and $\gamma > 0$, the norm $||\cdot||_Y$ is (μ-γ)-quasi-isometric to $||\cdot||_X$, we have, for $x \neq e$,

$$\mu^{-1} - \frac{\gamma}{||x||_X} \leq \frac{p(x)}{||x||_X} \leq \mu - \frac{\gamma}{||x||_X}.$$

So for y fixed

$$\frac{p(xy)}{p(x)} = \frac{p(xy)}{||xy||} \cdot \frac{||xy||}{||x||} \cdot \frac{||x||}{p(x)} \leq \left(\mu - \frac{\gamma}{||xy||_X}\right) \cdot \frac{||xy||}{||x||} \cdot \left(\mu^{-1} - \frac{\gamma}{||x||_X}\right)^{-1},$$

giving, by Theorem 2.8 and because $||xy|| \geq ||x|| - ||y||$,

$$p^*(y) := \lim\sup_{x \to \infty} \frac{p(xy)}{p(x)} \leq \mu^2.$$

The left-sided inequality is proved dually (interchanging the roles of the upper and lower bounds on $||\pi(x)||_Y$). □

Remarks. 1. In the case of the *general (μ-γ)-quasi-isometry,* p exhibits the normed-groups O-analogue of slow-variation; compare [BGT] Cor. 2.0.5 p. 65.

2. When $X = \mathbb{R}$ the weaker boundedness property: "$p^*(y) < \infty$ on a large enough set of y's" implies that p satisfies

$$z^d \leq p_*(z) \leq p^*(z) \leq z^c, \qquad (z \geq Z)$$

for some constants c, d, Z (so is *extended regularly varying* in the sense of [BGT] Ch. 2, 2.2 p. 65). Some generalizations are given in Theorems 6.8 and 6.9.

3. We pause to consider briefly some classical examples. If $X = H = \mathbb{R}$ is construed additively, so that $e_H = e_X = 0$ and $||x|| := |x - 0| = |x|$ in both cases, and with the action tx denoting $t + x$, the function $f(x) := |x|$ is not slowly varying, because $(x + t) - x = t \nrightarrow 0 = e_H$. On the other hand a multiplicative construction on $H = \mathbb{R}_+^*$, for which $e_H = 1$ and $||h||_H := |\log h|$, but with $X = \mathbb{R}$ still additive and tx still meaning $t + x$, yields f as having slow variation (as in the Theorem 2.8), as

$$f(tx)f(x)^{-1} = (x + t)/x \to 1 = e_H \text{ as } x \to \infty.$$

We note that in this context the regularly varying functions h on X have $h(tx)h(x)^{-1} = h(t + x) - h(x) \to at$, for some constant a.

Note that, for $X = H = \mathbb{R}_+^*$, and with tx meaning $t \cdot x$, since $||x|| = |\log x|$ (as just noted) is the group-norm, we have here

$$f(tx)f(x)^{-1} = ||tx||/||x|| = \frac{|\log tx|}{|\log x|} = \frac{|\log t + \log x|}{|\log x|} \to 1 = e_H, \text{ as } x \to \infty,$$

which again illustrates the content of Theorem 2.7. Here the regularly varying functions $h(tx)h(x)^{-1} \to e^{at}$, for some constant a. See [BGT] Ch. 1 for background on additive and multiplicative formulations of regular variation in the classical setting of functions $f : G \to H$ with $G, H = \mathbb{R}$ or $\mathbb{R}_+$.

Definition. 1. Say that $\xi \in X$ is *infinitely divisible* if, for each positive integer n, there is x with $x^n = \xi$. (Compare Section 3.)

2. Say that the infinitely divisible element ξ is *embeddable* if, for some one-parameter subgroup T in X, we have $\xi \in T$. When such a T exists it is unique (the elements $\xi^{m/n}$, for m, n integers, are dense in T); we write $T(\xi)$ for it.

Clearly any element of a one-parameter subgroup is both infinitely divisible and embeddable. For results on this see Davies [D], Heyer [Hey], McCrudden [McC]. With these definitions, our previous analysis allows the First Limit Theorem for subadditive functions (cf. Th. 2.8 and [BOst5]) to be restated in the context of normed groups.

Proposition 2.10. *Let ξ be infinitely divisible and embeddable in $T(\xi)$, a one-parameter subgroup of X. Then for any Baire subadditive $p : X \to \mathbb{R}_+$*

and $t \in T(\xi)$,

$$\partial_{T(\xi)} p(t) := \lim_{s \in T,\ ||s|| \to \infty} \frac{p(ts)}{||s||} = ||p||_T,$$

i.e., treating the subgroup $T(\xi)$ *as a direction, the limit function is determined by the direction.*

Proof. By subadditivity, $p(s) = p(t^{-1}ts) \leq p(t^{-1}) + p(ts)$, so

$$p(s) - p(t^{-1}) \leq p(ts) \leq p(t) + p(s).$$

For $s \neq e$, divide through by $||s||$ and let $||s|| \to \infty$ (as in Th. 2.8):

$$||p||_T \leq \partial_T p(t) \leq ||p||_T. \qquad \square$$

Definition (Supremum metric, supremum norm). Let X have a metric d^X. As before $\mathcal{G}$ is a fixed subgroup of $Auth(X)$, for example $Tr_L(X)$ the group of left-translations λ_x (cf. Th. 3.10), defined by

$$\lambda_x(z) = xz.$$

(We consider this in detail in the Section 4.) For $g, h \in \mathcal{G}$, define the possibly infinite number

$$\hat{d}^X(g, h) := \sup_{x \in X} d^X(g(x), h(x)).$$

Put

$$\mathcal{H}(X) = \mathcal{H}(X, \mathcal{G}) := \{g \in \mathcal{G} : \hat{d}^X(g, id_X) < \infty\},$$

and call these the *bounded elements* of $\mathcal{G}$. For g, h in $\mathcal{H}(X)$, we call $\hat{d}^X(g, h)$ the *supremum metric* and the associated norm

$$||h||_{\mathcal{H}} = ||h||_{\mathcal{H}(X)} := \hat{d}^X(h, id_X) = \sup_{x \in X} d^X(h(x), x)$$

the *supremum norm*. This metric notion may also be handled in the setting of uniformities (cf. the notion of functions limited by a cover $\mathcal{U}$ arising in [AnB] Section 2; see also [BePe] Ch. IV Th. 1.2); in such a context excursions into invariant measures rather than use of Haar measure (as in Section 6) would refer to corresponding results established by Itzkowitz [Itz].

Our next result justifies the terminology of the definition above.

Proposition 2.11 (Group-norm properties in $\mathcal{H}(X)$).
If $||h|| = ||h||_{\mathcal{H}}$, then $||\cdot||$ is a group-norm: that is, for $h, h' \in \mathcal{H}(X)$,

$$||h|| = 0 \text{ iff } h = e, \; ||h \circ h'|| \leq ||h|| + ||h'|| \text{ and } ||h|| = ||h^{-1}||.$$

Proof. Evidently $\hat{d}(h, id_X) = \sup_{x \in X} d(h(x), x) = 0$ iff $h(x) = id_X$. We have

$$||h|| = \hat{d}(h, id_X) = \sup_{x \in X} d(h(x), x) = \sup_{y \in X} d(y, h^{-1}(y)) = ||h^{-1}||.$$

Next note that

$$\hat{d}(id_X, h \circ h') = \sup_{x \in X} d(hh'(x), x) = \sup_{y \in X} d(h(y), h'^{-1}(y)) = \hat{d}(h, h'^{-1}). \qquad (2)$$

But

$$\hat{d}(h, h') = \sup_{x \in X} d(h(x), h'(x)) \leq \sup_{x \in X}[d(h(x), x) + d(x, h'(x))] \leq \hat{d}(h, id) + \hat{d}(h', id) < \infty.$$

□

Theorem 2.12. *The set $\mathcal{H}(X)$ of bounded auto-homeomorphisms of a metric group X is a group under composition, metrized by the right-invariant supremum metric $\hat{d}^X$.*

Proof. The identity, id_X, is bounded. For right-invariance (cf. (2)),

$$\hat{d}(g \circ h, g' \circ h) = \sup_{x \in X} d(g(h(x)), g'(h(x)) = \sup_{y \in X} d(g(y), g'(y)) = \hat{d}(g, g'). \qquad \square$$

Theorem 2.13 ([BePe] Ch. IV Th 1.1**).** *Let d be a bounded metric on X. As a group under composition, $\mathcal{A} = Auth(X)$ is a topological group under the weak Δ-refinement topology for $\Delta := \{\hat{d}_\pi : \pi \in \mathcal{A}\}$.*

Proof. To prove continuity of inversion at F, write $H = F^{-1}$ and for any x put $y = f^{-1}(x)$. Then

$$d_\pi(f^{-1}(x), F^{-1}(x)) = d_\pi(H(F(y)), H(f(y))) = d_{\pi H}(F(y), f(y)),$$

and so

$$\hat{d}_\pi(f^{-1}, F^{-1}) = \sup_x d_\pi(f^{-1}(x), F^{-1}(x)) = \sup_y d_{\pi H}(F(y), f(y)) = \hat{d}_{\pi H}(f, F).$$

Thus f^{-1} is in any $\hat{d}_\pi$ neighbourhood of F^{-1} provided f is in any $\hat{d}_{\pi H}$ neighbourhood of F.

As for continuity of composition at F, G, we have for fixed x that

$$\begin{aligned} d_\pi(f(g(x)), F(G(x))) &\leq d_\pi(f(g(x)), F(g(x))) + d_\pi(F(g(x)), F(G(x))) \\ &= d_\pi(f(g(x)), F(g(x)) + d_{\pi F}(g(x), G(x)) \\ &\leq \hat{d}_\pi(f, F) + \hat{d}_{\pi F}(g, G). \end{aligned}$$

Hence

$$\hat{d}_\pi(fg, FG) \leq \hat{d}_\pi(f, F) + \hat{d}_{\pi F}(g, G),$$

so that fg is in the $\hat{d}_\pi$-ball of radius ε of FG provided f is in the $\hat{d}_\pi$-ball of radius $\varepsilon/2$ of F and g is in the $\hat{d}_{\pi H}$-ball of radius $\varepsilon/2$ of G. □

Remark: The compact-open topology. In similar circumstances, we show in Theorem 3.17 below that under the *strong* Δ-refinement topology $Auth(X)$ is a normed group and a topological group. Rather than use weak or strong refinement of metrics in $Auth(X)$, one may consider the compact-open topology (the topology of uniform convergence on compacts, introduced by Fox and studied by Arens in [Ar1], [Ar2]). However, in order to ensure the kind of properties we need (especially in flows), the metric space X would then need to be restricted to a special case. Recall some salient features of the compact-open topology. For composition to be continuous local compactness is essential ([Dug] Ch. XII.2, [Mc], [BePe] Section 8.2, or [vM] Ch.1). When T is compact the topology is admissible (i.e. $Auth(X)$ is a topological group under it), but the issue of admissibility in the non-compact situation is not currently fully understood (even in the locally compact case for which counter-examples with non-continuous inversion exist, and so additional properties such as local connectedness are usually invoked – see [Dij] for the strongest results). In applications the focus of interest may fall on separable spaces (e.g. function spaces), but, by a theorem of Arens, if X is separable metric and further the compact-open topology on $\mathcal{C}\ (X, \mathbb{R})$ is metrizable, then X is necessarily locally compact and σ-compact, and conversely (see e.g [Eng] p.165 and 266).

We will now apply the supremum-norm construction to deduce that right-invariance may be arranged if for every $x \in X$ the left translation λ_x has finite sup-norm:

$$||\lambda_x||_{\mathcal{H}} = \sup_{z \in X} d^X(xz, z) < \infty.$$

We will need to note the connection with conjugate norms.

Definition. Recall the g-conjugate norm is defined by

$$||x||_g := ||gxg^{-1}||.$$

The *conjugacy refinement norm* corresponding to the family of all the g-conjugate norms $\Gamma = \{||.||_g : g \in G\}$ will be denoted by

$$||x||_\infty := \sup_g ||x||_g,$$

in contexts where this is finite.

Clearly, for any g,

$$||x||_\infty = ||gxg^{-1}||_\infty,$$

and so $||x||_\infty$ is an abelian norm. Evidently, if the metric d_L^X is left-invariant we have

$$||x||_\infty = \sup_g ||x||_g = \sup_{z \in X} d_L^X(z^{-1}xz, e) = \sup_{z \in X} d_L^X(xz, z). \tag{shift}$$

One may finesse the left-invariance assumption, using (shift), as we will see in Proposition 2.14.

Example C. As $\mathcal{H}(X)$ is a group and $\hat{d}^X$ is right-invariant, the norm $||f||_{\mathcal{H}}$ gives rise to a conjugacy refinement norm. Working in $\mathcal{H}(X)$, suppose that $f_n \to f$ under the supremum norm $\hat{d}^X$. Let $g \in \mathcal{H}(X)$. Then pointwise

$$\lim_n f_n(g(x)) = f(g(x)).$$

Hence, as f^{-1} is continuous, we have for any $x \in X$,

$$f^{-1}(\lim_n f_n(g(x))) = \lim_n f^{-1} f_n(g(x)) = g(x).$$

Likewise, as g^{-1} is continuous, we have for any $x \in X$,

$$g^{-1}(\lim_n f^{-1} f_n(g(x))) = \lim_n g^{-1} f^{-1} f_n(g(x)) = x.$$

Thus

$$g^{-1} f^{-1} f_n g \to id_X \text{ pointwise.}$$

This result is generally weaker than the assertion $||f^{-1}f_n||_g \to 0$, which requires uniform rather than pointwise convergence.

We need the following notion of admissibility (with the norm $||.||_\infty$ in mind; compare also Section 3).

Definitions. 1. Say that the metric d^X satisfies the *metric admissibility condition* on $H \subset X$ if, for any $z_n \to e$ in H under d^X and arbitrary y_n,

$$d^X(z_n y_n, y_n) \to 0.$$

2. If d^X is left-invariant, the condition may be reformulated as a *norm admissibility condition* on $H \subset X$, since

$$||y_n^{-1} z_n y_n|| = d_L^X(y_n^{-1} z_n y_n, e) = d_L^X(z_n y_n, y_n) \to 0.$$

3. We will say that the group X satisfies the *topological admissibility condition* on $H \subset X$ if, for any $z_n \to e$ in H and arbitrary y_n

$$y_n^{-1} z_n y_n \to e.$$

Proposition 2.14 (Right-invariant sup-norm).
For any metric d^X on a group X, put

$$\begin{aligned} H_X \ &: = H = \{x \in X : \sup_{z \in X} d^X(xz, z) < \infty\}, \\ ||x||_{\mathcal{H}} \ &: = \sup d^X(xz, z), \text{ for } x \in H. \end{aligned}$$

For $x, y \in H$, let $\bar{d}_{\mathcal{H}}(x,y) := \hat{d}^X(\lambda_x, \lambda_y) = \sup_z d^X(xz, yz)$. Then:
(i) *$\bar{d}_{\mathcal{H}}$ is a right-invariant metric on H, and $\bar{d}_{\mathcal{H}}(x,y) = ||xy^{-1}||_{\mathcal{H}} = ||\lambda_x \lambda_y^{-1}||_{\mathcal{H}}$.*
(ii) *If d^X is left-invariant, then $\bar{d}_{\mathcal{H}}$ is bi-invariant on H, and so $||x||_\infty = ||x||_{\mathcal{H}}$ and the norm is abelian on H.*
(iii) *The $\bar{d}_{\mathcal{H}}$-topology on H is equivalent to the d^X-topology on H iff d^X satisfies the metric admissibility condition on H, i.e. for $z_n \to e$ in H and arbitrary $y_n \in X$,*

$$d^X(z_n y_n, y_n) \to 0.$$

(iv) *In particular, if d^X is right-invariant, then $H = X$ and $\bar{d}_{\mathcal{H}} = d^X$.*

(v) *If X is a compact topological group under d^X, then $\bar{d}_{\mathcal{H}}$ is equivalent to d^X.*

Proof. (i) The argument relies implicitly on the natural embedding of X in $Auth(X)$ as $Tr_L(X)$ (made explicit in the next section). For $x \in X$ we write

$$||\lambda_x||_{\mathcal{H}} := \sup_z d^X(xz, z).$$

For $x \neq e$, we have $0 < ||\lambda_x||_{\mathcal{H}} \leq \infty$. By Proposition 2.12, $\mathcal{H}(X) = \mathcal{H}(X, Tr_L(X)) = \{\lambda_x : ||\lambda_x||_{\mathcal{H}} < \infty\}$ is a subgroup of $\mathcal{H}(X, Auth(X))$ on which $|| \cdot ||_{\mathcal{H}}$ is thus a norm. Identifiying $\mathcal{H}(X)$ with the subset $H = \{x \in X : ||\lambda_x|| < \infty\}$ of X, we see that on H

$$\bar{d}_{\mathcal{H}}(x, y) := \sup_z d^X(xz, yz) = \hat{d}_{\mathcal{H}}(\lambda_x, \lambda_y)$$

defines a right-invariant metric, as

$$\bar{d}_{\mathcal{H}}(xv, yv) = \sup_z d^X(xvz, yvz) = \sup_z d^X(xz, yz) = \bar{d}_{\mathcal{H}}(x, y).$$

Hence with

$$||x||_{\mathcal{H}} = \bar{d}_{\mathcal{H}}(x, e) = ||\lambda_x||_{\mathcal{H}},$$

by Proposition 2.11

$$||\lambda_x \lambda_y^{-1}||_{\mathcal{H}} = \bar{d}_{\mathcal{H}}(x, y) = ||xy^{-1}||_{\mathcal{H}},$$

as asserted.

If d^X is left-invariant, then

$$\bar{d}_{\mathcal{H}}(vx, vy) = \sup_z d_L^X(vxz, vyz) = \sup_z d_L^X(xz, yz) = \bar{d}_{\mathcal{H}}(x, y),$$

and so $\bar{d}_{\mathcal{H}}$ is both left invariant and right-invariant.

Note that

$$||x||_{\mathcal{H}} = \bar{d}_{\mathcal{H}}(x, e) = \sup_z d_L^X(xz, z) = \sup_z d_L^X(z^{-1}xz, e) = \sup_z ||x||_z = ||x||_\infty.$$

(ii) We note that

$$d^X(z_n, e) \leq \sup_y d^X(z_n y, y).$$

Thus if $z_n \to e$ in the sense of $d_{\mathcal{H}}$, then also $z_n \to e$ in the sense of d^X. Suppose that the metric admissibility condition holds but the metric $d_{\mathcal{H}}$ is not equivalent to d^X. Thus for some $z_n \to e$ (in H and under d^X) and $\varepsilon > 0$,

$$\sup_y d^X(z_n y, y) \geq \varepsilon.$$

Thus there are y_n with

$$d^X(z_n y_n, y_n) \geq \varepsilon/2,$$

which contradicts the admissibility condition.

For the converse, if the metric $d_{\mathcal{H}}$ is equivalent to d^X, and $z_n \to e$ in H and under d^X, then $z_n \to e$ also in the sense of $d_{\mathcal{H}}$; hence for y_n given and any $\varepsilon > 0$, there is N such that for $n \geq N$,

$$\varepsilon > \bar{d}_{\mathcal{H}}(z_n, e) = \sup_y d^X(z_n y, y) \geq d^X(z_n y_n, y_n).$$

Thus $d^X(z_n y_n, y_n) \to 0$, as required.

(iii) If d^X is right-invariant, then $d^X(z_n y_n, y_n) = d^X(z_n, e) \to 0$ and the admissibility condition holds on H. Of course $||\lambda_x||_{\mathcal{H}} = \sup_z d^X(xz, z) = d^X(x, e) = ||x||_X$ and so $H = X$.

(iv) If X is compact, then $H = H_X$ as $z \to d^X(xz, z)$ is continuous. If $z_n \to e$ and y_n are arbitrary, suppose that the admissibility condition fails. Then for some $\varepsilon > 0$ we have w.l.o.g.

$$d^X(z_n y_n, y_n) \geq \varepsilon.$$

Passing down a subsequence $y_m \to y$ and assuming that X is a topological group we obtain

$$0 = d^X(ey, y) \geq \varepsilon,$$

a contradiction. □

As a corollary we obtain the following known result (cf. Theorem 3.3.4 in [vM] p. 101, for a different proof).

Proposition 2.15. *In a first countable topological group X the condition $y_n^{-1} z_n y_n \to e$ on X is equivalent to the existence of an abelian norm (equivalently, a bi-invariant metric).*

Proof. By the Birkhoff-Kakutani Theorem, the topology is induced by a left-invariant metric, d_L^X say, which is w.l.o.g. bounded (take $d = \max\{d_L^X, 1\}$, which is also left-invariant, cf. Example A6). Then $H_X = X$, and the topological admissibility condition $y_n^{-1}z_ny_n \to e$ on X implies the metric admissibility condition for d_L^X. The metric thus induces the norm $||x||_{\mathcal{H}}$, which is abelian, and, by Proposition 2.3, defines a bi-invariant metric. Conversely, if $||.||_X$ is abelian, then the topological admissibility condition follows from the observation that

$$||y_n^{-1}z_ny_n|| = ||y_ny_n^{-1}z_n|| = ||z_n|| \to 0. \qquad \square$$

Application. Let S, T be normed groups. For $\alpha : S \to T$ we define the possibly infinite number

$$||\alpha|| := \sup\{||\alpha(s)||_T/||s||_S : s \in S\} = \inf\{M : ||\alpha(s)|| \leq M||s|| \ (\forall s \in S)\}.$$

α is called bounded if $||\alpha||$ is finite. The bounded elements form a group G under the pointwise multiplication $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Clearly $||\alpha|| = 0$ implies that $\alpha(t) = e$, for all t. Symmetry is clear. Also

$$||\alpha(t)\beta(t)|| \leq ||\alpha(t)|| + ||\beta(t)|| \leq [||\alpha|| + ||\beta||]||t||,$$

so

$$||\alpha\beta|| \leq ||\alpha|| + ||\beta||.$$

We say that a function $\alpha : S \to T$ is *multiplicative* if α is bounded and

$$\alpha(ss') = \alpha(s)\alpha(s').$$

A function $\gamma : S \to T$ is *asymptotically multiplicative* if $\gamma = \alpha\beta$, where α is multiplicative and bounded and β is bounded. In the commutative situation with S, T normed vector spaces, the norm here reduces to the operator norm. This group norm is studied extensively in [CSC] in relation to Ulam's problem. We consider in Section 3 the case $S = T$ and functions α which are inner automorphisms.

Proposition 2.16 (Magnification metric). *Let $T = \mathcal{H}(X)$ with group norm $||t|| = d^T(t, e_T)$ and $\mathcal{A}$ a subgroup (under composition) of $Auth(T)$ (so, for $t \in T$ and $\alpha \in \mathcal{A}$, $\alpha(t) \in \mathcal{H}(X)$ is a homeomorphism of X). For any $\varepsilon \geq 0$, put*

$$d_{\mathcal{A}}^{\varepsilon}(\alpha, \beta) := \sup_{||t|| \leq \varepsilon} \hat{d}^T(\alpha(t), \beta(t)).$$

Suppose further that X distinguishes the maps $\{\alpha(e_{\mathcal{H}(X)}) : \alpha \in \mathcal{A}\}$, i.e., for $\alpha, \beta \in A$, there is $z = z_{\alpha,\beta} \in X$ with $\alpha(e_{\mathcal{H}(X)})(z) \neq \beta(e_{\mathcal{H}(X)})(z)$.

Then $d^{\varepsilon}_{\mathcal{A}}(\alpha, \beta)$ is a metric; furthermore, $d^{\varepsilon}_{\mathcal{A}}$ is right-invariant for translations by γ such that γ^{-1} maps the ε-ball to the ε-ball.

Proof. To see that this is a metric, note that for $t = e_{\mathcal{H}(X)} = id_T$ we have $||t|| = 0$ and

$$\begin{aligned}\hat{d}^T(\alpha(e_{\mathcal{H}(X)}), \beta(e_{\mathcal{H}(X)})) &= \sup_z d^X(\alpha(e_{\mathcal{H}(X)})(z), \beta(e_{\mathcal{H}(X)})(z)) \\ &\geq d^X(\alpha(e_{\mathcal{H}(X)})(z_{\alpha,\beta}), \beta(e_{\mathcal{H}(X)})(z_{\alpha,\beta})) > 0.\end{aligned}$$

Symmetry is clear. Finally the triangle inequality follows as usual:

$$\begin{aligned}d^{\varepsilon}_{\mathcal{A}}(\alpha, \beta) &= \sup_{||t|| \leq 1} \hat{d}^T(\alpha(t), \beta(t)) \leq \sup_{||t|| \leq 1} [\hat{d}^T(\alpha(t), \gamma(t)) + \hat{d}^T(\gamma(t), \beta(t))] \\ &\leq \sup_{||t|| \leq 1} \hat{d}^T(\alpha(t), \gamma(t)) + \sup_{||t|| \leq 1} \hat{d}^T(\gamma(t), \beta(t)) \\ &= d^{\varepsilon}_{\mathcal{A}}(\alpha, \gamma) + d^{\varepsilon}_{\mathcal{A}}(\gamma, \beta).\end{aligned}$$

One cannot hope for the metric to be right-invariant in general, but if γ^{-1} maps the ε-ball to the ε-ball, one has

$$\begin{aligned}d^{\varepsilon}_{\mathcal{A}}(\alpha\gamma, \beta\gamma) &= \sup_{||t|| \leq \varepsilon} \hat{d}^T(\alpha(\gamma(t)), \beta(\gamma(t)) \\ &= \sup_{||\gamma^{-1}(s)|| \leq \varepsilon} \hat{d}^T(\alpha(s), \beta(s)). \qquad \square\end{aligned}$$

In this connection we note the following.

Proposition 2.17. *In the setting of Proposition 2.16, denote by $||.||_{\varepsilon}$ the norm induced by $d^{\varepsilon}_{\mathcal{A}}$; then*

$$\sup_{||t|| \leq \varepsilon} ||\gamma(t)||_T - \varepsilon \leq ||\gamma||_{\varepsilon} \leq \sup_{||t|| \leq \varepsilon} ||\gamma(t)||_T + \varepsilon.$$

Proof. By definition, for t with $||t|| \leq \varepsilon$, we have

$$\begin{aligned}||\gamma||_{\varepsilon} &= \sup_{||t|| \leq \varepsilon} \hat{d}^T(\gamma(t), t) \leq \sup_{||t|| \leq \varepsilon} [\hat{d}^T(\gamma(t), e) + \hat{d}^T(e, t)] \leq \sup_{||t|| \leq \varepsilon} ||\gamma(t)||_T + \varepsilon, \\ ||\gamma(t)||_T &= \hat{d}^T(\gamma(t), e) \leq \hat{d}^T(\gamma(t), t) + \hat{d}^T(t, e) \\ &\leq ||t|| + ||\gamma||_{\varepsilon} \leq \varepsilon + ||\gamma||_{\varepsilon}. \qquad \square\end{aligned}$$

Theorem 2.18 (Invariance of Norm Theorem – for (b) cf. [Klee]).
(a) *The group-norm is abelian (and the metric is bi-invariant) iff*

$$||xy(ab)^{-1}|| \leq ||xa^{-1}|| + ||yb^{-1}||,$$

for all x, y, a, b, *or equivalently,*

$$||uabv|| \leq ||uv|| + ||ab||,$$

for all x, y, u, v.
(b) *Hence a metric* d *on the group* X *is bi-invariant iff the Klee property holds:*

$$d(ab, xy) \leq d(a, x) + d(b, y). \tag{Klee}$$

In particular, this holds if the group X *is itself abelian.*
(c) *The group norm is abelian iff the norm is preserved under conjugacy (inner automorphisms).*

Proof (a) If the group-norm is abelian, then by the triangle inequality

$$\begin{aligned} ||xyb^{-1} \cdot a^{-1}|| &= ||a^{-1}xyb^{-1}|| \\ &\leq ||a^{-1}x|| + ||yb^{-1}||. \end{aligned}$$

For the converse we demonstrate bi-invariance in the form $||ba^{-1}|| = ||a^{-1}b||$. In fact it suffices to show that $||yx^{-1}|| \leq ||x^{-1}y||$; for then bi-invariance follows, since taking $x = a, y = b$ we get $||ba^{-1}|| \leq ||a^{-1}b||$, whereas taking $x = b^{-1}, y = a^{-1}$ we get the reverse $||a^{-1}b|| \leq ||ba^{-1}||$. As for the claim, we note that

$$||yx^{-1}|| \leq ||yx^{-1}yy^{-1}|| \leq ||yy^{-1}|| + ||x^{-1}y|| = ||x^{-1}y||.$$

(b) Klee's result is deduced as follows. If d is a bi-invariant metric, then $|| \cdot ||$ is abelian. Conversely, for d a metric, let $||x|| := d(e, x)$. Then $||.||$ is a group-norm, as

$$d(ee, xy) \leq d(e, x) + d(e, y).$$

Hence d is right-invariant and $d(u, v) = ||uv^{-1}||$. Now we conclude that the group-norm is abelian since

$$||xy(ab)^{-1}|| = d(xy, ab) \leq d(x, a) + d(y, b) = ||xa^{-1}|| + ||yb^{-1}||.$$

Hence d is also left-invariant.

(c) Suppose the norm is abelian. Then for any g, by the cyclic property $||g^{-1}bg|| = ||gg^{-1}b|| = ||b||$. Conversely, if the norm is preserved under automorphism, then we have bi-invariance, since $||ba^{-1}|| = ||a^{-1}(ba^{-1})a|| = ||a^{-1}b||$. □

Remark. Note that, taking $b = v = e$, we have the triangle inequality. Thus the result (a) characterizes maps $||\cdot||$ with the positivity property as group pre-norms which are abelian. In regard to conjugacy, see also the Uniformity Theorem for Conjugation in Section 11. We close with the following classical result.

Theorem 2.19 (Normability Theorem for Groups – Kakutani-Birkhoff). *Let X be a first-countable group and let V_n be a balanced local base at e_X with*

$$V_{n+1}^4 \subseteq V_n.$$

Let $r = \sum_{n=1}^{\infty} c_n(r)2^{-n}$ be a terminating representation of the dyadic number r, and put

$$A(r) := \sum_{n=1}^{\infty} c_n(r)V_n.$$

Then

$$p(x) := \inf\{r : x \in A(r)\}$$

is a group-norm. If further X is locally compact and non-compact, then p may be arranged such that p is unbounded on X, but bounded on compact sets.

For a proof see that offered in [Ru-FA2] for Th. 1.24 (p. 18-19), which derives a metrization of a topological vector space in the form $d(x, y) = p(x-y)$ and makes no use of the scalar field, That proof may be rewritten verbatim with xy^{-1} substituting for the additive notation $x - y$ (cf. Proposition 2.2).

Remarks.

1. If the group-norm is abelian, then we have the *commutator inequality*

$$||[x, y]|| \leq 2||x^{-1}y||,$$

because

$$||[x, y]|| = ||x^{-1}y^{-1}xy|| \leq ||x^{-1}y|| + ||y^{-1}x|| = 2||x^{-1}y||.$$

The triangle inequality gives only

$$||[x,y]|| = ||x^{-1}y^{-1}xy|| \leq ||x^{-1}y^{-1}|| + ||xy|| = ||xy|| + ||yx||.$$

2. Take $u = f(tx), v = f(x)^{-1}$ etc.; then, assuming the Klee Property, we have

$$\begin{aligned} ||f(tx)g(tx)[f(x)g(x)]^{-1}|| &= ||f(tx)g(tx)g(x)^{-1}f(x)^{-1}|| \\ &\leq ||f(tx)f(x)^{-1}|| + ||g(tx)g(x)^{-1}||, \end{aligned}$$

showing that the product of two slowly varying functions is slowly varying, since

$$f(tx)f(t)^{-1} \to e \text{ iff } ||f(tx)f(t)^{-1}|| \to 0.$$

3 Normed versus topological groups

By the Birkhoff-Kakutani Theorem above (Th. 2.19) any *metrizable* topological group has a right-invariant equivalent metric, and hence is a normed group. We show below that a normed group is a topological group provided its shifts are continuous, i.e. the group is *semitopological* (see [ArRez]). This is not altogether surprising: assuming that a group T is metrizable, non-meagre and analytic in the metric, and that both shifts are continuous, then T is a topological group (see e.g. [THJ] in [Rog2] p. 352; compare also [Ell]).

As we have seen in Th. 2.3, a group-norm defines two metrics: the right-invariant metric which we denote in *this* section by $d_R(x,y) := ||xy^{-1}||$ and the conjugate left-invariant metric, here to be denoted $d_L(x,y) := d_R(x^{-1},y^{-1})$ $= ||x^{-1}y||$. There is correspondingly a right and left metric topology which we term the *right* or *left norm topology*. We write $\to_R$ for convergence under d_R etc. Recall that both metrics give rise to the same norm, since $d_L(x,e) = d_R(x^{-1},e) = d_R(e,x) = ||x||$, and hence define the same balls centered at the origin e:

$$B_R^d(e,r) := \{x : d(e,x) < r\} = B_L^d(e,r).$$

Denoting this commonly determined set by $B(r)$, we have seen in Proposition 2.5 that

$$\begin{aligned} B_R(a,r) &= \{x : x = ya \text{ and } d_R(a,x) = d_R(e,y) < r\} = B(r)a, \\ B_L(a,r) &= \{x : x = ay \text{ and } d_L(a,x) = d_L(e,y) < r\} = aB(r). \end{aligned}$$

Thus the open balls are right- or left-shifts of the norm balls at the origin. This is best viewed in the current context as saying that under d_R the right-shift $\rho_a : x \to xa$ is right uniformly continuous, as

$$d_R(xa, ya) = d_R(x, y),$$

and likewise that under d_L the left-shift $\lambda_a : x \to ax$ is left uniformly continuous, as

$$d_L(ax, ay) = d_L(x, y).$$

In particular, under d_R we have $y \to_R b$ iff $yb^{-1} \to_R e$, as $d_R(e, yb^{-1}) = d_R(y, b)$. Likewise, under d_L we have $x \to_L a$ iff $a^{-1}x \to_L e$, as $d_L(e, a^{-1}x) = d_L(x, a)$.

Thus either topology is determined by the neighbourhoods of the identity (origin) and according to choice makes the appropriately sided shift continuous. We noted earlier that the triangle inequality implies that multiplication is jointly continuous at the identity e. Inversion is also continuous at the identity by the symmetry axiom. To obtain similar results elsewhere one needs to have continuous conjugation, and this is linked to the equivalence of the two norm topologies. The conjugacy map under $g \in G$ (inner automorphism) is defined by

$$\gamma_g(x) := gxg^{-1}.$$

Recall that the inverse of γ_g is given by conjugation under g^{-1} and that γ_g is a homomorphism. Its continuity is thus determined by behaviour at the identity, as we verify below. We work with the right topology (under d_R).

Lemma 3.1. *The homomorphism γ_g is right-continuous at any point iff it is right-continuous at e.*

Proof. This is immediate since $x \to_R a$ iff $xa^{-1} \to_R e$ and $\gamma_g(x) \to_R \gamma_g(a)$ iff $\gamma_g(xa^{-1}) \to_R \gamma_g(e)$, since

$$||gxg^{-1}(gag^{-1})^{-1}|| = ||gxa^{-1}g^{-1}||. \qquad \square$$

We note that by the Generalized Darboux Theorem (see Section 10) if γ_g is locally norm-bounded and the norm is $\mathbb{N}$-subhomogeneous (i.e. there are constants $\kappa_n \to \infty$ with $\kappa_n||z|| \leq ||z^n||$), then γ_g is continuous. Working under d_R, we will relate inversion to left shift. We begin with the following.

Lemma 3.2. *If inversion is right-to-right continuous, then*
$x \to_R a$ *iff* $a^{-1}x \to_R e$.

Proof. For $x \to_R a$, we have, assuming continuity, that $d_R(e, a^{-1}x) = d_R(x^{-1}, a^{-1}) \to 0$. Conversely, for $a^{-1}x \to_R e$ we have $d_R(a^{-1}x, e) \to 0$, i.e. $d_R(x^{-1}, a^{-1}) \to 0$. So since inversion is right-continuous and $(x^{-1})^{-1} = x$, etc, we have $d_R(x, a) \to 0$. □

Expansion of the last argument yields the following.

Theorem 3.3. *The following are equivalent:*
(i) *inversion is right-to-right continuous,*
(ii) *left-open sets are right-open,*
(iii) *for each* g *the conjugacy* γ_g *is right-continuous at* e*, i.e. for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$gB(\delta)g^{-1} \subset B(\varepsilon),$$

(iv) *left-shifts are right-continuous.*
Proof. We show that (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv).
Assume (i). For any a and any $\varepsilon > 0$, by continuity of inversion at a, there is $\delta > 0$ such that, for x with $d_R(x, a) < \delta$, we have $d_R(x^{-1}, a^{-1}) < \varepsilon$, i.e. $d_L(x, a) < \varepsilon$. Thus

$$B(\delta)a = B_R(a, \delta) \subset B_L(a, \varepsilon) = aB(\varepsilon), \qquad \text{(incl)}$$

i.e. left-open sets are right-open, giving (ii). For the converse, we just reverse the last argument. Let $\varepsilon > 0$. As $a \in B_L(a, \varepsilon)$ and $B_L(a, \varepsilon)$ is left open, it is right open and so there is $\delta > 0$ such that

$$B_R(a, \delta) \subset B_L(a, \varepsilon).$$

Thus for x with $d_R(x, a) < \delta$, we have $d_L(x, a) < \varepsilon$, i.e. $d_R(x^{-1}, a^{-1}) < \varepsilon$, i.e. inversion is right-to-right continuous, giving (i).
To show that (ii)$\Longleftrightarrow$(iii) note that the inclusion (incl) is equivalent to

$$a^{-1}B(\delta)a \subset B(\varepsilon),$$

i.e. to

$$\gamma_a^{-1}[B(\delta)] \subset B(\varepsilon).$$

that is, to the assertion that $\gamma_a(x)$ is continuous at $x = e$ (and so continuous, by Lemma 3.1). The property (iv) is equivalent to (iii) since the right shift is right-continuous and $\gamma_a(x)a = \lambda_a(x)$ is equivalent to $\gamma_a(x) = \lambda_a(x)a^{-1}$. □

We may now deduce the following characterization of metric topological groups.

Theorem 3.4 (Equivalence Theorem). *A normed group is a topological group under either the right (resp. left) norm topology iff each conjugacy*

$$\gamma_g(x) := gxg^{-1}$$

is right- (resp. left-) continuous at $x = e$ (and so everywhere), i.e. for $z_n \to_R e$ and any g

$$gz_ng^{-1} \to_R e. \tag{adm}$$

Equivalently, it is a topological group iff left/right shifts are continuous for the right/left norm topology, or iff the two norm topologies are themselves equivalent.

Proof. Only one direction needs proving. We work with the d_R topology, the right topology. By Theorem 3.3 we need only show that multiplication is jointly right-continuous. First we note that multiplication is right-continuous iff

$$d_R(xy, ab) = ||xyb^{-1}a^{-1}||, \text{ as } (x, y) \to_R (a, b).$$

Here, we may write $Y = yb^{-1}$ so that $Y \to_R e$ iff $y \to_R b$, and we obtain the equivalent condition:

$$d_R(xYb, ab) = d_R(xY, a) = ||xYa^{-1}||, \text{ as } (x, Y) \to_R (a, e).$$

By Theorem 3.3, as inversion is right-to-right continuous, Lemma 3.2 justifies re-writing the second convergence condition with $X = a^{-1}x$ and $X \to_R e$, yielding the equivalent condition

$$d_R(aXYb, ab) = d_R(aXY, a) = ||aXYa^{-1}||, \text{ as } (X, Y) \to_R (e, e).$$

But, by Lemma 3.1, this is equivalent to continuity of conjugacy. □

Corollary 3.5. *For X a topological group under its norm, the left shifts $\lambda_a(x) := ax$ are bounded and uniformly continuous in norm.*

Proof. We have $||\lambda_a|| = ||a||$ as

$$\sup_x d_R(x, ax) = d_R(e, a) = ||a||.$$

We also have

$$d_R(ax, ay) = d_R(axy^{-1}a^{-1}, e) = ||\gamma_a(xy^{-1})||.$$

Hence, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for $||z|| < \delta$

$$||\gamma_a(z)|| \leq \varepsilon.$$

Thus provided $d_R(x, y) = ||xy^{-1}|| < \gamma$ we have, $d_R(ax, ay) < \varepsilon$. □

Remarks.

1 **(Klee property).** If the group has an abelian norm (in particular if the group is abelian), then the norm has the Klee property (see [Klee] for the original metric formulation, or Th. 2.18), and then it is a *topological group* under the norm-topology. Indeed the Klee property is that

$$||xyb^{-1}a^{-1}|| \leq ||xa^{-1}|| + ||yb^{-1}||,$$

and so if $x \to_R a$ and $y \to_R b$, then $xy \to_R ab$. This may also be deduced from the observation that γ_g is continuous, since here

$$||gxg^{-1}|| = ||gxeg^{-1}|| \leq ||gg^{-1}|| + ||xe|| = ||x||.$$

Compare [vM] Section 3.3, especially Example 3.3.6 of a topological group of real matrices which fails to have an abelian norm.

2. For T a normed group with right-invariant metric d_R one is led to study the associated supremum metric on the group of bounded homeomorphisms h from T to T (i.e. having $\sup_T d(h(t), t) < \infty$) with composition $\circ$ as group operation:

$$d_{\mathcal{A}}(h, h') = \sup_T d(h(t), h'(t)).$$

This is a right-invariant metric which generates the norm

$$||h||_{\mathcal{A}} := d_{\mathcal{A}}(h, e_{\mathcal{A}}) = \sup_T d(h(t), t).$$

It is of interest from the perspective of topological flows, in view of the following observation.

Lemma 3.6. *Under $d_{\mathcal{A}}$ on $\mathcal{A} = Auth(T)$ and d^T on T, the evaluation map $(h,t) \to h(t)$ from $\mathcal{A}{\times}T$ to T is continuous.*

Proof. Fix h_0 and t_0. The result follows from continuity of h_0 at t_0 via

$$\begin{aligned} d^T(h_0(t_0), h(t)) &\leq d^T(h_0(t_0), h_0(t)) + d^T(h_0(t), h(t)) \\ &\leq d^T(h_0(t_0), h_0(t)) + d_{\mathcal{A}}(h, h_0). \qquad \square \end{aligned}$$

3. Since the conjugate metric of a right-invariant metric need not be continuous, one is led to consider the *symmetrization refinement of a metric* d, given by

$$d^S(g,h) = \max\{d(g,h), d(g^{-1}, h^{-1})\}. \qquad \text{(sym)}$$

This metric need not be translation invariant on either side (cf. [vM] Example 1.4.8); however, it is inversion-invariant:

$$d^S(g,h) = d^S(g^{-1}, h^{-1}),$$

so one expects to induce topological group structure with it, as we do in Th. 3.11 below. When $d = d_R^X$ is right-invariant and so induces the group-norm $||x|| := d(x,e)$ and $d(x^{-1}, y^{-1}) = d_L^X(x,y)$, we may use (sym) to define

$$||x||_S := d_S^X(x,e).$$

Then

$$||x||_S = \max\{d_R^X(x,e), d_R^X(x^{-1},e)\} = ||x||,$$

which is a group-norm, even though d_S^X need not be either left- or right-invariant. This motivates the following result, which follows from the Equivalence Theorem (Th. 3.4) and Example A4 (Topological permutations).

Theorem 3.7 (Ambidextrous Refinement). *For X a normed group with norm $||.||$, put*

$$d_S^X(x,y) := \max\{||xy^{-1}||, ||x^{-1}y||\} = \max\{d_R^X(x,y), d_L^X(x,y)\}.$$

Then X is a topological group under the right (or left) norm topology iff X is a topological group under the symmetrization refinement metric d_S^X.

Proof. Suppose that under the right-norm topology X is a topological group. Then d_L^X is d_R^X-continuous, by Th. 3.4, and hence d_S^X is also d_R^X-continuous. Thus if $x_n \to x$ under d_R^X, then also, by continuity, $x_n \to x$

under d_S^X. Now if $x_n \to x$ under d_S^X, then also $x_n \to x$ under d_R^X, as $d_R^X \leq d_S^X$. Thus d_S^X generates the topology and so X is a topological group under d_S^X.

Conversely, suppose that X is a topological group under d_S^X. As X is a topological group, its topology is generated by the neighbourhoods of the identity. But as already noted,

$$d_S^X(x, e) := ||x||,$$

so the d_S^X-neighbourhoods of the identity are also generated by the norm; in particular left-open sets $aB(\varepsilon)$ are d_S^X-open and so right-open. Hence by Th. 3.4 (or Th. 3.3) X is a topological group under either norm topology. □

Thus, according to the Ambidextrous Refinement Theorem, a symmetrization that creates a topological group structure from a norm structure is in fact *redundant*. We are about to see such an example in the next theorem.

Given a metric space (X, d), we let $\mathcal{H}_{unif}(X)$ denote the subgroup of uniformly continuous homeomorphisms (relative to d), i.e. homeomorphisms α satisfying the condition that, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$d(\alpha(x), \alpha(x')) < \varepsilon, \text{ for } d(x, x') < \delta. \tag{3}$$

Lemma 3.8 (Compare [dGMc] Cor. 2.13).

(i) *For fixed* $\xi \in \mathcal{H}(X)$*, the mapping* $\rho_\xi : \alpha \to \alpha\xi$ *is continuous.*

(ii) *For fixed* $\alpha \in \mathcal{H}_{unif}(X)$*, the mapping* $\lambda_\alpha : \beta \to \alpha\beta$ *is in* $\mathcal{H}_{unif}(X)$ *– i.e. is uniformly continuous.*

(iii) *The mapping* $(\alpha, \beta) \to \alpha\beta$ *is continuous from* $\mathcal{H}_{unif}(X) \times \mathcal{H}_{unif}(X)$ *to* $\mathcal{H}(X)$ *under the supremum norm.*

Proof. (i) We have

$$\hat{d}(\alpha\xi, \beta\xi) = \sup d(\alpha(\xi(t)), \beta(\xi(t))) = \sup d(\alpha(s), \beta(s)) = \hat{d}(\alpha, \beta).$$

(ii) For $\alpha \in \mathcal{H}_{unif}(X)$ and given $\varepsilon > 0$, choose $\delta > 0$, so that (3) holds. Then, for β, γ with $\hat{d}(\beta, \gamma) < \delta$, we have $d(\beta(t), \gamma(t)) < \delta$ for each t, and hence

$$\hat{d}(\alpha\beta, \alpha\gamma) = \sup d(\alpha(\beta(t)), \alpha(\gamma(t))) \leq \varepsilon.$$

(iii) Again, for $\alpha \in \mathcal{H}_{unif}(X)$ and given $\varepsilon > 0$, choose $\delta > 0$, so that (3) holds. Thus, for β, η with $\hat{d}(\beta, \eta) < \delta$, we have $d(\beta(t), \eta(t)) < \delta$ for each t.

Hence for ξ with $\hat{d}(\alpha, \xi) < \varepsilon$ we obtain

$$\begin{aligned} d(\alpha(\beta(t)), \xi(\eta(t))) &\leq d(\alpha(\beta(t)), \alpha(\eta(t))) + d(\alpha(\eta(t)), \xi(\eta(t))) \\ &\leq \varepsilon + \hat{d}(\alpha, \xi) \leq \varepsilon + \varepsilon. \end{aligned}$$

Consequently, we have

$$\hat{d}(\alpha\beta, \xi\eta) = \sup d(\alpha(\beta(t)), \xi(\eta(t))) \leq 2\varepsilon.$$

□

Comment. See also [AdC] for a discussion of the connection between choice of metric and uniform continuity. The following result is of interest

Proposition 3.9 (**deGroot-McDowell Lemma,** [dGMc], Lemma 2.2). *Given Φ, a countable family of auto-homeomorphism of X closed under composition (i.e. a semigroup in $Auth(X)$), the metric on X may be replaced by a topologically equivalent one such that each $\alpha \in \Phi$ is uniformly continuous.*

Definition. Say that a homeomorphism h is **bi-uniformly continuous** if both h and h^{-1} are uniformly continuous. Write

$$\mathcal{H}_u = \{h \in \mathcal{H}_{unif} : h^{-1} \in \mathcal{H}_{unif}\}.$$

Proposition 3.10 (Group of left shifts). *For a normed topological group X with right-invariant metric d^X, the group $Tr_L(X)$ of left shifts is (under composition) a subgroup of $\mathcal{H}_u(X)$ that is isometric to X, hence has the same norm.*

Proof. As X is a topological group, we have $Tr_L(X) \subseteq \mathcal{H}_u(X)$ by Cor. 3.5; $Tr_L(X)$ is a subgroup and $\lambda : X \to Tr_L(X)$ is an isomorphism, because

$$\lambda_x \circ \lambda_y(z) = \lambda_x(\lambda_y(z)) = x(\lambda_y(z) = xyz = \lambda_{xy}(z).$$

Moreover, λ is an isometry, as d^X is right-invariant; indeed, we have

$$d^T(\lambda_x, \lambda_y) = \sup_z d^X(xz, yz) = d^X(x, y). \qquad \square$$

We now offer a generalization which motivates the duality considerations of Section 11.

Theorem 3.11 *The family $\mathcal{H}_u(T)$ of bi-uniformly continuous bounded homeomorphisms of a complete metric space T is a complete topological group under the symmetrized supremum metric. It is a topological group under the supremum metric.*

Proof. Suppose that T is metrized by a complete metric d. The bounded homeomorphisms of T, i.e. those homeomorphisms h for which $\sup d(h(t),t) < \infty$, form a group $\mathcal{H} = \mathcal{H}(T)$ under composition. The subgroup

$$\mathcal{H}_u = \{h \in \mathcal{H} : h \text{ and } h^{-1} \text{ is uniformly continuous}\}$$

is complete under the supremum metric $\hat{d}(h,h') = \sup d(h(t),h'(t))$, by the standard 3ε argument. It is a topological semigroup since the composition map $(h,h') \to h \circ h'$ is continuous. Indeed, as in Proposition 2.13, in view of the inequality

$$\begin{aligned} d(h \circ h'(t), H \circ H'(t)) &\leq d(h \circ h'(t), H \circ h'(t)) + d(H \circ h'(t), H \circ H'(t)) \\ &\leq \hat{d}(h,H) + d(H \circ h'(t), H \circ H'(t)), \end{aligned}$$

for each $\varepsilon > 0$ there is $\delta = \delta(H,\varepsilon) < \varepsilon$ such that for $\hat{d}(h',H') < \delta$ and $\hat{d}(h,H) < \varepsilon$,

$$\hat{d}(h \circ h', H \circ H') \leq 2\varepsilon.$$

Likewise, mutatis mutandis, for their inverses; to be explicit, writing $g = h'^{-1}, G = H'^{-1}$ etc, for each $\varepsilon > 0$ there is $\delta' = \delta(G,\varepsilon) = \delta(H'^{-1},\varepsilon)$ such that for $\hat{d}(g',G') < \delta'$ and $\hat{d}(g,G) < \varepsilon$,

$$\hat{d}(g \circ g', G \circ G') \leq 2\varepsilon.$$

Set $\eta = \min\{\delta,\delta'\} < \varepsilon$. So for $\hat{d}(h',H')+\hat{d}(g,G) < \eta$ and $\hat{d}(h,H)+\hat{d}(g',G') < \eta$, we have $\hat{d}(h',H') < \delta, \hat{d}(h,H) < \delta < \varepsilon$, and $\hat{d}(g',G') < \delta$ and $\hat{d}(g,G) < \varepsilon$, so

$$\hat{d}(h \circ h', H \circ H') + \hat{d}(g \circ g', G \circ G') \leq 4\varepsilon.$$

So composition is continuous under the symmetrized metric

$$d_S(g,h) = \hat{d}(g,h,) + \hat{d}(g^{-1},h^{-1}).$$

But as this metric is inversion-invariant, i.e.

$$d_S(g,h) = d_S(g^{-1},h^{-1}),$$

this gives continuity of inversion. This means that $\mathcal{H}_u$ is a complete metric topological group under the symmetrized supremum metric.

The final assertion follows from the Ambidextrous Refinement Theorem, Th. 3.7. □

We now deduce a corollary with important consequences for the Uniform Convergence Theorem of topological regular variation (for which see [BOst13]). We need the following definitions and a result due to Effros (for a proof and related literature see [vM]).

Definition. A group $G \subset \mathcal{H}(X)$ acts *transitively* on a space X if for each x, y in X there is g in X such that $g(x) = y$.

The group acts *micro-transitively* on X if for U open in G and $x \in X$ the set $\{h(x) : h \in U\}$ is a neighbourhood of x.

Theorem 3.12 (Effros' Open Mapping Principle, [Eff]**).** *Let G be a Polish group acting transitively on a separable metrizable space X. The following are equivalent.*

(i) *G acts micro-transitively on X,*

(ii) *X is Polish,*

(iii) *X is of second category.*

Remark. van Mill [vM1] gives the stronger result that for G an analytic group (iii) implies (i). See Section 10 for definitions, references and the related classical Open Mapping Theorem (which follows from Th. 3.12: see [vM1]).

Theorem 3.13 (Crimping Theorem). *Let T be a Polish space with a complete metric d. Suppose that a closed subgroup $\mathcal{G}$ of $\mathcal{H}_u(T)$ acts on T transitively, i.e. for any s, t in T there is h in $\mathcal{G}$ such that $h(t) = s$. Then for each $\varepsilon > 0$ and $t \in T$, there is $\delta > 0$ such that for any s with $d^T(s, t) < \delta$, there exists h in $\mathcal{G}$ with $||h||_{\mathcal{H}} < \varepsilon$ such that $h(t) = s$.*

Consequently:

(i) *if y, z are in $B_\delta(t)$, then there exists h in $\mathcal{G}$ with $||h||_{\mathcal{H}} < 2\varepsilon$ such that $h(y) = z$,*

(ii) *Moreover, for each $z_n \to t$ there are h_n in $\mathcal{G}$ converging to the identity such that $h_n(t) = z_n$.*

Proof. As $\mathcal{G}$ and T are Polish, by Effros' Theorem, $\mathcal{G}$ acts micro-transitively on T; that is, for each t in T and each $\varepsilon > 0$ the set $\{h(t) :$

$h \in \mathcal{H}_u(T)$ and $||h||_{\mathcal{H}} < \varepsilon\}$ is a neighbourhood of t, i.e. for some $\delta = \delta(\varepsilon) > 0$, $B_\delta(t) \subset \{h(t) : ||h|| < \varepsilon\}$. Hence if $d^T(s,t) < \delta$ we have for some h in $\mathcal{G}$ with $||h||_{\mathcal{H}} < \varepsilon$ that $h(t) = s$.

If $y, z \in B_\delta(t)$, there is h, k in $\mathcal{G}$ with $||h|| < \varepsilon$ and $||k|| < \varepsilon$ such that $h(t) = y$ and $k(t) = z$. Thus $kh^{-1}$is in $\mathcal{G}$, $kh^{-1}(y) = z$ and

$$||kh^{-1}|| \leq ||k|| + ||h^{-1}|| = ||k|| + ||h|| \leq 2\varepsilon,$$

as the norm is inversion symmetric.

For the final conclusion, taking for ε successively the values $\varepsilon_n = 1/n$, we define $\delta_n = \delta(\varepsilon_n)$. Let $z_n \to t$. By passing to a subsequence we may assume that $d^T(z_n, t) < \delta_n$. Now there exists h_n in $\mathcal{G}$ such that $||h_n|| < 2\varepsilon_n$ and $h_n(t) = z_n$. As $h_n \to id$, we have constructed the 'crimping sequence' of homeomorphisms asserted. □

Remarks. By Proposition 3.10, this result applies also to the closed subgroup of left translations on T.

The Crimping Theorem implies the following classical result.

Theorem 3.14 (Ungar's Theorem, [Ung], [vM] Th. 2.4.1, p.78**).** *Let $\mathcal{G}$ be a subgroup of $\mathcal{H}(X)$. Let X be a compact metric space on which $\mathcal{G}$ acts transitively. For each $\varepsilon > 0$, there is $\delta > 0$ such that for x, y with $d(x, y) < \delta$ there is $h \in \mathcal{G}$ such that $h(x) = y$ and $||h|| < \varepsilon$.*

Proof. X is a Polish space, and $\mathcal{H}(X) = \mathcal{H}_u(X)$, as X is compact. Let $\varepsilon > 0$. By the Crimping Theorem for each $x \in X$ there is $\delta = \delta(x, \varepsilon) > 0$ such that for $y, z \in B_\delta(x)$ there is $h \in \mathcal{G}$ with $h(y) = z$ and $||h|| < \varepsilon$. Thus $\{B_{\delta(x,\varepsilon)}(x) : x \in X\}$ covers X. By compactness, for some finite set $F = \{x_1, ..., x_N\}$, the space X is covered by $\{B_{\delta(x,\varepsilon)}(x) : x \in F\}$. The conclusion of the theorem follows on taking $\delta = \min\{\delta(x, e) : x \in F\}$. □

Definition. Let G be a normed group normed by $||.||$. For $g \in G$, recall that the *g-conjugate norm* is defined by

$$||x||_g := ||\gamma_g(x)|| = ||gxg^{-1}||.$$

If left or right shifts are continuous in G (in particular if G is a semitopological group), then $||z_n|| \to 0$ iff $||z_n||_g \to 0$.

Example. For X a normed group with metric d^X take $G = \mathcal{H}_u(X)$ normed by $||h|| := ||h||_{\mathcal{H}}$. Then

$$||h||_g = \sup_x d^X(ghg^{-1}(x), x) = \sup_z d^X(g(h(z)), z).$$

We now give an explicit construction of a equivalent bi-invariant metric on G when one exists (compare [HR] Section 8.6), namely

$$||x||_\infty := \sup\{||x||_g : g \in G\}.$$

We recall from Section 2 that the group norm satisfies the **norm admissibility condition** if, for $z_n \to e$ and g_n arbitrary,

$$||g_n z_n g_n^{-1}||_G \to 0. \tag{n-adm}$$

Evidently, this is a sharper version of (adm).

Theorem 3.15. *Suppose that* $||.||_\infty$ *is finite on* G. *Then* $||x||_\infty$ *is an equivalent norm iff the* $||.||_G$ *meets the norm admissibility condition* (n-adm).

In particular, for the bounded norm $|x| := \min\{||x||, 1\}$ *the corresponding norm* $|x|_\infty := \sup\{|x|_g : g \in G\}$ *is an equivalent abelian norm iff the admissibility condition* (n-adm) *holds.*

Proof. First assume (n-adm) holds. As $||x|| = ||x||_e \leq ||x||_\infty$ we need to show that if $z_n \to e$ then $||z_n||_\infty \to 0$. Suppose otherwise; then for some $\varepsilon > 0$, w.l.o.g. $||z_n||_\infty \geq \varepsilon$, and so there is for each n an element g_n such that

$$||g_n z_n g_n^{-1}|| \geq \varepsilon/2.$$

But this contradicts the admissibility condition (n-adm)

As to the abelian property of the norm, we have

$$||yzy^{-1}||_\infty = \sup\{||gyzy^{-1}g^{-1}|| : g \in G\} = \sup\{||gyz(gy)^{-1}|| : g \in G\} = ||z||_\infty,$$

and so taking $z = xy$ we have $||yx|| = ||xy||$.

For the converse, assume $||x||_\infty$ is an equivalent norm. For g_n arbitrary, suppose that $||z_n|| \to 0$ and $\varepsilon > 0$. For some N and all $n \geq N$ we thus have $||z_n||_\infty < \varepsilon$. Hence for $n \geq N$,

$$||g_n z_n g_n^{-1}|| \leq ||z_n||_\infty < \varepsilon,$$

verifying the condition (n-adm). □

Theorem 3.16. *Let G be a normed topological group which is compact and normed by $||.||_G$. Then*

$$||x||_\infty := \sup\{||x||_g : g \in G\}$$

is an abelian (hence bi-invariant) norm topologically equivalent to $||x||$.

Proof. We write $||.||$ for $||.||_G$. Suppose, for some x, that $\{||x||_g : g \in G\}$ is unbounded. We may select g_n with

$$||g_n x g_n^{-1}|| \to \infty.$$

Passing to a convergent subsequence we obtain a contradiction. Thus $||x||_\infty$ is finite and hence a norm. We verify the admissibility condition. Suppose to the contrary that for some $z_n \to e$, arbitrary g_n, and some $\varepsilon > 0$ we have

$$||g_n z_n g_n^{-1}|| > \varepsilon.$$

Using compactness, we may pass to a convergent subsequence, $g_m \to g$ (in the norm $||.||_G$). Since multiplication is jointly continuous in G we obtain the contradiction that $||geg^{-1}|| = ||e|| = 0 > \varepsilon$. □

Remarks. 1. Suppose as usual that d_R is a right-invariant metric on a group G. The right-shift $\rho_a(x) = xg$ is uniformly continuous, as

$$d_R(xg, yg) = d_R(x, y).$$

However, it is not necessarily bounded, as

$$||\rho_g|| = \sup_x d_R(xg, x) = \sup_x ||g||_x = ||g||_\infty.$$

But on the subgroup $\{\rho_g : ||g||_\infty < \infty\}$, the norm $||\rho_g||$ is bi-invariant, since $||g||_\infty$ is bi-invariant.

2. The condition we used in Theorem 3.15 to check *admissibility* of the supremum norm may be reformulated, without reference to the group norm, topologically thus:

$$g_n z_n g_n^{-1} \to e \text{ for } z_n \to e,$$

with g_n arbitrary. In a first-countable topological group this condition is equivalent to the *existence* of a bi-invariant metric (see Proposition 2.15; cf. Theorem 3.3.4 in [vM] p. 101). We will see a related condition in Theorem 3.21 below.

3. Note that the set of 2×2 real matrices under matrix multiplication and with the subspace topology of $\mathbb{R}^4$ forms a topological group with no equivalent bi-invariant metric; for details see e.g. [vM] Example 3.3.6 (p.103), where matrices a_n, g_n are exhibited with $z_n := a_n g_n \to e$ and $g_n a_n \nrightarrow e$, so that $g_n(a_n g_n)g_n^{-1} \nrightarrow e$.

We now apply the last theorem and earlier results to an example of our greatest interest.

Example. Let X be a normed group with right-invariant metric d^X. Give the group $G = \mathcal{H}(X)$ the usual group-norm

$$||f||_{\mathcal{H}} := \sup_x d^X(f(x), x).$$

Finally, for $f, g \in G$ recall that the g-conjugate norm and the conjugacy refinement norm are

$$||f||_g := ||gfg^{-1}||_{\mathcal{H}}, \text{ and } ||f||_\infty := \sup\{||f||_g : g \in G\}.$$

Thus

$$||f||_\infty = \sup_x \sup_g d_g^X(f(x), x).$$

Theorem 3.17 (**Abelian normability of** $\mathcal{H}(X)$ – cf. [BePe] Ch. IV Th 1.1). *Assume that* $||f||_\infty$ *is finite for each* f *in* $\mathcal{H}(X)$ *– for instance if* d^X *is bounded, and in particular if* X *is compact.*

Then:

(i) $\mathcal{H}(X)$ *under the abelian norm* $||f||_\infty$ *is a topological group.*

(ii) *The norm* $||f||_\infty$ *is equivalent to* $||f||_{\mathcal{H}}$ *iff the admissibility condition* (n-adm) *holds, which here reads: for* $||f_n||_{\mathcal{H}} \to 0$ *and any* g_n *in* $\mathcal{H}(X)$,

$$||g_n f_n g_n^{-1}||_{\mathcal{H}} \to 0.$$

Equivalently, for $||z_n||_{\mathcal{H}} \to 0$ *(i.e.* z_n *converging to the identity), any* g_n *in* $\mathcal{H}(X)$, *and any* $y_n \in X$,

$$||g_n(z_n(y_n))g_n(y_n)^{-1}||_X \to 0.$$

(iii) *In particular, if* X *is compact,* $\mathcal{H}(X) = \mathcal{H}_u(X)$ *is under* $||f||_{\mathcal{H}}$ *a topological group.*

Proof. (i) and the first part of (ii) follow from Th. 3.15 (cf. Remarks 1 on the Klee property, after Cor. 3.5); as to (iii), this follows from Th. 3.12 and 3.7. Turning to the second part of (ii), suppose first that

$$||g_n z_n g_n^{-1}||_{\mathcal{H}} \to 0,$$

and let y_n be given. For any $\varepsilon > 0$ there is N such that, for $n \geq N$,

$$\varepsilon > ||g_n z_n g_n^{-1}||_{\mathcal{H}} = \sup_x d(g_n z_n g_n^{-1}(x), x).$$

Taking x here as $x_n = g_n(y_n)$, we obtain

$$\varepsilon > d(g_n z_n(y_n), g_n(y_n)) = d(g_n z_n(y_n) g_n(y_n)^{-1}, e_X), \text{ for } n \geq N.$$

Hence $||g_n(z_n(y_n))g_n(y_n)^{-1}||_X \to 0$, as asserted.

For the converse direction, suppose next that

$$||g_n z_n g_n^{-1}||_{\mathcal{H}} \nrightarrow 0.$$

Then w.l.o.g. there is $\varepsilon > 0$ such that for all n

$$||g_n z_n g_n^{-1}||_{\mathcal{H}} = \sup_x d(g_n z_n g_n^{-1}(x), x) > \varepsilon.$$

Hence, for each n, there exists x_n such that

$$d(g_n z_n g_n^{-1}(x_n), x_n) > \varepsilon.$$

Equivalently, setting $y_n = g_n^{-1}(x_n)$ we obtain

$$d(g_n(z_n(y_n))g_n(y_n)^{-1}, e_X) = d(g_n(z_n(y_n)), g_n(y_n)) > \varepsilon.$$

Thus, for this sequence y_n we have

$$||g_n(z_n(y_n))g_n(y_n)^{-1}||_X \nrightarrow 0. \qquad \square$$

Remark. To see the need for the refinement norm in verifying continuity of composition in $\mathcal{H}(X)$, we work with metrics and recall the permutation metric $d_g^X(x,y) := d^X(g(x),g(y))$. Recall also that the metric defined by the norm $||f||_g$ is the supremum metric $\hat{d}_g$ on $\mathcal{H}(X)$ arising from d_g on X. Indeed

$$\begin{aligned} d_g(h',h) &= ||h'h^{-1}||_g = \sup_z d^X(gh'h^{-1}g^{-1}(z),z) = \sup_x d^X(g(h'(x)),g(h(x))) \\ &= \sup_x d_g^X(h'(x)),h(x)). \end{aligned}$$

Now, as in Proposition 2.13

$$\hat{d}_g(F_1G_1,FG) \leq \hat{d}_g(F_1,F) + \hat{d}_{gF}(G_1,G) \leq \hat{d}_\infty(F_1,F) + \hat{d}_\infty(G_1,G),$$

we may conclude that

$$\hat{d}_\infty(F_1G_1,FG) \leq \hat{d}_\infty(F_1,F) + \hat{d}_\infty(G_1,G).$$

This reconfirms that composition is continuous. When $g = e$, the term $\hat{d}_F$ arises above and places conditions on how 'uniformly' close G_1 needs to be to G (as in Th. 3.11).

For these reasons we find ourselves mostly concerned with $\mathcal{H}_u(X)$.

Below we weaken the Klee property, characterized by the condition $||gxg^{-1}|| \leq ||x||$, by considering instead the existence of a real-valued function $g \to M_g$ such that

$$||gxg^{-1}|| \leq M_g||x||, \text{ for all } x.$$

Remark. Under these circumstances, on writing xy^{-1} for x and with d^X the right-invariant metric defined by the norm, one has

$$d^X(gxg^{-1},gyg^{-1}) = d^X(gx,gy) \leq M_g d^X(x,y),$$

so that the inner-automorphism γ_g is uniformly continuous (and a homeomorphism). Moreover, M_g is related to the *Lipschitz-1 norms* $||g||_1$ and $||\gamma_g||_1$, where

$$||g||_1 := \sup_{x\neq y} \frac{d^X(gx,gy)}{d^X(x,y)}, \text{ and } ||\gamma_g||_1 := \sup_{x\neq y} \frac{d^X(gxg^{-1},gyg^{-1})}{d^X(x,y)},$$

cf. [Ru-FA2] Ch. I, Exercise 22. This motivates the following terminology.

Definition. 1. Say that an automorphism $f : G \to G$ of a normed group has the *Lipschitz property* if there is $M > 0$ such that

$$||f(x)|| \leq M||x||, \text{ for all } x \in G. \quad \text{(mag)}$$

2. Say that a group-norm has the *Lipschitz property* , or that the group is *Lipschitz-normed*, if each *continuous* automorphism has the Lipschitz property under the group-norm.

Definitions. 1. By analogy with the definition in Section 2, call a group G *infinitely divisible* if for each $x \in G$ and $n \in \mathbb{N}$ there is some $\xi \in G$ with $x = \xi^n$. We may write $\xi = x^{1/n}$ (without implying uniqueness).

2. Recall that a group-norm is $\mathbb{N}$-*homogeneous* if it is n-homogeneous for each $n \in \mathbb{N}$, i.e. for each $n \in \mathbb{N}$, $||x^n|| = n||x||$ for each x. Thus if $\xi^n = x$, then $||\xi|| = \frac{1}{n}||x||$ and, as $\xi^m = x^{m/n}$, we have $\frac{m}{n}||x|| = ||x^{m/n}||$, i.e. for rational $q > 0$ we have $q||x|| = ||x^q||$.

Theorem 3.18 below relates the Lischitz property of a norm to local behaviour. One should expect local behaviour to be critical, as asymptotic properties are trivial, since by the triangle inequality

$$\lim_{||x||\to\infty} \frac{||x||_g}{||x||} = 1.$$

As this asserts that $||x||_g$ is slowly varying (see Section 2) and $||x||_g$ is continuous, the Uniform Convergence Theorem (UCT) applies (see [BOst13]; for the case $G = \mathbb{R}$ see [BGT]), and so this limit is uniform on compact subsets of G. Theorem 3.19 identifies circumstances when a group-norm on G has the Lipschitz property and Theorem 3.20 considers the Lipschitz property of the supremum norm in $\mathcal{H}_u(X)$.

On a number of occasions, the study of group-norm behaviour is aided by the presence of the following property. Its definition is motivated by the notion of a 'invariant connected metric' as defined in [Var] Ch. III.4 (see also [Nag]). The property expresses scale-comparability between word-length and distance, in keeping with the key notion of quasi-isometry.

Definition (Word-net). Say that a normed group G has a group-norm $||.||$ with a *vanishingly small word-net* (which may be also *compactly generated,* as appropriate) if, for any $\varepsilon > 0$, there is $\eta > 0$ such that, for all δ with $0 < \delta < \eta$ there is a set (a compact set) of generators Z_δ in $B_\delta(e)$ and a constant M_δ such that, for all x with $||x|| > M_\delta$, there is some word $w(x) = z_1 ... z_{n(x)}$ using generators in Z_δ with $||z_i|| = \delta(1 + \varepsilon_i)$, with $|\varepsilon_i| < \varepsilon$, where

$$d(x, w(x)) < \delta$$

and

$$1 - \varepsilon \leq \frac{n(x)\delta}{||x||} \leq 1 + \varepsilon.$$

Say that the word-net is *global* if $M_\delta = 0$.

Remarks. 1. $\mathbb{R}^d$ has a vanishingly small compactly generated global word-net and hence so does the sequence space l_2.

2. An infinitely divisible group X with an $\mathbb{N}$-homogenous norm has a vanishingly small global word-net. Indeed, given $\delta > 0$ and $x \in X$ take $n(x) = ||x||/\delta$, then if $\xi^n = x$ we have $||x|| = n||\xi||$, and so $||\xi|| = \delta$ and $n(x)\delta/||x|| = 1$.

Theorem 3.18. *Let G be a locally compact topological group with a norm having a compactly generated, vanishingly small global word-net. For f a continuous automorphism (e.g. $f(x) = gxg^{-1}$), suppose*

$$\beta := \lim \sup_{||x|| \to 0+} \frac{||f(x)||}{||x||} < \infty.$$

Then

$$M = \sup_x \frac{||f(x)||}{||x||} < \infty.$$

We defer the proof to Section 4 as it relies on the development there of the theory of subadditive functions.

Theorem 3.19. *If G is an infinitely divisible group with an $\mathbb{N}$-homogeneous norm, then its norm has the Lipschitz property, i.e. if $f : G \to G$ is a continuous automorphism, then for some $M > 0$*

$$||f(x)|| \leq M||x||.$$

Proof. Suppose that $\delta > 0$. Fix $x \neq e$. Define

$$p_\delta(x) := \sup\{q \in \mathbb{Q}^+ : ||x^q|| < \delta\} = \delta/||x||.$$

Let f be a continuous automorphism. As $f(e) = e$, there is $\delta > 0$ such that, for $||z|| \leq \delta$,

$$||f(z)|| < 1.$$

If $||x^q|| < \delta$, then

$$||f(x^q)|| < 1.$$

Thus for each $q < p_\delta(x)$ we have

$$||f(x)|| < 1/q.$$

Taking limits, we obtain, with $M = 1/\delta$,

$$||f(x)|| \leq 1/p_\delta(x) = M||x||. \qquad \square$$

Definitions. 1. Let G be a Lipschitz-normed topological group. We may now take $f(x) = \gamma_g(x) := gxg^{-1}$, since this homomorphism is continuous. The *Lipschitz norm* is defined by

$$M_g := \sup_{x \neq e} ||\gamma_g(x)||/||x|| = \sup_{x \neq e} ||x||_g/||x||.$$

(As noted before the introduction of the Lipschitz property this is the Lipschitz-1 norm.) Thus

$$||x||_g := ||gxg^{-1}|| \leq M_g||x||.$$

2. For X a normed group with right-invariant metric d^X and $g \in \mathcal{H}_u(X)$ denote the following (inverse) modulus of continuity by

$$\delta(g) = \delta_1(g) := \sup\{\delta > 0 : d^X(g(z), g(z')) \leq 1, \text{ for all } d^X(z, z') \leq \delta\}.$$

Theorem 3.20 (Lipschitz property in $\mathcal{H}_u$). *Let X be a normed group with a right-invariant metric d^X having a vanishingly small global word-net. Then, for $g, h \in \mathcal{H}_u(X)$*

$$||h||_g \leq \frac{2}{\delta(g)}||h||,$$

and so $\mathcal{H}_u(X)$ has the Lipschitz property.

Proof. We have for $d(z, z') < \delta(g)$ that

$$d(g(z), g(z')) < 1$$

For given x put $y = h(x)x^{-1}$. In the definition of the word-net take $\varepsilon < 1$. Now suppose that $w(y) = w_1...w_{n(y)}$ with $||z_i|| = \frac{1}{2}\delta(1 + \varepsilon_i)$ and $|\varepsilon_i| < \varepsilon$, where $n(y) = n(y, \delta)$ satisfies

$$1 - \varepsilon \leq \frac{n(y)\delta(g)}{||y||} \leq 1 + \varepsilon.$$

Put $y_0 = e$,

$$y_{i+1} = w_i y_i$$

for $0 < i < n(y)$, and $y_{n(x)+1} = y$; the latter is within δ of y. Now

$$d(y_i, y_{i+1}) = d(e, w_i) = ||w_i|| < \delta.$$

Finally put $z_i = y_i x$, so that $z_0 = x$ and $z_{n(y)+1} = h(x)$. As

$$d(z_i, z_{i+1}) = d(y_i x, y_{i+1} x) = d(y_i, y_{i+1}) < \delta,$$

we have

$$d(g(z_i), g(z_{i+1})) \leq 1.$$

Hence

$$\begin{aligned} d(g(x), g(h(x))) &\leq n(y) + 1 < 2||y||/\delta(g) \\ &= \frac{2}{\delta(g)} d(h(x), x). \end{aligned}$$

Thus

$$||h||_g = \sup_x d(g(x), g(h(x))) \leq \frac{2}{\delta(g)} \sup_x d(h(x), x) = \frac{2}{\delta(g)} ||h||. \qquad \square$$

Lemma 3.21 (Bi-Lipschitz property). *$M_e = 1$ and $M_g \geq 1$, for each g; moreover $M_{gh} \leq M_g M_h$ and for any g and all x in G,*

$$\frac{1}{M_{g^{-1}}} ||x|| \leq ||x||_g \leq M_g ||x||.$$

Thus in particular $||x||_g$ is an equivalent norm.

Proof. Evidently $M_e = 1$. For $g \neq e$, as $\gamma_g(g) = g$, we see that

$$||g|| = ||g||_g \leq M_g ||g||,$$

and so $M_g \geq 1$, as $||g|| > 0$. Now for any g and all x,

$$||g^{-1} x g|| \leq M_{g^{-1}} ||x||.$$

So with gxg^{-1} in place of x, we obtain

$$||x|| \leq M_{g^{-1}}||gxg^{-1}||, \text{ or } \frac{1}{M_{g^{-1}}}||x|| \leq ||x||_g. \qquad \square$$

Definition. In a Lipschitz-normed group, put $|\gamma_g| := \log M_g$ and define the *symmetrization pseudo-norm* $||\gamma_g|| := \max\{|\gamma_g|, |\gamma_g^{-1}|\}$. Furthermore, put

$$Z_\gamma(G) := \{g \in G : ||\gamma_g|| = 0\}.$$

Since $M_g \geq 1$ and $M_{gh} \leq M_g M_h$ the symmetrization in general yields, as we now show, a pseudo-norm (unless $Z_\gamma = \{e\}$) on the inner-automorphism subgroup

$$\mathcal{I}nn := \{\gamma_g : g \in G\} \subset Auth(G).$$

Evidently, one may adjust this deficiency, e.g. by considering $\max\{||\gamma_g||, ||g||\}$, as $\gamma_g(g) = g$ (cf. [Ru-FA2] Ch. I Ex. 22).

Theorem 3.22. *Let G be a Lipschitz-normed topological group. The set Z_γ is the subgroup of elements g characterized by*

$$M_g = M_{g^{-1}} = 1,$$

equivalently by the 'norm-central' property:

$$||gx|| = ||xg|| \text{ for all } x \in G,$$

and so $Z_\gamma(G) \subseteq Z(G)$, the centre of G.

Proof. The condition $\max\{|\gamma_g|, |\gamma_g^{-1}|\} = 0$ is equivalent to $M_g = M_{g^{-1}} = 1$. Thus Z_γ is closed under inversion; the inequality $1 \leq M_{gh} \leq M_g M_h = 1$ shows that Z_γ is closed under multiplication. For $g \in Z_\gamma$, as $M_g = 1$, we have $||gxg^{-1}|| \leq ||x||$ for all x, which on substitution of xg for x is equivalent to

$$||gx|| \leq ||xg||.$$

Likewise $M_{g^{-1}} = 1$ yields the reverse inequality:

$$||xg|| \leq ||g^{-1}x^{-1}|| \leq ||x^{-1}g^{-1}|| = ||gx||.$$

Conversely, if $||gx|| = ||xg||$ for all x, then replacing x either by xg^{-1} or $g^{-1}x$ yields both $||gxg^{-1}|| = ||x||$ and $||g^{-1}xg|| = ||x||$ for all x, so that $M_g = M_{g^{-1}} = 1$. □

Corollary 3.23. *$M_g = 1$ for all $g \in G$ iff the group norm is abelian iff $||ab|| \leq ||ba||$ for all $a, b \in G$.*

Proof. $Z_\gamma = G$ (cf. Th. 2.18). □

The condition $M_g \equiv 1$ is not necessary for the existence of an equivalent bi-invariant norm, as we see below. The next result is similar to Th. 3.15 (where the Lipschitz property is absent).

Theorem 3.24. *Let G be a Lipschitz-normed topological group. If $\{M_g : g \in G\}$ is bounded, then $||x||_\infty$ is an equivalent abelian (hence bi-invariant) norm.*

Proof. Let M be a bound for the set $\{M_g : g \in G\}$. Thus we have

$$||x||_\infty \leq M||x||,$$

and so $||x||_\infty$ is again a norm. As we have

$$||x|| = ||x||_e \leq ||x||_\infty \leq M||x||,$$

we see that $||z_n|| \to 0$ iff $||z_n||_\infty \to 0$. □

Theorem 3.25. *Let G be a compact, Lipschitz-normed, topological group. Then $\{M_g : g \in G\}$ is bounded, hence $||x||_\infty$ is an equivalent abelian (hence bi-invariant) norm.*

Proof. The mapping $|\gamma_.| := g \to \log M_g$ is subadditive. For G a compact metric group, $|\gamma_.|$ is Baire since

$$\{g : a < M_g < b\} = proj_1\{(g, x) \in G^2 : ||gxg^{-1}|| > a||x||\} \cap \{g : ||gxg^{-1}|| < b||x||\},$$

and so is analytic, hence by Nikodym's Theorem (see [Jay-Rog] p. 42) has the Baire property. As G is Baire, the subadditive mapping $|\gamma_.|$ is locally bounded (the proof of Prop. 1 in [BOst5] is applicable here; cf. Section 4), and so by the compactness of G, is bounded; hence Theorem 3.18 applies. □

Definition. Let G be a Lipschitz-normed topological group. Put

$$\begin{aligned} \mathcal{M}(g) &: = \{m : ||x||_g \leq m||x|| \text{ for all } x \in G\}, \text{ and then} \\ M_g &: = \inf\{m : m \in \mathcal{M}(g)\}, \\ \mu(g) &: = \{m > 0 : m||x|| \leq ||x||_g \text{ for all } x \in G\}, \text{ and then} \\ m_g &: = \sup\{m : m \in \mu(g)\}. \end{aligned}$$

Proposition 3.26. *Let G be a Lipschitz-normed topological group. Then*

$$m_g^{-1} = M_{g^{-1}}.$$

Proof. For $0 < m < m_g$ we have for all x that

$$||x|| \leq \frac{1}{m}||gxg^{-1}||.$$

Setting $x = g^{-1}zg$ we obtain, as in Lemma 3.20,

$$||g^{-1}zg|| \leq \frac{1}{m}||z||,$$

so $M_{g^{-1}} \leq 1/m$. □

Definitions (cf. [Kur-1] Ch. I §18 and [Kur-2] Ch. IV §43; [Hil] I.B.3, [Berg] Ch. 6 – where compact values are assumed – [Bor] Ch. 11; the first unification of these ideas is attributed to Fort [For]).

1. The correspondence $g \to \mathcal{M}(g)$ *has closed graph* means that if $g_n \to g$ and $m_n \to m$ with $m_n \in \mathcal{M}(g_n)$, then $m \in \mathcal{M}(g)$.

2. The correspondence is *upper semicontinuous* means that for any open U with $\mathcal{M}(g) \subset U$ there is a neighbourhood V of g such that $\mathcal{M}(g') \subset U$ for $g' \in V$.

3. The correspondence is *lower semicontinuous* means that for any open U with $\mathcal{M}(g) \cap U \neq \varnothing$ there is a neighbourhood V of g such that $\mathcal{M}(g') \cap U \neq \varnothing$ for $g' \in V$.

Theorem 3.27. *Let G be a Lipschitz-normed topological group. The mapping $g \to \mathcal{M}(g)$ has closed graph and is upper semicontinuous.*

Proof. For the closed graph property: suppose $g_n \to g$ and $m_n \to m$ with $m_n \in \mathcal{M}(g_n)$. Fix $x \in G$. We have

$$||g_n x g_n^{-1}|| \leq m_n ||x||,$$

so passing to the limit

$$||gxg^{-1}|| \leq m||x||.$$

As x was arbitrary, this shows that $m \in \mathcal{M}(g)$.

For the upper semicontinuity property: suppose otherwise. Then for some g and some open U with $\mathcal{M}(g) \subset U$ the property fails. We may thus suppose that $\mathcal{M}(g) \subset (m', \infty) \subset U$ for some $m' < M_g$ and that there are $g_n \to g$ and $m_n < m'$ with $m_n \in \mathcal{M}(g_n)$. Thus, for any n and all x,

$$||g_n x g_n^{-1}|| \leq m_n||x||.$$

As $1 \leq m_n \leq m'$, we may pass to a convergent subsequence $m_n \to m$, so that we have in the limit that

$$||gxg^{-1}|| \leq m||x||.$$

for arbitrary fixed x. Thus $m \in \mathcal{M}(g)$ and yet $m \leq m' < M_g$, a contradiction. □

Definition. Say that the group-norm is **nearly abelian** if for arbitrary $g_n \to e$ and $z_n \to e$

$$\lim_n ||g_n z_n g_n^{-1}||/||z_n|| = 1,$$

or equivalently

$$\lim_n ||g_n z_n||/||g_n z_n|| = 1. \tag{ne}$$

Theorem 3.28. *Let G be a Lipschitz-normed topological group. The following are equivalent:*

(i) *the mapping $g \to M_g$ is continuous,*

(ii) *the mapping $g \to M_g$ is continuous at e,*

(iii) *the norm is nearly abelian, i.e.* (ne) *holds.*

In particular, if in addition G is compact and condition (ne) *holds, then $\{M_g : g \in G\}$ is bounded, and so again Theorem 3.24 applies, confirming that $||x||_\infty$ is an equivalent abelian (hence bi-invariant) norm.*

Proof. Clearly (i)$\Longrightarrow$ (ii). To prove (ii)$\Longrightarrow$ (i), given continuity at e, we prove continuity at h as follows. Write $g = hk$; then $h = gk^{-1}$ and $g \to h$ iff $k \to e$ iff $k^{-1} \to e$. Now by Lemma 3.21,

$$M_h = M_{gk^{-1}} \leq M_g M_{k^{-1}},$$

so since $M_{k^{-1}} \to M_e = 1$, we have

$$M_h \leq \lim_{g \to h} M_g.$$

Since $M_k \to M_e = 1$ and

$$M_g = M_{hk} \leq M_h M_k,$$

we also have

$$\lim_{g \to h} M_g \leq M_h.$$

Next we show that (ii)$\Longrightarrow$(iii). By Lemma 3.21, we have

$$1/M_{g_n^{-1}} \leq ||g_n z_n g_n^{-1}||/||z_n|| \leq M_{g_n}.$$

By assumption, $M_{g_n} \to M_e = 1$ and $M_{g_n^{-1}} \to M_e = 1$, so

$$\lim_n ||g_n^{-1} z_n g_n||/||z_n|| = 1.$$

Finally we show that (iii)$\Longrightarrow$(ii). Suppose that the mapping is not continuous at e. As $M_e = 1$ and $M_g \geq 1$, for some $\varepsilon > 0$ there is $g_n \to e$ such that $M_{g_n} > 1 + \varepsilon$. Hence there are $x_n \neq e$ with

$$(1 + \varepsilon)||x_n|| \leq ||g_n x_n g_n^{-1}||.$$

Suppose that $||x_n||$ is unbounded. We may suppose that $||x_n|| \to \infty$. Hence

$$(1 + \varepsilon) \leq \frac{||g_n x_n g_n^{-1}||}{||x_n||} \leq \frac{||g_n|| + ||x_n|| + ||g_n^{-1}||}{||x_n||},$$

and so as $||g_n|| \to 0$ and $||x_n|| \to \infty$ we have

$$(1 + \varepsilon) \leq \lim_{n \to \infty} \left(\frac{||g_n|| + ||x_n|| + ||g_n||}{||x_n||} \right) = \lim_{n \to \infty} \left(1 + \frac{2}{||x_n||} \cdot ||g_n|| \right) = 1,$$

again a contradiction. We may thus now suppose that $||x_n||$ is bounded and so w.l.o.g. convergent, to $\xi \geq 0$ say. If $\xi > 0$, we again deduce the contradition that

$$(1 + \varepsilon) \leq \lim_{n \to \infty} \frac{||g_n|| + ||x_n|| + ||g_n^{-1}||}{||x_n||} = \frac{0 + \xi + 0}{\xi} = 1.$$

Thus $\xi = 0$, and hence $x_n \to e$. So our assumption of (iii) yields

$$(1+\varepsilon) \leq \lim_{n\to\infty} \frac{||g_n x_n g_n^{-1}||}{||x_n||} = 1,$$

a final contradiction. □

We note the following variant on Theorem 3.28.

Theorem 3.29. *Let G be a Lipschitz-normed topological group. The following are equivalent:*
(i) *the mapping $g \to \mathcal{M}(g)$ is continuous,*
(ii) *the mapping $g \to \mathcal{M}(g)$ is continuous at e,*
(iii) *for arbitrary $g_n \to e$ and $z_n \to e$*

$$\lim_n ||g_n z_n g_n^{-1}||/||z_n|| = 1.$$

Proof. Clearly (i)$\Longrightarrow$ (ii). To prove (ii)$\Longrightarrow$ (iii), suppose the mapping is continuous at e, then by the continuity of the maximization operation (cf. [Bor] Ch.12, [Hil] I.B.III) $g \to M_g$ is continuous at e, and Theorem 3.28 applies.

To prove (iii)$\Longrightarrow$ (ii), assume the condition; it now suffices by Theorem 3.28 to prove lower semicontinuity (lsc) at $g = e$. So suppose that, for some open U, $U \cap \mathcal{M}(e) \neq \varnothing$. Thus $U \cap (1,\infty) \neq \varnothing$. Choose $m' < m''$ with $1 < m$ such that $(m', m'') \subset U \cap \mathcal{M}(e)$. If $\mathcal{M}$ is not lsc at e, then there is $g_n \to e$ such

$$(m', m'') \cap \mathcal{M}(g_n) = \varnothing.$$

Take, e.g., $m := \frac{1}{2}(m' + m'')$. As $m' < m < m''$, there is $x_n \neq e$ such that

$$m||x_n|| < ||g_n x_n g_n^{-1}||.$$

As before, if $||x_n||$ is unbounded we may assume $||x_n|| \to \infty$, and so obtain the contradiction

$$1 < m \leq \lim_{n\to\infty} \frac{||g_n|| + ||x_n|| + ||g_n^{-1}||}{||x_n||} = 1.$$

Now assume $||x_n|| \to \xi \geq 0$. If $\xi > 0$ we have the contradiction

$$m \leq \lim_{n\to\infty} \frac{||g_n|| + ||x_n|| + ||g_n^{-1}||}{||x_n||} = \frac{0+\xi+0}{\xi} = 1.$$

Thus $\xi = 0$. So we obtain $x_n \to 0$, and now deduce that

$$1 < m \leq \lim_{n\to\infty} \frac{||g_n x_n g_n^{-1}||}{||x_n||} = 1,$$

again a contradiction. □

Remark. On the matter of continuity a theorem of Mueller ([Mue] Th. 3, see Th. 4.6 below) asserts that in a locally compact group a subadditive p satisfying

$$\liminf_{x\to e} (\limsup_{y\to x} p(y)) \leq 0$$

is continuous almost everywhere.

4 Subadditivity

Definition. Let X be a normed group. A function $p : X \to \mathbb{R}$ is *subadditive* if

$$p(xy) \leq p(x) + p(y).$$

Thus a norm $||x||$ and so also any g-conjugate norm $||x||_g$ are examples. Recall from [Kucz] p. 140 the definitions of *upper and lower hulls* of a function p :

$$\begin{aligned} M_p(x) &= \lim_{r\to 0+} \sup\{p(z) : z \in B_r(x)\}, \\ m_p(x) &= \lim_{r\to 0+} \inf\{p(z) : z \in B_r(x)\}. \end{aligned}$$

(Usually these are of interest for convex functions p.) These definitions remain valid for a normed group. (Note that e.g. $\inf\{p(z) : z \in B_r(x)\}$ is a decreasing function of r.) We understand the balls here to be defined by a *right-invariant* metric, i.e.

$$B_r(x) := \{y : d(x,y) < r\} \text{ with } d \text{ right-invariant.}$$

These are subadditive functions if the group G is $\mathbb{R}^d$. We reprove some results from Kuczma [Kucz], thus verifying the extent to which they may be generalized to normed groups. Only our first result appears to need the Klee property (bi-invariance of the metric); fortunately this result is not needed in the sequel. The Main Theorem below concerns the behaviour of $p(x)/||x||$.

Lemma 4.1 (cf. [Kucz] L. 1 p. 403). *For a normed group G with the Klee property, m_p and M_p are subadditive.*

Proof. For $a > m_p(x)$ and $b > m_p(y)$ and $r > 0$, let $d(u,x) < r$ and $d(v,y) < r$ satisfy

$$\inf\{p(z) : z \in B_r(x)\} \le p(u) < a, \text{ and } \inf\{p(z) : z \in B_r(y)\} \le p(v) < b.$$

Then, by the Klee property,

$$d(xy, uv) \le d(x,u) + d(y,v) < 2r.$$

Now

$$\inf\{p(z) : z \in B_{2r}(xy)\} \le p(uv) \le p(u) + p(v) < a + b,$$

hence

$$\inf\{p(z) : z \in B_{2r}(xy)\} \le \inf\{p(z) : z \in B_r(x)\} + \inf\{p(z) : z \in B_r(x)\},$$

and the result follows on taking limits as $r \to 0+$. □

Lemma 4.2 (cf. [Kucz] L. 2 p. 403). *For a normed group G, if $p : G \to \mathbb{R}$ is subadditive, then*

$$m_p(x) \le M_p(x) \text{ and } M_p(x) - m_p(x) \le M_p(e).$$

Proof. Only the second assertion needs proof. For $a > m_p(x)$ and $b < M_p(x)$, there exist $u, v \in B_r(x)$ with

$$a > p(u) \ge m_p(x), \text{ and } b < p(v) \le M_p(x).$$

So

$$b - a < p(v) - p(u) \le p(vu^{-1}u) - p(u) \le p(vu^{-1}) + p(u) - p(u) = p(vu^{-1}).$$

Now

$$||vu^{-1}|| \le ||v|| + ||u|| < 2r,$$

so $vu^{-1} \in B_{2r}(e)$, and hence

$$p(vu^{-1}) \le \sup\{p(z) : z \in B_{2r}(e)\}.$$

Hence, with r fixed, taking a, b to their respective limits,

$$M_p(x) - m_p(x) \leq \sup\{p(z) : z \in B_{2r}(e)\}.$$

Taking limits as $r \to 0+$, we obtain the second inequality. □

Lemma 4.3. *For any subadditive function $f : G \to \mathbb{R}$, if f is locally bounded above at a point, then it is locally bounded at every point.*

Proof. We repeat the proof in [Kucz] p. 404 Th. 2, thus verifying that it continues to hold in a normed group.

Supppose that p is locally bounded above at t_0 by K. We first show that f is locally bounded above at e. Suppose otherwise that for some $t_n \to e$ we have $p(t_n) \to \infty$. Now $t_n t_0 \to e t_0 = t_0$ and so

$$p(t_n) = p(t_n t_0 t_0^{-1}) \leq p(t_n t_0) + p(t_0^{-1}) \leq K + p(t_0^{-1}),$$

a contradiction. Hence p is locally bounded above at e, i.e. $M_p(e) < \infty$. But $0 \leq M_p(x) - m_p(x) \leq M_p(e)$, hence both $M_p(x)$ and $m_p(x)$ are finite for every x. That is, p is locally bounded above and below at each x. □

Proposition 4.4 (cf. [Kucz] p 404 Th 3). *For a Baire group G and a Baire function $f : G \to \mathbb{R}$, if f is subadditive, then f is locally bounded.*

Proof. By the Baire assumptions, for some k $H^k = \{x : |f(x)| < k\}$ is non-meagre. Suppose that f is not locally bounded; then it is not locally bounded above at some point u, i.e. there exists $u_n \to u$ with

$$f(u_n) \to +\infty.$$

By the Category Embedding Theorem ([BOst11], and Section 5), for some $k \in \omega$, $t \in H^k$ and an infinite $\mathbb{M}$, we have

$$\{u_n t : n \in \mathbb{M}\} \subseteq H^k.$$

For n in $\mathbb{M}$, we have

$$f(u_n) = f(u_n t t^{-1}) \leq f(u_n t) + f(t^{-1}) \leq k + f(t^{-1}),$$

which contradicts $f(u_n) \to +\infty$. □

We recall that *vanishingly small word-nets* were defined in Section 3.

Theorem 4.5. *Let G be a normed group with a vanishingly small word-net. Let $p : G \to \mathbb{R}_+$ be Baire, subadditive with*

$$\beta := \lim \sup_{||x|| \to 0+} \frac{p(x)}{||x||} < \infty.$$

Then

$$\lim \sup_{||x|| \to \infty} \frac{p(x)}{||x||} \leq \beta < \infty.$$

Proof. Let $\varepsilon > 0$. Let $b = \beta + \varepsilon$. Hence on $B_\delta(e)$ for δ small enough to gurantee the existence of Z_δ and M_δ we have also

$$\frac{p(x)}{||x||} \leq b.$$

By Proposition 4.4, we may assume that p is bounded by some constant K in $B_\delta(e)$. Let $||x|| > M_\delta$.

Choose a word $w(x) = z_0 z_1 ... z_n$ with $||z_i|| = \delta(1 + \varepsilon_i)$ with $|\varepsilon_i| < \varepsilon$, with

$$p(x_i) < b||x_i|| = b\delta(1 + \varepsilon_i)$$

and

$$d(x, w(x)) < \delta,$$

i.e.

$$x = w(x)s$$

for some s with $||s|| < \delta$ and

$$1 - \varepsilon \leq \frac{n(x)\delta}{||x||} \leq 1 + \varepsilon.$$

Now

$$\begin{aligned} p(x) &= p(ws) \leq p(w) + p(r) = \sum p(z_i) + p(s) \\ &\leq \sum b\delta(1 + \varepsilon_i) + p(s) \\ &= nb\delta(1 + \varepsilon) + K. \end{aligned}$$

So

$$\frac{p(x)}{||x||} \leq \frac{n\delta}{||x||} b(1+\varepsilon) + \frac{M}{||x||}.$$

Hence we obtain

$$\frac{p(x)}{||x||} \leq b(1+\varepsilon)^2 + \frac{M}{||x||}.$$

So in the limit

$$\lim \sup_{||x|| \to \infty} \frac{p(x)}{||x||} < \beta,$$

as asserted. □

We note a related result, which requires the following definition. For p subadditive, put (for this section only)

$$p_*(x) = \lim \inf_{y \to x} p(y), \qquad p^*(x) := \lim \sup_{y \to x} p(y).$$

These are subadditive and lower (resp. upper) semicontinuous with $p_*(x) \leq p(x) \leq p^*(x)$.

We now return to the result announced as Theorem 3.18.

Proof of Theorem 3.18. Apply Theorem 4.5 to the subadditive function $p(x) := ||f(x)||$ which is continuous and so Baire. Thus there is X such that, for $||x|| \geq X$,

$$||f(x)|| \leq \beta ||x||.$$

Taking $\varepsilon = 1$ in the definition of a word-net, there is $\delta > 0$ small enough such that $B_\delta(e)$ is pre-compact and there exists a compact set of generators Z_δ such that each x there is a word of length $n(x)$ employing generators of Z_δ with $n(x) \leq 2||x||/\delta$. Hence if $||x|| \leq X$ we have $n(x) \leq 2M/\delta$. Let $N := [2M/\delta]$, the least integer greater than $2M/\delta$. Note that $Z_\delta^N := Z_\delta \cdot \ldots \cdot Z_\delta$ (N times) is compact. The set $B_K(e)$ is covered by the compact swelling $K :=\mathrm{cl}[Z_\delta^N B_\delta(e)]$. Hence, we have

$$\sup_{x \in K} \frac{||f(x)||}{||x||} < \infty,$$

(referring to $\beta_g < \infty$, and continuity of $||x||_g/||x||$ away from e), and so

$$M \leq \max\{\beta, \sup_{x \in K} ||f(x)||/||x||\} < \infty. \qquad \square$$

Theorem 4.6 (Mueller's Theorem – [Mue] Th. 3). *Let p be subadditive on a locally compact group G and suppose*

$$\liminf_{x\to e} p^*(x) \leq 0.$$

Then p is continuous almost everywhere.

5 Theorems of Steinhaus type and Dichotomy

If ψ_n converges to the identity, then, for large n, each ψ_n is almost an isometry. Indeed, as we shall see in Section 11, by Proposition 11.2, we have

$$d(x,y) - 2||\psi_n|| \leq d(\psi_n(x), \psi_n(y)) \leq d(x,y) + 2||\psi_n||.$$

This motivates our next result; we need to recall a definition and the Category Embedding Theorem from [BOst11], whose proof we reproduce here for completeness. In what follows, the words *quasi everywhere (q.e.)*, or *for quasi all points* mean *for all points off a meagre set* (see [Kah]).

Definition (**weak category convergence**). A sequence of homeomorphisms ψ_n satisfies the *weak category convergence* condition (wcc) if:

For any non-empty open set U, there is a non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n\geq k} V \backslash \psi_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ \psi_n(t) \in V.$$

Theorem 5.1 (Category Embedding Theorem). *Let X be a Baire space. Suppose given homeomorphisms $\psi_n : X \to X$ for which the weak category convergence condition* (wcc) *is met. Then, for any non-meagre Baire set T, for locally quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{\psi_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Suppose T is Baire and non-meagre. We may assume that $T = U\backslash M$ with U non-empty and M meagre. Let $V \subseteq U$ satisfy (wcc).

Since the functions h_n are homeomorphisms, the set

$$M' := M \cup \bigcup_n h_n^{-1}(M)$$

is meagre. Put

$$W = \mathbf{h}(V) := \bigcap_{k\in\omega} \bigcup_{n\geq k} V \cap h_n^{-1}(V) \subseteq V \subseteq U.$$

Then $V \cap W$ is co-meagre in V. Indeed

$$V\backslash W = \bigcup_{k\in\omega} \bigcap_{k\geq n} V\backslash h_n^{-1}(V),$$

which by assumption is meagre.

Take $t \in V \cap W\backslash M' \subseteq U\backslash M = T$, as $V \subseteq U$ and $M \subseteq M'$. Thus $t \in T$. Now there exists an infinite set $\mathbb{M}_t$ such that, for $m \in \mathbb{M}_t$, there are points $v_m \in V$ with $t = h_m^{-1}(v_m)$. Since $h_m^{-1}(v_m) = t \notin h_m^{-1}(M)$, we have $v_m \notin M$, and hence $v_m \in T$. Thus $\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T$ for t in a co-meagre set, as asserted. □

Examples. In $\mathbb{R}$ we may consider $\psi_n(t) = t + z_n$ with $z_n \to z_0 := 0$. It is shown in [BOst11] that for this sequence the condition (wcc) is satisfied in both the usual topology and the density topology on $\mathbb{R}$. This remains true in $\mathbb{R}^d$, where the specific instance of the theorem is referred to as the Kestelman-Borwein-Ditor Theorem; see the next section ([Kes], [BoDi]; compare also the Oxtoby-Hoffmann-Jørgensen zero-one law for Baire groups, [HJ] p. 356, [Oxt], cf. [RR-1]). In fact in any metrizable group$X$ with right-invariant metric d^X, for a null sequence tending to the identity $z_n \to z_0 := e_X$, the mapping defined by $\psi_n(x) = z_n x$ converges to the identity (see [BOst13], Corollary to Ford's Theorem); here too (wcc) holds. This follows from the next result, which extends the proof of [BOst11]; cf. Theorem 6.5.

Theorem 5.2 (First Verification Theorem for weak category convergence). *For d a metric on X, if ψ_n converges to the identity under $\hat{d}$, then ψ_n satisfies the weak category convergence condition* (wcc).

Proof. It is more convenient to prove the equivalent statement that ψ_n^{-1} satisfies the category convergence condition.

Put $z_n = \psi_n(z_0)$, so that $z_n \to z_0$. Let k be given.

Suppose that $y \in B_\varepsilon(z_0)$, i.e. $r = d(y, z_0) < \varepsilon$. For some $N > k$, we have $\varepsilon_n = d(\psi_n, id) < \frac{1}{3}(\varepsilon - r)$, for all $n \geq N$. Now

$$\begin{aligned} d(y, z_n) &\leq d(y, z_0) + d(z_0, z_n) \\ &= d(y, z_0) + d(z_0, \psi_n(z_0)) \leq r + \varepsilon_n. \end{aligned}$$

For $y = \psi_n(x)$ and $n \geq N$,

$$\begin{aligned} d(z_0, x) &\leq d(z_0, z_n) + d(z_n, y) + d(y, x) \\ &= d(z_0, z_n) + d(z_n, y) + d(x, \psi_n(x)) \\ &\leq \varepsilon_n + (r + \varepsilon_n) + \varepsilon_n < \varepsilon. \end{aligned}$$

So $x \in B_\varepsilon(z_0)$, giving $y \in \psi_n(B_\varepsilon(z_0))$. Thus

$$y \notin \bigcap_{n \geq N} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)) \supseteq \bigcap_{n \geq k} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)).$$

It now follows that

$$\bigcap_{n \geq k} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)) = \varnothing.$$

□

As a first corollary we have the following topological result; we deduce as corollaries also measure-theoretic variants in Theorems 6.6 and 10.11.

Corollary (Topological Kestelman-Borwein Theorem Theorem). *In a normed group X let $\{z_n\} \to e_T$ be a null sequence. If T is Baire, then for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{z_m t : m \in \mathbb{M}_t\} \subseteq T.$$

Likewise, for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that

$$\{t z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Apply Th. 5.2, taking for d a right-invariant metric, d_R^X say; the maps $\psi_n(t) = z_n t$ satisfy $d_R^X(z_n t, t) = ||z_n||_{\mathcal{H}} \to 0$, so converge to the identity.

Likewise taking for d a left-invariant metric d_L^X say, the maps $\psi_n(t) = tz_n$ satisfy $d_R^X(tz_n, t) = ||z_n||_{\mathcal{H}} \to 0$, so again converge to the identity. □

As a corollary we have the following important result known for topological groups (see [RR-TG], Rogers [Jay-Rog] p. 48, and [Kom1] for the topological vector space setting) and here proved in the metric setting.

Theorem 5.3 (Piccard-Pettis Theorem – Piccard [Pic1], [Pic2], Pettis [Pet1], [RR-TG] cf. [BOst14]). *In a normed group, for A non-meagre, the sets AA^{-1} and $A^{-1}A$ have non-empty interior.*

Proof. Suppose otherwise. We consider the set AA^{-1} and refer to the right-invariant metric $d(x, y) = ||xy^{-1}||$. Suppose the condition fails: then, for each integer $n = 1, 2, ...$ there is $z_n \in B_{1/n}(e) \backslash AA^{-1}$; hence $z_n \to z_0 = e$. By Proposition 2.11(iv), $\psi_n(x) := z_n x$ converges to the identity (as the metric is right-invariant and $d(z_n x, x) = ||z_n||$), and so satisfies (wcc) by Th. 5.2; hence, there is $a \in A$ such that for infinitely many n

$$\psi_n(a) \in A, \text{ i.e. } z_n a \in A, \text{ or } z_n \in AA^{-1},$$

a contradiction. Reference to the conjugate metric secures the same result for $A^{-1}A$. □

One says that a set A is *thick* if e is an interior point of AA^{-1} (see e.g. [HJ] Section 3.4). The next result (proved essentially by the same means) applied to the additive group $\mathbb{R}$ implies the Kesteman-Borwein-Ditor ([BOst11]) theorem on the line. The name used here refers to a similar (weaker) property studied in Probability Theory (in the context of probabilities regarded as a semigroup under convolution, for which see [PRV], or [Par] 3.2 and 3.5, [BlHe], [Hey]). We need a definition.

Definition. Say that a set A in G is *right-shift compact* if, for any sequence of points a_n in A, there is a point t and a subsequence $\{a_n : n \in \mathbb{M}_t\}$ such that $a_n t$ converges through $\mathbb{M}_t$ to a point $a_0 t$ in A; similarly for *left-shift compact*. Evidently, finite Cartesian products of shift-compact sets are shift-compact. Thus a right-shift compact set A is pre-compact. (If the subsequence $a_m t$ converges to $a_0 t$, for m in $\mathbb{M}_t$, then likewise a_m converges to a_0, for m in $\mathbb{M}_t$.)

Proposition 5.4. *In a normed group, if a subgroup S is locally right-shift compact, then S is closed and locally compact. Conversely, a closed, locally compact subgroup is locally right-shift compact.*

Proof. Suppose that $a_n \to a_0$ with $a_n \in S$. If $a_m t \to a_0 t \in S$ down a subset M then $a_0 t(a_m t)^{-1} = a_0 a_m^{-1} \in S$ for $m \in M$. Hence also $a_0 = a_0 a_m^{-1} a_m \in S$ for $m \in M$. Thus S is closed. □

Remark. Suppose that $a_n = (a_n^i) \in A = \prod A_i$. Pick t_i and inductively infinite $\mathbb{M}_i \subseteq \mathbb{M}_{i-1}$ so that $a_n^i t_i \to a_0^i t_i$ along $n \in \mathbb{M}_i$ with $a_n^i t_i \in A_i$ for $n \in \omega$. Diagonalize $\mathbb{M}_i$ by setting $\mathbb{M} := \{m_i\}$, where $m_{n+1} = \min\{m \in \mathbb{M}_{n+1} : m > m_n\}$. Then the subsequence $\{a_m : m \in \mathbb{M}\}$ satisfies, for each J finite,

$$\mathrm{pr}_J t a_m \subseteq \prod_{j \in J} A_j \text{ for eventually all } m \in \mathbb{M}.$$

Theorem 5.5 (Shift-Compactness Theorem). *In a normed group G, for A precompact, Baire and non-meagre, the set A is right-shift compact, i.e., for any sequence $a_n \in A$, there are $t \in G$ and $a \in A$ such that $a_n t \in A$ and $a_n t \to a$ down a subsequence. Likewise the set A is left-shift compact.*

Proof. Suppose $a_n \in A \subseteq \bar{A}$ with $\bar{A}$ compact. W.l.o.g. $a_n \to a_0 \in \bar{A}$. Hence $z_n := a_n a_0^{-1} \to e_G$. By Theorem 5.2 (the First Verification Theorem), $\psi_n(x) := z_n x$ converges to the identity. Hence, for some $a \in A$ and infinite $\mathbb{M}$, we have $\{z_m a : m \in \mathbb{M}\} \subseteq A$. Taking $t = a_0^{-1} a$, we thus have $a_n t \in A$ and $a_n t \to a \in A$ along $\mathbb{M}$. Replace A by A^{-1} to obtain the other-handed result. □

The following theorem asserts that a 'covering property modulo shift' is satisfied by bounded shift-compact sets. It will be convenient to make the following

Definitions. 1. Say that $\mathcal{D} := \{D_1, ..., D_h\}$ *shift-covers* X, or is a *shifted-cover* of X if, for some $d_1, ..., d_h$ in G,

$$(D_1 - d_1) \cup ... \cup (D_h - d_h) = X.$$

Say that X is *compactly shift-covered* if every open cover $\mathcal{U}$ of X contains a finite subfamily $\mathcal{D}$ which shift-covers X.

2. We say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *strongly shift-covers* A, or is a *strong shifted-cover* of A, if there are *arbitrarily small* $d_1, ..., d_h$ in $\mathbb{R}$ such that

$$(D_1 - d_1) \cup ... \cup (D_h - d_h) \supseteq A.$$

Say that A is *compactly strongly shift-covered* if every open cover $\mathcal{U}$ of A contains a finite subfamily $\mathcal{D}$ which strongly shift-covers A.

Example. Note that $A \subseteq \mathbb{R}$ is density-open (open in the density topology) iff each point of A is a density point of A. Suppose a_0 is a limit point of such a set A in the usual topology; then, for any $\varepsilon > 0$, we may find a point $\alpha \in A$ within $\varepsilon/2$ of a_0 and hence some $t \in A$ within $\varepsilon/2$ of the point α such that some subsequence $t + a_m$ is included in A, with limit $t + a_0$ and with $|t| < \varepsilon$. That is, a dense-open set is strongly shift-compact.

Theorem 5.6 (Compactness Theorem – modulo shift, [BOst8]**).** *Let A be a right-shift compact subset of a separable normed group G. Then A is compactly shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, and for each member of $\mathcal{V}$ a translator, such that the corresponding translates of $\mathcal{V}$ cover A.*

Proof. Let $\mathcal{U}$ be an open cover of A. Since G is second-countable we may assume that $\mathcal{U}$ is a countable family. Write $\mathcal{U} = \{U_i : i \in \omega\}$. Let $Q = \{q_j : j \in \omega\}$ enumerate a dense subset of G. Suppose, contrary to the assertion, that there is no finite subset $\mathcal{V}$ of $\mathcal{U}$ such that translates of $\mathcal{V}$, each translated by one element of Q, cover A. For each n, choose $a_n \in A$ not covered by $\{U_i - q_j : i, j < n\}$. As A is precompact, we may assume, by passing to a subsequence (if necessary), that a_n converges to some point a_0, and also that, for some t, the sequence $a_n t$ lies entirely in A. Let U_i in $\mathcal{U}$ cover $a_0 t$. Without loss of generality we may assume that $a_n t \in U_i$ for all n. Thus $a_n \in U_i t^{-1}$ for all n. Thus we may select $V := U_i q_j$ to be a translation of U_i such that $a_n \in V = U_i q_j$ for all n. But this is a contradiction, since a_n is not covered by $\{U_{i'} q_{j'} : i', j' < n\}$ for $n > \max\{i, j\}$. $\square$

The above proof of the compactness theorem for shift-covering may be improved to strong shift-covering, with only a minor modification (replacing Q with a set $Q^\varepsilon = \{q_j^\varepsilon : j \in \omega\}$ which enumerates, for given $\varepsilon > 0$, a dense subset of the ε ball about e), yielding the following.

Theorem 5.7 (Strong Compactness Theorem – modulo shift, cf. [BOst8])**.** *Let A be a strongly right-shift compact subset of a separable normed group G. Then A is compactly strongly shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, and for each member of $\mathcal{V}$ an arbitrarily small translator, such that the corresponding translates of $\mathcal{V}$ cover A.*

Next we turn to the Steinhaus theorem, which we will derive in Section 8 more directly as a corollary of the Category Embedding Theorem. For completeness we recall in the proof below its connection with the Weil topology introduced in [We].

Definition ([Hal-M] Section 72, p. 257 and 273).

1. A *measurable group* $(X, \mathcal{S}, m)$ is a σ-finite measure space with X a group and m a non-trivial measure such that both $\mathcal{S}$ and m is left-invariant and the mapping $x \to (x, xy)$ is measurability preserving.

2. A measurable group X is *separated* if for each $x \neq e_X$ in X, there is a measurable $E \subset X$ of finite, positive measure such that $\mu(E \triangle xE) > 0$.

Theorem 5.8 (Steinhaus Theorem – cf. Comfort [Com] Th. 4.6 p. 1175). *Let X be a normed locally compact group which is separated under its Haar measure. For measurable A having positive finite Haar measure, the sets AA^{-1} and $A^{-1}A$ have non-empty interior.*

Proof. For X separated, we recall (see [Hal-M] Sect. 62 and [We]) that the Weil topology on X, under which X is a topological group, is generated by the neighbourhood base at e_X comprising sets of the form $N_{E,\varepsilon} := \{x \in X : \mu(E \triangle xE) < \varepsilon\}$, with $\epsilon > 0$ and E measurable and of finite positive measure. Recall from [Hal-M] Sect. 62 the following results: (Th. F) a measurable set with non-empty interior has positive measure; (Th. A) a set of positive measure contains a set of the form GG^{-1}, with G measurable and of finite, positive measure; and (Th. B) for such G, $N_{G\varepsilon} \subseteq GG^{-1}$ for all small enough $\varepsilon > 0$. Thus a measurable set has positive measure iff it is non-meagre in the Weil topology. Thus if A is measurable and has positive measure it is non-meagre in the Weil topology. Moreover, by [Hal-M] Sect 61, Sect. 62 Ths. A and B, the metric open sets of X are generated by sets of the form $N_{E,\varepsilon}$ for some Borelian-($\mathcal{K}$) set E of positive, finite measure. By the Piccard-Pettis Theorem, Th. 5.3 (from the Category Embedding Theorem, Th. 5.1) AA^{-1} contains a non-empty Weil neighbourhood $N_{E,\varepsilon}$. □

Remark. See Section 7 below for an alternative proof via the density topology drawing on Mueller's Haar measure density theorem [Mue] and a category-measure theorem of Martin [Mar] (and also for extensions to products AB).

Theorem 5.9 (The Subgroup Dichotomy Theorem, Banach-Kuratowski Theorem – [Ban-G] Satz 1, [Kur-1] Ch. VI. 13. XII; cf. [Kel] Ch. 6 Pblm P; cf. [BGT] Cor. 1.1.4 and also [BCS] and [Be] for the measure variant).

Let X be a normed group which is non-meagre and let A any Baire subgroup. Then A is either meagre or clopen in X.

Proof. Suppose that A is non-meagre. We show that e is an interior point of A, from which it follows that A is open. Suppose otherwise. Then there is a sequence $z_n \to e$ with $z_n \in B_{1/n}(e)\backslash A$. Now for some $a \in A$ and infinite M we have $z_n a \in A$ for all $n \in M$. But A is a subgroup, hence $z_n = z_n a a^{-1} \in A$ for $n \in M$, a contradiction.

Now suppose that A is not closed. Let a_n be a sequence in A with limit x. Then $a_n x^{-1} \to e$. Now for some $a \in A$ and infinite M we have $z_n x^{-1} a \in A$ for all $n \in M$. But A is a subgroup, so z_n^{-1} and a^{-1} are in A and hence, for all $n \in M$, we have $x^{-1} = z_n^{-1} z_n x^{-1} a a^{-1} \in A$. Hence $x \in A$, as A is a subgroup. □

Remark. Banach's proof is purely topological, so applies to topological groups (though originally stated for metric groups), and relies on the mapping $x \to ax$ being a homeomorphism, likewise Kuratowski's proof, which proceeds via another dichotomy as detailed below.

Theorem 5.10 (Kuratowski Dichotomy – [Kur-B], [Kur-1], [McSh] Cor. 1)**.** *Suppose $H \subseteq Auth(X)$ acts transitively on X, and $Z \subseteq X$ is Baire and has the property that for each $h \in H$*

$$Z = h(Z) \text{ or } Z \cap h(Z) = \varnothing,$$

i.e. under each $h \in H$, either Z is invariant or Z and its image are disjoint. Then, either H is meagre or it is clopen.

The result below generalizes the category version of the Steinhaus Theorem [St] of 1920, first stated explicitly by Piccard [Pic1] in 1939, and restated in [Pet1] in 1950; in the current form it may be regarded as a 'localized-refinement' of [RR-TG].

Theorem 5.11 (Generalized Piccard-Pettis Theorem – [Pic1], [Pic2], [Pet1], [Pet2], [BGT] Th. 1.1.1, [BOst3], [RR-TG], cf. [Kel] Ch. 6 Prb. P). *Let X be a homogenous space. Suppose that ψ_u converges to the identity as $u \to u_0$, and that A is Baire and non-meagre. Then, for some $\delta > 0$, we have*

$$A \cap \psi_u(A) \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta,$$

or, equivalently, for some $\delta > 0$

$$A \cap \psi_u^{-1}(A) \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta.$$

Proof. We may suppose that $A = V \backslash M$ with M meagre and V open. Hence, for any $v \in V \backslash M$, there is some $\varepsilon > 0$ with

$$B_\varepsilon(v) \subseteq U.$$

As $\psi_u \to id$, there is $\delta > 0$ such that, for u with $d(u, u_0) < \delta$, we have

$$\hat{d}(\psi_u, id) < \varepsilon/2.$$

Hence, for any such u and any y in $B_{\varepsilon/2}(v)$, we have

$$d(\psi_u(y), y) < \varepsilon/2.$$

So

$$W := \psi_u(B_{\varepsilon/2}(z_0)) \cap B_{\varepsilon/2}(z_0) \neq \varnothing,$$

and

$$W' := \psi_u^{-1}(B_{\varepsilon/2}(z_0)) \cap B_{\varepsilon/2}(z_0) \neq \varnothing.$$

For fixed u with $d(u, u_0) < \delta$, the set

$$M' := M \cup \psi_u(M) \cup \psi_u^{-1}(M)$$

is meagre. Let $w \in W \backslash M'$ (or $w \in W' \backslash M'$, as the case may be). Since $w \in B_\varepsilon(z_0) \backslash M \subseteq V \backslash M$, we have

$$w \in V \backslash M \subseteq A.$$

Similarly, $w \in \psi_u(B_\varepsilon(z_0)) \backslash \psi_u(M) \subseteq \psi_u(V) \backslash \psi_u(M)$. Hence

$$\psi_u^{-1}(w) \in V \backslash M \subseteq A.$$

In this case, as asserted,

$$A \cap \psi_u^{-1}(A) \neq \varnothing.$$

In the other case ($w \in W' \backslash M'$), one obtains similarly

$$\psi_u(w) \in V \backslash M \subseteq A.$$

Here too

$$A \cap \psi_u^{-1}(A) \neq \varnothing.$$

□

Remarks.

1. In the theorem above it is possible to work with a weaker condition, namely local convergence at z_0, where one demands that for some neighbourhood $B_\eta(z_0)$ and some K,

$$d(\psi_u(z), z) \leq K d(u, u_0), \text{ for } z \in B_\eta(z_0).$$

This implies that, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for $z \in B_\delta(z_0)$,

$$d(\psi_u(z), z) < \varepsilon, \text{ for } z \in B_\delta(z_0).$$

2. The Piccard-Pettis Theorem for topological groups (named by Kelley, [Kel] Ch. 6 Pblm P-(b), the Banach-Kuratowski-Pettis Theorem, say BKPT for short) asserts the category version of the Steinhaus Theorem [St] that, for A Baire and non-meagre, the set $A^{-1}A$ is a neighbourhood of the identity; our version of the Piccard theorem as stated implies this albeit only in the context of metric groups. Let d^X be a right-invariant metric on X and take $\psi_u(x) = ux$ and $u_0 = e$. Then ψ_u converges to the identity (see [BOst13] Section 4), and so the theorem implies that $B_\delta(e) \subseteq A^{-1} \cap A$ for some $\delta > 0$; indeed $a' \in A \cap \psi_u(A)$ for $u \in B_\delta(e)$ means that $a' \in A$ and, for some $a \in A$, also $ua = a'$ so that $u = a^{-1}a' \in A^{-1}A$. It is more correct to name the following important and immediate corollary the BKPT, since it appears in this formulation in [Ban-G], [Kur-1], derived by different means, and was used by Pettis in [Pet1] to deduce his Steinhaus-type theorem.

Theorem 5.12 (McShane's Interior Points Theorem – [McSh] Cor. 3). *Let $T : X^2 \to X$ be such that $T_a(x) := T(x, a)$ is a self-homeomorphism for each $a \in X$ and such that for each pair (x_0, y_0) there is a homeomorphism $\varphi : X \to X$ with $y_0 = \varphi(x_0)$ satisfying*

$$T(x, \varphi(x)) = T(x_0, y_0), \text{ for all } x \in X.$$

Let A and B be second category with B Baire. Then the image $T(A,B)$ has interior points and there are $A_0 \subseteq A, B_0 \subseteq B$, with $A\backslash A_0$ and $B\backslash B_0$ meagre and $T(A_0,B_0)$ open.

6 The Kestelman-Borwein-Ditor Theorem: a bitopological approach

Definition (Genericity). Suppose Γ is $\mathcal{L}$ or $\mathcal{B}a$, the class of measurable sets or Baire sets (i.e. sets with the Baire property). We will say that $P \in \Gamma$ holds *for generically all t* if $\{t : t \notin P\}$ is null/meagre according as Γ is $\mathcal{L}$ or $\mathcal{B}a$.

In this section we develop a bi-topological approach to a generalization of the following result. An alternative approach is given in the next section.

Theorem 6.1 (Kestelman-Borwein-Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null/Baire and non-meagre, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

A stronger form still is derived in [BOst9] (the Generic Reflection Theorem); see also [BOst3] Section 3.1 Note 3, [BOst4] Section 3.1 Note 1. For proofs see the original papers [Kes] and [BoDi]; for a unified treatment in the real-variable case see [BOst9].

Let $(X, \mathcal{S}, m)$ be a probability space which is totally-finite. Let m^* denote the outer measure

$$m^*(E) := \inf\{m(F) : E \subset F \in \mathcal{S}\}.$$

Let the family $\{K_n(x) : x \in X\} \subset \mathcal{S}$ satisfy
(i) $x \in K_n(x)$,
(ii) $m(K_n(x)) \to 0$.

Relative to a fixed family $\{K_n(x) : x \in X\}$ define the upper and lower (outer) density at x of any set E by

$$\begin{aligned}\overline{D}^*(E,x) &= \sup \limsup_n m^*(E \cap K_n(x))/m(K_n(x)),\\ \underline{D}^*(E,x) &= \inf \liminf_n m^*(E \cap K_n(x))/m(K_n(x)).\end{aligned}$$

By definition $\overline{D}^*(E,x) \geq \underline{D}^*(E,x)$. When equality holds, one says that the density of E exists at x, and the common value is denoted by $D^*(E,x)$. If E is measurable the star associated with the outer measure m^* is omitted. If the density is 1 at x, then x is a density point; if the density is 0 at x then x is a dispersion point of E.

Say that a *(weak) density theorem* holds for $\{K_n(x) : x \in X\}$ if for every set (every measurable set) A almost every point of A is a density (an outer density) point of A.

Martin [Mar] shows that the family

$$\mathcal{U} = \{U : \overline{D}^*(X \backslash U, x) = 0, \text{ for all } x \in U\}$$

forms a topology, the density topology on X, with the following property.

Theorem 6.2 (Density Topology Theorem). *If a density theorem holds for $\{K_n(x) : x \in X\}$ and U is d-open, then every point of U is a density point of U and so U is measurable. Furthermore, a measurable set such that each point is a density point is d-open.*

We note that the idea of a density topology was introduced slightly earlier by Goffman ([GoWa],[GNN]); see also Tall [T]. It can be traced to the work of Denjoy [Den] in 1915. Recall that a function is approximately continuous in the sense of Denjoy iff it is continuous under the density topology: [LMZ], p.1.

Theorem 6.3 (Category-Measure Theorem – [Mar] Th. 4.11). *Suppose X is a probability space and a density theorem holds for $\{K_n(x) : x \in X\}$. A necessary and sufficient condition that a set be nowhere dense in the d-topology is that it have measure zero. Hence a necessary and sufficient condition that a set be meagre is that it have measure zero. In particular the topological space $(X, \mathcal{U})$ is a Baire space.*

We now see that the preceeding theorem is applicable to a Haar measure on a locally compact group X by reference to the following result. Here bounded means pre-compact (covered by a compact set).

Theorem 6.4 (Haar measure density theorem – [Mue]; cf. [Hal-M] p. 268). *Let A be a σ-bounded subset and μ a left-invariant Haar measure of a locally compact group X. Then there exists a sequence U_n of bounded measurable neigbourhoods of e_X such that $m^*(A \cap U_n x)/m^*(U_n x) \to 1$ for almost all x out of a measurable cover of A.*

Corollary. *In the setting of Theorem 6.4 with A of positive, totally-finite Haar measure, let $(A, \mathcal{S}_A, m_A)$ be the induced probability subspace of X with $m_A(T) = m(S \cap A)/m(A)$ for $T = S \cap A \in \mathcal{S}_A$. Then the density theorem holds in A.*

We now offer a generalization of a result from [BOst11]; cf. Theorem 5.2.

Theorem 6.5 (Second Verification Theorem for weak category convergence). *Let X be a normed locally compact group with left-invariant Haar measure m. Let V be m-measurable and non-null. For any null sequence $\{z_n\} \to e$ and each $k \in \omega$,*

$$H_k = \bigcap_{n \geq k} V \backslash (V \cdot z_n) \text{ is of } m\text{-measure zero, so meagre in the } d\text{-topology.}$$

That is, the sequence $h_n(x) := x z_n^{-1}$ satisfies the weak category convergence condition (wcc)

Proof. Suppose otherwise. We write $V z_n$ for $V \cdot z_n$, etc. Now, for some k, $m(H_k) > 0$. Write H for H_k. Since $H \subseteq V$, we have, for $n \geq k$, that $\emptyset = H \cap h_n^{-1}(V) = H \cap (V z_n)$ and so a fortiori $\emptyset = H \cap (H z_n)$. Let u be a metric density point of H. Thus, for some bounded (Borel) neighbourhood $U_\nu u$ we have

$$m[H \cap U_\nu u] > \frac{3}{4} m[U_\nu u].$$

Fix ν and put

$$\delta = m[U_\nu u].$$

Let $E = H \cap U_\nu u$. For any z_n, we have $m[(E z_n) \cap U_\nu u z_n] = m[E] > \frac{3}{4}\delta$. By Theorem A of [Hal-M] p. 266, for all large enough n, we have

$$m(U_\nu u \triangle U_\nu u z_n) < \delta/4.$$

Hence, for all n large enough we have $m[(Ez_n)\backslash U_\nu u] \leq \delta/4$. Put $F = (Ez_n) \cap U_\nu u$; then $m[F] > \delta/2$.

But $\delta \geq m[E \cup F] = m[E] + m[F] - m[E \cap F] \geq \frac{3}{4}\delta + \frac{1}{2}\delta - m[E \cap F]$. So

$$m[H \cap (Hz_n)] \geq m[E \cap F] \geq \frac{1}{4}\delta,$$

contradicting $\emptyset = H \cap (Hz_n)$. This establishes the claim. □

As a corollary of the Category Embedding Theorem, Theorem 6.5 and its Corollary now yield the following result (compare also Th. 10.11).

Theorem 6.6 (First Generalized Measurable Kestelman-Borwein-Ditor Theorem). *Let X be a normed locally compact group, $\{z_n\} \to e_X$ be a null sequence in X. If T is Haar measurable and non-null, resp. Baire and non-meagre, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq T.$$

This theorem in turn yields an important conclusion.

Theorem 6.7 (Kodaira's Theorem – [Kod] Corollary to Satz 18. p. 98, cf. [Com] Th. 4.17 p. 1182). *Let X be a normed locally compact group and $f : X \to Y$ a homorphism into a separable normed group Y. Then f is Haar-measurable iff f is Baire under the density topology iff f is continuous under the norm topology.*

Proof. Suppose that f is measurable. Then under the d-topology f is a Baire function. Hence by the classical Baire Continuity Theorem (see, e.g. Section 8 below), since Y is second-countable, f is continuous on some co-meagre set T. Now suppose that f is not continuous at e_X. Hence, for some $\varepsilon > 0$ and some $z_n \to z_0 = e_X$ (in the sense of the norm on X), we have $||f(z_n)|| > \varepsilon$, for all n. By the Kestelman-Borwein-Ditor Theorem (Th. 6.1), there is $t \in T$ and an infinite $\mathbb{M}_t$ such that $tz_n \to t = tz_0 \in T$. Hence, for n in $\mathbb{M}_t$, we have

$$f(t)f(z_n) = f(tz_n) \to f(tz_0) = f(t),$$

i.e. $f(z_n) \to e_Y$, a contradiction. □

Remarks.

1. Comfort [Com] Th. 4.17 proves this result for both X and Y locally compact, with the hypothesis that Y is σ-compact and f measurable with respect to the two Haar measures on X and Y. That proof employs Steinhaus' Theorem and the Weil topology. (Under the density topology, Y will not be second-countable.) When Y is metrizable this implies that Y is separable; of course if f is a continuous surjection, Y will be locally compact.

2. The theorem reduces measurability to the Baire property and in so doing resolves a long-standing issue in the foundations of regular variation; hitherto the theory was established on two alternative foundations employing either measurable functions, or Baire functions, for its scope, with historical preference for measurable functions in connection with integration. We refer to [BGT] for an exposition of the theory which characterizes regularly varying functions of either type by a reduction to an underlying homomorphism of the corresponding type relying on its continuity and then represents either type by very well-behaved functions. Kodaira's Theorem shows that the broader topological class may be given priority. See in particular [BGT] p. 5,11 and [BOst11].

3. The Kestelman-Borwein-Ditor Theorem inspires the following definitions, which we will find useful in the next section

Definitions. Call a set T *subuniversal* if for any null sequence $z_n \to e_G$ there is $t \in G$ and infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T.$$

Call a set T *generically subuniversal* if for any null sequence $z_n \to e_G$ there is $t \in G$ and infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \text{ and } t \in T.$$

Thus the Kestelman-Borwein-Ditor Theorem asserts that a set T which is Baire non-meagre, or measurable non-null, is (generically) subuniversal. The term subuniversal is coined from Kestelman's definition of set being 'universal for null sequences' ([Kes] Th. 2), which required $\mathbb{M}_t$ above to be co-finite rather than infinite. By Theorem 6.5 (Shift-compactness Theorem), a generically subuniversal subset of a normed group is shift-compact (Section 5).

Our final results follow from the First Generalized KBD Theorem and are motivated by the literature of *extended* regular variation in which one assumes only that

$$h^*(u) := \lim \sup_{||x|| \to \infty} h(ux)(x)^{-1}$$

is finite on a 'large enough' domain set (see [BOst-RVWL], [BGT] Ch. 2,3 for the classical context of $\mathbb{R}_+^*$). We need the following definitions generalizing their $\mathbb{R}$ counterparts (in [BOst-RVWL]) to the normed group context.

Definitions. 1. Say that $\mathbf{NT}^*(\{T_k\})$ holds, in words No Trumps holds *generically*, if for any null sequence $z_n \to e_X$ there is $k \in \omega$ and an infinite $\mathbb{M}$ such that

$$\{tz_m : m \in \mathbb{M}\} \subset T_k \text{ and } t \in T_k.$$

For the definition of **NT** see [BOst1], [BOst4] where bounded, rather than null sequences z_n appear and the location of the translator t need not be in T_k. [Of course $\mathbf{NT}^*(\{T_k : k \in \omega\})$ implies $\mathbf{NT}(\{T_k : k \in \omega\})$.]

2. For X a normed group, $h : T \to Y$ or $\mathbb{R}_+$, with $T \subset X$, where Y is a normed group and $\mathbb{R}_+$ refers to the set of positive reals, for $x = \{x_n\}$ with $||x_n|| \to \infty$, put

$$T_k(x) := \bigcap_{n>k} \{t \in T : h(tx_n)h(x_n)^{-1} < n\}$$

or

$$T_k^Y(x) := \bigcap_{n>k} \{t \in T : ||h(tx_n)h(x_n)^{-1}||_Y < n\},$$

according to whether h takes values in $\mathbb{R}_+$ or Y.

Let us say that h is $\mathbf{NT}^*$ on T if for any $x_n \to \infty$ and any null sequence $z_n \to 0$, $\mathbf{NT}^*(\{T_k(x)\})$, resp. $\mathbf{NT}^*(\{T_k^Y(x)\})$, holds.

Theorem 6.8 (**Generic No Trumps Theorem or No Trumps* Theorem**). *In a normed group X, for T Baire non-meagre/measurable non-null and h Baire/measurable with $h^*(t) < +\infty$ on T, h is* $\mathbf{NT}^*$ *on T.*

Proof. The sets $T_k(x)$ are Baire/measurable. Fix $t \in T$. Since $h^*(t) < \infty$ suppose that $h^*(t) < k \in \mathbb{N}$. Then w.l.o.g., for all $n > k$, we have $h(tx_n)h(x_n)^{-1} < n$ and so $t \in T_k(x)$. Thus

$$T = \bigcup_k T_k(x),$$

and so for some k, the set $T_k(x)$ is Baire non-meagre/measurable non-null. The result now follows from the topological or measurable Kestelman-Borwein-Ditor Theorem (Cor to Th. 5.2 or Th. 6.6). □

We now have two variant generalizations of Theorem 7 of [BOst-RVWL].

Theorem 6.9A (Combinatorial Uniform Boundedness Theorem – cf. [Ost-knit]). *In a normed group X, for $h : X \to \mathbb{R}_+$ suppose that $h^*(t) < \infty$ on a set T on which h is* $\mathbf{NT}^*$. *Then for compact $K \subset T$*

$$\lim_{||x||\to\infty} \sup \sup_{u\in K} h(ux)h(x)^{-1} < \infty.$$

Proof. Suppose not: then for some $\{u_n\} \subset K \subset T$ and $||x_n||$ unbounded we have, for all n,

$$h(u_n x_n)h(x_n)^{-1} > n^3. \tag{4}$$

W.l.o.g. $u_n \to u \in K$. Now $||ux_n|| \to \infty$, as $||x_n|| - ||u|| \le ||ux_n||$, by the triangle inequality. Thus we may put $y_n := ux_n$; then

$$T_k(y) := \bigcap_{n>k} \{t \in T : h(tux_n)h(ux_n)^{-1} < n\},$$

and $\mathbf{NT}^*(T_k(y))$ holds. Now $z_n := u_n u^{-1}$ is null. So for some $k \in \omega$, $t \in T_k(y)$ and infinite $\mathbb{M}$,

$$\{t(u_m u^{-1}) : m \in \mathbb{M}\} \in T_k(y).$$

So

$$h(tu_m u^{-1}ux_m)h(ux_m)^{-1} < m \text{ and } t \in T.$$

Now $||u_n x_n|| \to \infty$, as $||x_n|| - ||u_n|| \le ||u_n x_n||$ and $||u_n||$ is bounded. But $t \in T$ so, as before since $h^*(t) < \infty$, for all n large enough

$$h(tu_n x_n)h(u_n x_n)^{-1} < n.$$

Now also $u \in K \subset T$. So for all n large enough

$$h(ux_n)h(x_n)^{-1} < n.$$

But

$$\begin{aligned} h(u_n x_n)h(x_n)^{-1} &= h(u_n x_n)h(tu_n x_n)^{-1} \\ &\quad \times h(tu_n x_n)h(ux_n)^{-1} \\ &\quad \times h(ux_n)h(x_n)^{-1}. \end{aligned}$$

Then for m large enough and in $\mathbb{M}_t$ we have

$$h(u_m x_m)h(x_m)^{-1} < m^3,$$

a contradiction for such m to (4). □

We note a generalization with an almost verbatim proof (requiring, mutatis mutandis, the replacement of $h(ux)h(x)^{-1}$ by $||h(ux)h(x)^{-1}||$). Note that one cannot deduce Th. 6.7A from this variant by referring to the normed group $Y = \mathbb{R}_+^*$, because the natural norm on $\mathbb{R}_+^*$ is $||x||_Y = |\log x|$ (cf. Remarks to Corollary 2.9).

Theorem 6.9B (Combinatorial Uniform Boundedness Theorem). *For $h : X \to Y$ a mapping between normed groups, put*

$$h^*(u) := \limsup ||h(ux)h(x)^{-1}||_Y,$$

and suppose that $h^*(t) < \infty$ on a set T on which h is **NT***. *Then for compact $K \subset T$*

$$\lim \sup_{||x|| \to \infty} \sup_{u \in K} ||h(ux)h(x)^{-1}|| < \infty.$$

We may now deduce the result referred to in the remarks to Corollary 2.9, regarding $\pi : X \to Y$ a group homomorphism,.by reference to the case $h(x) = \pi(x)$ treated in the Lemma below.

Theorem 6.10 (NT* property of quasi-isometry). *If X is a Baire normed group and $\pi : X \to Y$ a group homomorphism, where $||.||_Y$ is (μ-γ)-quasi-isometric to $||.||_X$ under the mapping π, then for any non-meagre Baire set T, π is* **NT*** *on T.*

Proof. Note that

$$||h(tx_n)h(x_n)^{-1}|| = ||\pi(tx_n)\pi(x_n)^{-1}|| = ||\pi(t)||.$$

Hence, as $\pi(e) = e$ (see Examples A4),

$$\{t \in T : h(tx_n)h(x_n)^{-1} < n\} = \{t \in T : ||\pi(t)|| < n\} = B_n^\pi(e),$$

and so

$$\bigcap_{n \geq k} T_n(x_n) = \{t \in T : ||\pi(t)|| < k\} = B_k^\pi(e).$$

Now

$$\frac{1}{\mu}||t||_X - \gamma \le ||\pi(t)||_Y \le \frac{1}{\mu}||t||_X + \gamma,$$

hence $B_n^\pi(e)$ is approximated from above and below by the closed sets $T_n^{\pm}$:

$$T_n^+ := \{t \in T : \frac{1}{\mu}||t||_X + \gamma \le n\} \subset T(x_n) = B_n^\pi(e) \subset T_n^- := \{t \in T : \frac{1}{\mu}||t||_X - \gamma \le n\},$$

which yields the equivalent approximation:

$$\begin{aligned} \bar{B}_{\mu(k-\gamma)} \cap T &= \{t \in T : ||t||_X \le \mu(k-\gamma)\} = \bigcap_{n \ge k} T_n^+ \\ &\subset T_k(x) \subset \bigcap_{n \ge k} T_n^- = \{t \in T : ||t||_X \le \mu(k+\gamma)\} = T \cap \bar{B}_{\mu(k+\gamma)}. \end{aligned}$$

Hence,

$$T = \bigcup_k T_k(x) = \bigcup_k T \cap \bar{B}_{\mu(k+\gamma)}.$$

Hence, by the Baire Category Theorem, for some k the set $T_k(x)$ contains a Baire non-meagre set $\bar{B}_{\mu(k-\gamma)} \cap T$ and the proof of Th. 6.8 applies. Indeed if $T \cap \bar{B}_{\mu(k'+\gamma)}$ is non-meagre for some k', then so is $T \cap \bar{B}_{\mu(k'+\gamma)}$ for $k \ge k' + 2\gamma$ and hence also $T_k(x)$ is so. □

Theorem 6.11 (Global bounds at infinity – Global Bounds Theorem). *Let X be a locally compact group with with norm having a vanishingly small global word-net.*

For $h : X \to \mathbb{R}_+$, if h^* is globally bounded, i.e.

$$h^*(u) = \lim \sup_{||x|| \to \infty} h(ux)h(x)^{-1} < B \qquad (u \in X)$$

for some positive constant B, independent of u, then there exist constants K, L, M such that

$$h(ux)h(x)^{-1} < ||u||^K \qquad (u \ge L, ||x|| \ge M).$$

Hence h is bounded away from ∞ on compact sets sufficiently far from the identity.

Proof. As X is locally compact, it is a Baire space (see e.g. [Eng] Section 3.9). Thus, by Th. 6.8, the Combinatorial Boundedness Theorem Th. 6.8A may be applied with $T = X$ to a compact closed neighbourhood $K = \bar{B}_\varepsilon(e_X)$ of the identity e_X, where w.l.o.g. $0 < \varepsilon < 1$; hence we have

$$\lim \sup_{||x|| \to \infty} \sup_{u \in K} h(ux)h(x)^{-1} < \infty.$$

Now we argue as in [BGT] page 62-3, though with a normed group as the domain. Choose X_1 and $\kappa > \max\{M, 1\}$ such that

$$h(ux)h(x)^{-1} < \kappa \qquad (u \in K,\ ||x|| \geq X_1).$$

Fix v. Now there is some word $w(v) = w_1...w_{m(v)}$ using generators in the compact set Z_δ with $||w_i|| = \delta(1 + \varepsilon_i) < 2\delta$, as $|\varepsilon_i| < 1$ (so $||w_i|| < 2\delta < \varepsilon$), where

$$d(v, w(v)) < \delta$$

and

$$1 - \varepsilon \leq \frac{m(v)\delta}{||x||} \leq 1 + \varepsilon,$$

and so

$$m + 1 < 2\frac{||v||}{\delta} + 1 < A||v|| + 1, \text{ where } A = 2/\delta.$$

Put $w_{m+1} = w^{-1}v$, $v_0 = e$, and for $k = 1, ...m + 1$,

$$v_k = w_1...w_k,$$

so that $v_{m+1} = v$. Now $(v_{k+1}x)(v_k x)^{-1} = w_{k+1} \in K$. So for $||x|| \geq X_1$ we have

$$\begin{aligned} h(vx)h(x)^{-1} &= \prod_{k=1}^{m+1} [h(v_k x)h(v_{k-1}x)]^{-1} \\ &\leq \kappa^{m+1} \leq ||v||^K \end{aligned}$$

(for large enough $||v||$), where

$$K = (A \log \kappa + 1).$$

Indeed, for $||v|| > \log \kappa$, we have

$$(m+1) \log \kappa < (A||v||+1) \log \kappa < ||v||(A \log \kappa + (\log \kappa)||v||^{-1}) < \log ||v||(A \log \kappa + 1).$$

For x_1 with $||x_1|| \geq M$ and with t such that $||tx_1^{-1}|| > L$, take $u = tx_1^{-1}$; then since $||u|| > L$ we have

$$h(ux_1)h(x_1)^{-1} = h(t)h(x_1)^{-1} \leq ||u||^K = ||tx_1^{-1}||^K,$$

i.e.

$$h(t) \leq ||tx_1^{-1}||^K h(x_1),$$

so that $h(t)$ is bounded away from ∞ on compact t-sets sufficiently far from the identity. □

Remarks. 1. The one-sided result in Th. 6.11 can be refined to a two-sided one (as in [BGT] Cor. 2.0.5): taking $s = t^{-1}$, $g(x) = h(x)^{-1}$ for $h : X \to \mathbb{R}_+$, and using the substitution $y = tx$, yields

$$g^*(s) = \sup_{||y|| \to \infty} g(sy)g(y)^{-1} = \inf_{||x|| \to \infty} h(tx)h(x)^{-1} = h_*(s).$$

2. A variant of Th. 6.11 holds with $||h(ux)h(x)^{-1}||_Y$ replacing $h(ux)h(x)^{-1}$.

3. Generalizations of Th. 6.11 along the lines of [BGT] Theorem 2.0.1 may be given for h^* *finite* on a 'large set' (rather than globally *bounded*), by use of the Semigroup Theorem (Th. 8.5).

Taking $h(x) := ||\pi(x)||_Y$, Lemma 2.9, Th. 6.10 and Th. 6.11 together immediately imply the following.

Corollary 6.12. *If X is a Baire normed group and $\pi : X \to Y$ a group homomorphism, where $||.||_Y$ is (μ-γ)-quasi-isometric to $||.||_X$ under the mapping π, then there exist constants K, L, M such that*

$$||\pi(ux)||_Y / ||\pi(x)||_Y < ||u||_X^K \qquad (u \geq L, ||x||_X \geq M).$$

7 The Subgroup Theorem

In this section G is a normed locally compact group with left-invariant Haar measure. We shall be concerned with two topologies on G : the norm topology and the density topology. Under the latter the *binary* group operation need not be jointly continuous (see Heath and Poerio [HePo]); nevertheless a right-shift $x \to xa$, for a constant, is continuous, and so we may say that the density topology is *right-invariant.* We note that if S is measurable and non-null then S^{-1} is measurable and non-null under the corresponding right-invariant Haar and hence also under the original left-invariant measure. We may thus say that both the norm and the density topologies are *inversion-invariant.* Likewise the First and Second Verification Theorems (Theorems 5.2 and 6.5) assert that under both these topologies shift homeomorphisms satisfy (wcc). This motivates a theorem that embraces both topologies as two instances.

Theorem 7.1 (Topological, or Category, Interior Point Theorem). *Let $\{z_n\} \to e$ be a null sequence (in the norm topology). Let G be given a right-invariant and inversion-invariant topology τ, under which it is a Baire space and for which the shift homeomorphisms $h_n(x) = xz_n$ satisfy* (wcc)*. For S Baire and non-meagre in τ, the difference set $S^{-1}S$, and likewise SS^{-1}, is an open neighbourhood of e.*

Proof. Suppose otherwise. Then for each positive integer n we may select

$$z_n \in B_{1/n}(e)\backslash(S^{-1}S).$$

Since $\{z_n\} \to e$ (in the norm topology), the Category Embedding Theorem (Th. 5.1) applies, and gives an $s \in S$ and an infinite $\mathbb{M}_s$ such that

$$\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S.$$

Then for any $m \in \mathbb{M}_s$,

$$sz_m \in S \text{ , i.e. } z_m \in S^{-1}S,$$

a contradiction. Replacing S by S^{-1} we obtain the corresponding result for SS^{-1}. □

Corollary 7.2 (Piccard Theorem, Piccard [Pic1], [Pic2]). *For S Baire and non-meagre in the norm topology, the difference sets SS^{-1} and $S^{-1}S$ have e as interior point.*

First Proof. Apply the preceeding Theorem , since by the First Verification Theorem (Th. 5.2), the condition (wcc) holds. □

Second Proof. Suppose otherwise. Then, as before, for each positive integer n we may select $z_n \in B_{1/n}(e)\backslash(S^{-1}S)$. Since $z_n \to e$, by the Kestelman-Borwein-Ditor Theorem (Th. 6.1), for quasi all $s \in S$ there is an infinite $\mathbb{M}_s$ such that $\{sz_m : m \in \mathbb{M}_s\} \subseteq S$. Then for any $m \in \mathbb{M}_s$, $sz_m \in S$, i.e. $z_m \in SS^{-1}$, a contradiction. □

Corollary 7.3 (Steinhaus' Theorem, [St], [We]; cf. Comfort [Com] Th. 4.6 p. 1175, Beck et al. [BCS]). *For S of positive measure, the difference sets $S^{-1}S$ and SS^{-1} have e as interior point.*

Proof. Arguing as in the first proof above, by the Second Verification Theorem (Th. 6.5), the condition (wcc) holds and S, in the density topology, is Baire and non-meagre (by the Category-Measure Theorem, Th. 6.3). The measure-theoretic form of the second proof above also applies. □

The following corollary to the Steinhaus Theorem Th. 5.7 (and its Baire category version) have important consequences in the Euclidean case. We will say that the group G is (weakly) *Archimedean* if for each $r > 0$ and each $g \in G$ there is $n = n(g)$ such that $g \in B^n$ where $B := \{x : ||x|| < r\}$ is the r-ball.

Theorem 7.4 (Category [Measure] Subgroup Theorem). *For a Baire [measurable] subgroup S of a weakly Archimedean locally compact group G, the following are equivalent:*
(i) $S = G$,
(ii) *S is non-meagre [non-null].*

Proof. By Th. 7.1, for some r-ball B,

$$B \subseteq SS^{-1} \subseteq S,$$

and hence $G = \bigcup_n B^n = S$. □

We will see in the next section a generalization of the Pettis extension of Piccard's result asserting that, for S, T Baire non-meagre, the product ST contains interior points. As our approach will continue to be bitopological, we will deduce also the Steinhaus result that, for S, T non-null and measurable, ST contains interior points.

8 The Semigroup Theorem

In this section G is a normed group which is locally compact. The aim here is to prove a generalization to the normed group setting of the following classical result due to Hille and Phillips [H-P] Th. 7.3.2 (cf. Beck et al. [BCS]

Th. 2, [Be]) in the measurable case, and to Bingham and Goldie [BG] in the Baire case; see [BGT] Cor. 1.1.5.

Theorem 8.1 (Category [Measure] Semigroup Theorem). *For an additive Baire [measurable] subsemigroup S of $\mathbb{R}_+$, the following are equivalent:*

(i) *S contains an interval,*

(ii) *$S \supseteq (s, \infty)$, for some s,*

(iii) *S is non-meagre [non-null].*

We will need a strengthening of the Kestelman-Borwein-Ditor Theorem, Th. 6.1. First we capture a key similarity (their topological 'common basis', adapting a term from logic) between the Baire and measure cases. Recall ([Rog2] p. 460) the usage in logic, whereby a set B is a basis for a class $\mathcal{C}$ of sets whenever any member of $\mathcal{C}$ contains a point in B.

Theorem 8.2 (Common Basis Theorem). *For V, W Baire non-meagre in G equipped with either the norm or the density topology, there is $a \in G$ such that $V \cap (aW)$ contains a non-empty open set modulo meagre sets common to both, up to translation. In fact, in both cases, up to translation, the two sets share a norm $\mathcal{G}_\delta$ subset which is non-meagre in the norm case and non-null in the density case.*

Proof. In the norm topology case if V, W are Baire non-meagre, we may suppose that $V = I \backslash M_0 \cup N_0$ and $W = J \backslash M_1 \cup N_1$, where I, J are open sets. Take $V_0 = I \backslash M_0$ and $W_0 = J \backslash M_1$. If v and w are points of V_0 and W_0, put $a := vw^{-1}$. Thus $v \in I \cap (aJ)$. So $I \cap (aJ)$ differs from $V \cap (aW)$ by a meagre set. Since $M_0 \cup N_0$ may be expanded to a meagre $\mathcal{F}_\sigma$ set M, we deduce that $I \backslash M$ and $J \backslash M$ are non-meagre $\mathcal{G}_\delta$-sets.

In the density topology case, if V, W are measurable non-null let V_0 and W_0 be the sets of density points of V and W. If v and w are points of V_0 and W_0, put $a := vw^{-1}$. Then $v \in T := V_0 \cap (aW_0)$ and so T is non-null and v is a density point of T. Hence if T_0 comprises the density points of T, then $T \backslash T_0$ is null, and so T_0 differs from $V \cap (aW)$ by a null set. Evidently T_0 contains a non-null closed, hence $\mathcal{G}_\delta$-subset (as T_0 is measurable non-null, by regularity of Haar measure). □

Theorem 8.3 (Conjunction Theorem). *For V, W Baire non-meagre/measurable non-null, there is $a \in G$ such that $V \cap (aW)$ is Baire non-meagre/ measur-*

able non-null and for any null sequence $z_n \to e_G$ and quasi all (almost all) $t \in V \cap (aW)$ there exists an infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset V \cap (aW).$$

Proof. In either case applying Theorem 8.2, for some a the set $T := V \cap (aW)$ is Baire non-meagre/measurable non-null. We may now apply the Kestelman-Borwein-Ditor Theorem to the set T. Thus for almost all $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \subset V \cap (aW). \qquad \square$$

See [BOst-KC] for other forms of countable conjuction theorems. The last result motivates a further strengthening of generic subuniversality (compare Section 6).

Definitions. Let S be generically subuniversal.

1. Call T *similar* to S if for every null sequence $z_n \to e_G$ there is $t \in S \cap T$ and $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset S \cap T.$$

Thus S is similar to T and both are generically subuniversal.

Call T *weakly similar* to S if if for every null sequence $z_n \to 0$ there is $s \in S$ and $\mathbb{M}_s$ such that

$$\{sz_m : m \in \mathbb{M}_s\} \subset T.$$

Thus again T is subuniversal.

2. Call S *subuniversally self-similar,* or just *self-similar* (up to inversion-translation), if for some $a \in G$ and some $T \subset S$, S is similar to aT^{-1}.

Call S *weakly self-similar* (up to inversion-translation) if for some $a \in G$ and some $T \subset S$, S is weakly similar to aT^{-1}.

Theorem 8.4 (Self-similarity Theorem). *For S Baire non-meagre/measurable non-null, S is self-similar.*

Proof. Fix a null sequence $z_n \to 0$. If S is Baire non-meagre/measurable non-null then so is S^{-1}; thus we have for some a that $T := S \cap (aS^{-1})$ is likewise Baire non-meagre/measurable non-null and so for quasi all (almost all) $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \subset S \cap (aS^{-1}),$$

as required. □

Theorem 8.5 (Semigroup Theorem – cf. [BCS], [Be]**).** *If S, T are generically subuniversal with T (weakly) similar to S, then ST^{-1} contains a ball about the identity e_G. Hence if S is generically subuniversal and (weakly) self-similar, then SS has interior points. Hence for $G = \mathbb{R}^d$, if additionally S is a semigroup, then S contains an open sector.*

Proof. For S, T (weakly) similar, we claim that ST^{-1} contains $B_\delta(e)$ for some $\delta > 0$. Suppose not: then for each positive n there is z_n with

$$z_n \in B_{1/n}(e) \backslash (ST^{-1}).$$

Now z_n^{-1} is null, so there is s in S and infinite $\mathbb{M}_s$ such that

$$\{z_m^{-1}s : m \in \mathbb{M}_t\} \subset T.$$

For any m in $\mathbb{M}_t$ pick $t_m \in T$ so that $z_m^{-1}s = t_m$; then we have

$$z_m^{-1} = t_m s^{-1}, \quad \text{so} \quad z_m = st_m^{-1},$$

a contradiction. Thus for some $\delta > 0$ we have $B_\delta(e) \subset ST^{-1}$.

For S self-similar, say S is similar to $T := aS^{-1}$, for some a, then $B_\delta(e)a \subset ST^{-1}a = S(aS^{-1})^{-1}a = SSa^{-1}a$, i.e. SS has non-empty interior. □

For information on the structure of semigroups see also [Wr]. For applications see [BOst-RVWL]. By the Common Basis Theorem (Th. 8.2), replacing T by T^{-1}, we obtain as an immediate corollary of Theorem 8.5 a new proof of two classical results, extending the Steinhaus and Piccard Theorem and Kominek's Vector Sum Theorem.

Theorem 8.6 (**Product Set Theorem**, Steinhaus [St] measure case, Pettis [Pet2] Baire case, cf. [Kom1] in the setting of topological vector spaces and [Be] and [BCS] in the group setting).

If S, T are Baire non-meagre/measurable non-null, then ST contains interior points.

9 Convexity

This section begins by developing natural conditions under which the Portmanteau theorem of convex functions (cf. [BOst6]) remains true when reformulated for a normed group setting, and then deduces generalizations of classical automatic continuity theorems for convex functions on a group.

Definitions.

1. A group G will be called 2-*divisible* (or quadratically closed) if the equation $x^2 = g$ for $g \in G$ always has a unique solution in the group to be denoted $g^{1/2}$. See [Lev] for a proof that any group may be embedded as a subgroup in an overgroup where the equations over G are soluble (compare also [Lyn1]).

2. In an arbitrary group, say that a subset C is $\frac{1}{2}$-*convex* if, for all x, y

$$x, y \in C \Longrightarrow \sqrt{xy} \in C,$$

where $\sqrt{xy}$ signifies some element z with $z^2 = xy$. We recall the following results.

Theorem 9.1 (**Eberlein-McShane Theorem**, [Eb], [McSh]). *Let X be a 2-divisible topological group of second category. Then any $\frac{1}{2}$-convex Baire set has a non-empty interior. If X is abelian and each sequence defined by $x_{n+1}^2 = x_n$ converges to e_X then the interior of a $\frac{1}{2}$-convex set C is dense in C.*

Theorem 9.2 (**Convex Minorant Theorem**, [McSh]). *Let X be 2-divisible abelian topological group. Let f and g be real-valued functions defined on a non-meagre subset C with f convex and g Baire such that*

$$f(x) \leq g(x), \text{ for } x \in C.$$

Then f is continuous on the interior of C.

Definition. We say that the function $h : G \to R$ is $\frac{1}{2}$-*convex* on the $\frac{1}{2}$–convex set C if, for $x, y \in C$,

$$h(\sqrt{xy}) \leq \frac{1}{2}\left(h(x) + h(y)\right),$$

with $\sqrt{xy}$ as above.

Example. For $G = R_+^*$ the function $h(x) = x$ is $\frac{1}{2}$-convex on G, since

$$2xy \leq x^2 + y^2.$$

Lemma 9.3 (Averaging Lemma). *A non-meagre set T is 'averaging', that is, for any given point $u \in T$ and for any sequence $\{u_n\} \to u$, there are $v \in G$ (a right-averaging translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have*

$$u_n^2 = v_n v.$$

There is likewise a left-averaging translator such that for some $\{w_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have

$$u_n^2 = w w_n.$$

Proof. Define null sequences by

$$z_n = u_n u^{-1}, \text{ and } \tilde{z}_n = u^{-1} u_n.$$

We are to solve

$$\begin{aligned} u_n^2 v^{-1} &= v_n \in T, \\ u \tilde{z}_n z_n u v^{-1} &= v_n \in T, \\ \tilde{z}_n z_n u v^{-1} &= u^{-1} v_n \in T' = u^{-1} T. \end{aligned}$$

Now put $\psi_n(x) := \tilde{z}_n z_n x$; then

$$d(x, \tilde{z}_n z_n x) = d(e, \tilde{z}_n z_n) = ||\tilde{z}_n z_n|| \leq ||\tilde{z}_n|| + ||z_n|| \to 0.$$

By the Category Embedding Theorem (Th. 5.1), for some $\lambda \in T' = u^{-1}T$, we have with $\lambda = u^{-1}t$ and for infinitely many n

$$\begin{aligned} u^{-1} v_n &:= \tilde{z}_n z_n \lambda \in T' = u^{-1} T, \\ u \tilde{z}_n z_n \lambda &= v_n \in T, \\ u \tilde{z}_n z_n u u^{-1} \lambda &= v_n \in T, \\ u_n^2 u^{-1} \lambda &= v_n \in T, \\ u_n^2 &= v_n \lambda^{-1} u = v_n v \end{aligned}$$

(with $v = \lambda^{-1} u = t^{-1} u^2 \in T^{-1} u^2$).

As for the remaining assertion, note that $u_n^{-1} \to u^{-1}, v_n^{-1} \in T^{-1}$ and

$$u_n^{-2} = v^{-1} v_n^{-1}.$$

Thus noting that T^{-1} is non-meagre (since inversion is a homeomorphism) and replacing T^{-1} by T we obtain the required assertion by a right-averaging translator. □

Note the connection between the norms of the null sequences is only by way of the conjugate metrics:

$$||z_n|| = d(e, u_n u^{-1}) = d(u, u_n), \text{ and } ||\tilde{z}_n|| = d(e, u^{-1} u_n) = d(u_n^{-1}, u^{-1}) = \tilde{d}(u_n, u).$$

Whilst we may make no comparisons between them, both norms nevertheless converge to zero.

Definition. We say that $f : G \to H$ is *locally Lipschitz* at g if, for some neighbourhood N_g of g and for some constants K_g and all x, y in N_g,

$$\left|\left|f(x)f(y)^{-1}\right|\right|_H \leq K_g ||xy^{-1}||_G.$$

We say that $f : G \to H$ is *locally bi-Lipschitz* at g if, for some neighbourhood N_g of g and for some positive constants K_g, κ_g, and all x, y in N_g,

$$\kappa_g ||xy^{-1}||_G \leq \left|\left|f(x)f(y)^{-1}\right|\right|_H \leq K_g ||xy^{-1}||_G.$$

If $f : G \to H$ is invertible, this asserts that both f and its inverse f^{-1} are locally Lipschitz at g and $f(g)$ respectively.

We say that the *norm* on G is *n-Lipschitz* if the function $f_n(x) := x^n$ from G to G is *locally Lipschitz* at all $g \neq e$, i.e. for each there is a neighbourhood N_g of g and positive constants κ_g, K_g so that

$$\kappa_g ||xy^{-1}||_G \leq \left|\left|x^n y^{-n}\right|\right|_G \leq K_g ||xy^{-1}||_G.$$

In an abelian context the power function is a homomorphism; we note that [HJ] p. 381 refers to a semigroup being *modular* when each f_n (defined as above) is an injective homomorphism. The condition on the right with $K = n$ is automatic, and so one need require only that for some positive constant κ

$$\kappa ||g|| \leq ||g^n||.$$

Note that, in the general context of an n-Lipschitz norm, if $x^n = y^n$, then as $(x^n y^{-n}) = e$, we have $\kappa_g ||xy^{-1}||_G \leq ||x^n y^{-n}||_G = ||e|| = 0$, and so $||xy^{-1}||_G = 0$, i.e. the power function is injective. If, moreover, the group is n-divisible, then the power function $f_n(x)$ is an isomorphism.

We note that in the additive group of reals x^2 fails to be locally bi-Lipschitz at the origin (since its derivative there is zero): see [Bart]. However, the following are bi-Lipschitz.

Examples.

1. In $\mathbb{R}^d$ with additive notation, we have $||x^2|| := ||2x|| = 2||x||$, so the norm is 2-Lipschitz.

2. In $\mathbb{R}_+^*$ we have $||x^2|| := |\log x^2| = 2|\log x| = 2||x||$ and again the norm is 2-Lipschitz.

3. In a Klee group the mapping $f(x) := x^n$ is uniformly (locally) Lipschitz, since

$$\left|\left|x^n y^{-n}\right|\right|_G \leq n||xy^{-1}||_G,$$

proved inductively from the Klee property (Th. 2.18) via the observation that

$$\left|\left|x^{n+1}y^{-(n+1)}\right|\right|_G = \left|\left|xx^n y^{-n} y^{-1}\right|\right|_G \leq \left|\left|x^n y^{-n}\right|\right|_G + \left|\left|xy^{-1}\right|\right|_G.$$

Lemma 9.4 (Reflecting Lemma). *Suppose the norm is everywhere locally 2-Lipschitz. Then, for T non-meagre, T is reflecting i.e. there are $w \in G$ (a right-reflecting translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have*

$$v_n^2 = u_n w.$$

There is likewise a left-reflecting translator.

Proof. Let $T^2 := \{g : g = t^2 \text{ for some } t \in T\}$. By assumption, T^2 is non-meagre. With $u_n = uz_n$, put $S = T^2$ and notice that $u_n w \in S$ iff $uz_n w \in S$ iff $z_n w \in u^{-1}S$. Now $u^{-1}S$ is non-meagre and $\psi_n(x) := z_n x$ as usual converges to the identity, so the existence of $w \in u^{-1}S$ is assured such that $z_n w = u^{-1}v_n^2$. $\square$

Remarks. 1. Note that the assertion here is

$$u_n^{-1} v_n = w v_n^{-1},$$

so that

$$d(v_n, w) = d(v_n^{-1}, u_n^{-1}) = \tilde{d}(v_n, u_n) \approx \tilde{d}(v_n, u),$$

or

$$d(v_n, w) \approx \tilde{d}(v_n, u),$$

suggesting the terminology of reflection.

2. Boundedness theorems for reflecting and averaging sets follow as in [BOst6] since the following are true in any group, as we see below.

Theorem 9.5. *For f a $\frac{1}{2}$-convex function, if f is locally bounded above at x_0 then it is locally bounded below at x_0 (and hence locally bounded at x_0).*

Proof. Say f is bounded above in $B := B_\delta(x_0)$ by M. Consider $u \in \tilde{B}_\delta(x_0)$. Thus $\tilde{d}(x_0, u) = ||u^{-1}x_0|| < \delta$. Put $t = u^{-1}x_0^2$; then $tx_0^{-1} = u^{-1}x_0$, and so

$$d(t, x_0) = ||tx_0^{-1}|| = ||u^{-1}x_0|| = \tilde{d}(u, x_0) < \delta.$$

Then $t \in B$, and since $x_0^2 = ut$ we have

$$2f(x_0) \leq f(u) + f(t) \leq f(u) + M,$$

or

$$f(u) \geq 2f(x_0) - M.$$

Thus $2f(x_0) - M$ is a lower bound for f on the open set $\tilde{B}_\delta(x_0)$. □

As a corollary a suitably rephrased Bernstein-Doetsch Theorem ([Kucz], [BOst6]) is thus true.

Theorem 9.6 (Bernstein-Doetsch Theorem). *For f a $\frac{1}{2}$-convex function, if f is locally bounded above at x_0, then f is continuous at at x_0.*

Proof. We repeat the 'Second proof' of [Kucz] p. 145. Choose $y_n \to x_0$ with $f(y_n) \to m_f(x_0)$ and $z_n \to x_0$ with $f(z_n) \to M_f(x_0)$. Let $u_n := y_n^2 x_n^{-1}$. Thus $y_n^2 = u_n x_n$ and so

$$2f(y_n) \leq f(u_n) + f(z_n),$$

i.e. $f(u_n) \geq 2f(y_n) - f(z_n)$. Hence in the limit we obtain

$$M_f(x_0) \geq \liminf f(u_n) \geq 2M_f(x_0) - m_f(x_0).$$

One thus has that $M_f(x_0) \leq m_f(x_0)$. But $m_f(x_0) \leq f(x_0) \leq M_f(x_0)$, and both hull values are finite (by the result above). Thus $m_f(x_0) = f(x_0) = M_f(x_0)$, from which continuity follows. □

We now consider the transferability of upper and lower local boundedness. Our proofs work directly with definitions (so are not modelled after those in Kuczma [Kucz]). We do not however consider domains other than the whole metric group. For clarity of proof structure we give separate proofs for the two cases, first when G is abelian and later for general G.

Theorem 9.7 (Local upper boundedness). *For f a $\frac{1}{2}$-convex function defined on a group G, if f is locally bounded above at some point x_0, then f is locally bounded above at all points.*

Proof. *Case* (i) *The Abelian case.* Say f is bounded above in $B := B_\delta(x_0)$ by M. Given a fixed point t, put $z = z_t := x_0^{-1}t^2$, so that $t^2 = x_0 z$. Consider any $u \in B_{\delta/2}(t)$. Write $u = st$ with $||s|| < \delta/2$. Now put $y = s^2$; then $||y|| = ||s^2|| \leq 2||s|| < \delta$. Hence $yx_0 \in B_\delta(x_0)$. Now

$$u^2 = (st)^2 = s^2t^2 = yx_0z,$$

as the group is abelian. So

$$f(u) \leq \frac{1}{2}f(yx_0) + \frac{1}{2}f(z) \leq \frac{1}{2}M + \frac{1}{2}f(z_t).$$

That is, $\frac{1}{2}(M + f(z_t))$ is an upper bound for f in $B_{\delta/2}(x_0)$.

Case (ii) *The general case.* Now we consider the general case. As before, suppose f is bounded above in $B := B_\delta(x_0)$ by M, and let t be a given a fixed point; put $z = z_t := x_0^{-1}t^2$ so that $t^2 = x_0 z$.

For this fixed t the mapping $y \to \alpha(y) := ytyt^{-1}y^{-2}$ is continuous with $\alpha(e) = e$, so $\alpha(y)$ is $o(y)$ as $||y|| \to 0$. Now

$$sts = [stst^{-1}s^{-2}]s^2t = \alpha(s)s^2t,$$

and we may suppose that, for some $\eta < \delta/2$, we have $||\alpha(s)|| < \delta/2$, for $||s|| < \eta$. Note that

$$stst = \alpha(s)s^2t^2.$$

Consider any $u \in B_r(t)$ with $r = \min\{\eta, \delta/2\}$. Write $u = st$ with $||s|| < r \leq \delta/2$. Now put $y = s^2$. Then $||y|| = ||s^2|| \leq 2||s|| < \delta$ and $||o(s)y|| \leq \eta + \delta/2 < \delta$. Hence $o(s)yx_0 \in B_\delta(x_0)$. Now

$$u^2 = stst = \alpha(s)s^2t^2 = \alpha(s)yx_0z.$$

Hence, by convexity,

$$f(u) \leq \frac{1}{2}f(o(s)yx_0) + \frac{1}{2}f(z) \leq \frac{1}{2}M + \frac{1}{2}f(z_t). \qquad \square$$

As an immediate corollary of the last theorem and the Bernstein-Doetsch Theorem (Th. 9.6) we have the following result.

Theorem 9.8 (Dichotomy Theorem for convex functions – [Kucz] p. 147). *For $\frac{1}{2}$-convex f (so in particular for additive f) either f is continuous everywhere, or it is discontinuous everywhere.*

The definition below requires continuity of 'square-rooting' – taken in the form of an algebraic closure property of degree 2 in a group G, expressed as the solvability of certain 'quadratic equations' over the group. Its status is clarified later by reference to Bartle's Inverse Function Theorem. We recall that a group is n-divisible if $x^n g = e$ is soluble for each $g \in G$. (In the absence of algebraic closure of any degree an extension of G may be constructed in which these equations are solvable – see for instance Levin [Lev].)

Definition. We say that the normed group G is *locally convex* at $\lambda = t^2$ if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for all g with $||g|| < \varepsilon$, the equation

$$xtxt = gt^2,$$

equivalently $xtxt^{-1} = g$, has its solutions satisfying $||x|| < \delta$.

Thus G is locally convex at e if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for all g with $||g|| < \varepsilon$, the equation

$$x^2 = g$$

has its solutions with $||x|| < \delta$.

Remark. Putting $u = xt$ the local convexity equation reduces to $u^2 = gt^2$, asserting the local existence of square roots (local 2-divisibility). If G is abelian the condition at t reduces to the condition at e.

Theorem 9.9 (Local lower boundedness). *Let G be a locally convex group with a 2-Lipschitz norm, i.e. $g \to g^2$ is a bi-Lipschitz isomorphism such that, for some $\kappa > 0$,*

$$\kappa ||g|| \leq ||g^2|| \leq 2||g||.$$

For f a $\frac{1}{2}$-convex function, if f is locally bounded below at some point, then f is locally bounded below at all points.

Proof. *Case* (i) *The Abelian case.* We change the roles of t and x_0 in the preceeding abelian theorem, treating t as a reference point, albeit now for lower boundedness, and x_0 as some arbitrary other fixed point. Suppose that f is bounded below by L on $B_\delta(t)$. Let $yx_0 \in B_{\kappa\delta}(x_0)$, so that $0 < ||y|| < \kappa\delta$. Choose s such that $s^2 = y$. Then,

$$\kappa ||s|| \leq ||y|| < \kappa\delta,$$

so $||s|| < \delta$. Thus $u = st \in B_\delta(t)$. Now the identity $u^2 = s^2t^2 = yx_0z$ implies that

$$\begin{aligned} L &\leq f(u) \leq \frac{1}{2} f(yx_0) + \frac{1}{2} f(z_t), \\ 2L - f(z_t) &\leq f(yx_0), \end{aligned}$$

i.e. that $2L - f(z_t)$ is a lower bound for f on $B_{\kappa\delta}(x_0)$.

Case (ii) *The general case.* Now we consider the general case. Suppose as before that f is bounded below by L on $B_\delta(t)$. Since the map $\alpha(\sigma) := \sigma t \sigma t^{-1} \sigma^{-2}$ is continuous and $\alpha(e) = e$, we may choose η such that $||\alpha(\sigma)|| < \kappa\delta/2$, for $||\sigma|| < \eta$. Now choose $\varepsilon > 0$ such that, for each y with $||y|| < \varepsilon$, the solution $u = \sigma t$ to

$$u^2 = yt^2$$

has $||\sigma|| < \eta$. Let $r = \min\{\kappa\delta/2, \varepsilon\}$.

Let $yx_0 \in B_r(x_0)$; then $0 < ||y|| < \kappa\delta/2$ and $||y|| < \varepsilon$. As before put $z = z_t := x_0^{-1}t^2$ so that $t^2 = x_0z$. Consider $u = \sigma t$ such that $u^2 = yx_0z$; thus we have

$$u^2 = \sigma t \sigma t = yx_0z = yx_0x_0^{-1}t^2 = yt^2.$$

Hence $||\sigma|| < \eta$ (as $||y|| < \varepsilon$). Now we write

$$u^2 = \sigma t \sigma t = [\sigma t \sigma t^{-1} \sigma^{-2}] \sigma^2 t^2 = \alpha(\sigma) \sigma^2 t^2 = y t^2.$$

We compute that

$$y = \alpha(\sigma) \sigma^2$$

and

$$\kappa\delta/2 \geq ||y|| = ||\alpha(\sigma)\sigma^2|| \geq ||\sigma^2|| - ||\alpha(\sigma)|| \geq \kappa||\sigma|| - ||\alpha(\sigma)||,$$

so

$$||\sigma|| \leq \delta/2 + ||\alpha(\sigma)||/\kappa < \delta/2 + \delta/2 < \delta.$$

Thus $u \in B_\delta(t)$. Now the identity $u^2 = y x_0 z$ together with convexity implies as usual that

$$\begin{aligned} L &\leq f(u) \leq \frac{1}{2} f(y x_0) + \frac{1}{2} f(z_t), \\ 2L - f(z_t) &\leq f(y x_0), \end{aligned}$$

i.e. $2L - f(z_t)$ is a lower bound for f on $B_{\kappa\delta}(x_0)$. □

The local 2-divisibility assumption at t^2 asserts that $f_t(\sigma) := \sigma t \sigma t^{-1}$ is invertible locally at e. Bartle's theorem below guarantees that f_t has uniform local inverse under a smoothness assumption, i.e. that for $||\sigma|| = ||f_t^{-1}(y)|| < \delta$, for all small enough y, say for $||y|| < \kappa\delta$. To state the theorem we need some definitions.

Definitions.

1. f is said to have a *derivative* at x_0 if there is a continuous homomorphism $f'(x_0)$ such that

$$\lim_{||u|| \to 0+} \frac{1}{||u||} ||f(u x_0) f(x_0)^{-1} [f'(x_0)(u)]^{-1}|| = 0.$$

2. f is *of class C'* on the open set U if it has a derivative at each point u in U and, for each x_0 and each $\varepsilon > 0$, there is $\delta > 0$ such that, for all x_1, x_2 in $B_\delta(x_0)$ both

$$||f'(x_1)(u)[f'(x_2)(u)]^{-1}|| < \varepsilon||u||$$

and

$$||f(x_1) f(x_2)^{-1} f'(x_0)(x_1 x_2^{-1})^{-1}|| < \varepsilon||x_1 x_2^{-1}||.$$

The two conditions may be rephrased relative to the right-invariant metric d on the group as

$$d(f'(x_1)(u), f'(x_2)(u)) < \varepsilon||u||,$$

and

$$d(f(x_1)f(x_2)^{-1}, f'(x_0)(x_1x_2^{-1}) < \varepsilon d(x_1, x_2).$$

3. Suppose that $y_0 = f(x_0)$. Then f is *smooth* at x_0 if there are positive numbers α, β such that if $0 < d(y, y_0) < \beta$ then there is x such that $y = f(x)$ and $d(x, x_0) \leq \alpha \cdot d(y, y_0)$. If f is invertible, then this asserts that

$$d(f^{-1}(y), f^{-1}(y_0)) \leq \alpha \cdot d(y, y_0).$$

Example. Let $f(x) = tx$ with t fixed. Here f is smooth at x_0 if there are positive numbers α, β such that

$$||xx_0^{-1}|| \leq \alpha||tx(tx_0)^{-1}|| = \alpha||txx_0^{-1}t^{-1}||.$$

Note that in a Klee group $||txx_0^{-1}t^{-1}|| = ||t^{-1}txx_0^{-1}|| = ||xx_0^{-1}||$.

Theorem 9.10 (**Bartle's Inverse Function Theorem**, [Bart] Th. 2.4). *Suppose that*

(i) *f is of class C' in the ball $B_r(x_0) = \{x \in G : ||xx_0^{-1}|| < r\}$, for some $r > 0$, and*

(ii) *$f'(x_0)$ is smooth (at e and so anywhere).*

Then f is smooth at x_0 and hence open.

If also the derivative $f'(x_0)$ is an isomorphism, then f has a uniformly continuous local inverse at x_0.

Corollary 9.11. *If $f_t(\sigma) := \sigma t \sigma t^{-1}$ is of class C' on $B_r(e)$ and $f_t'(e)$ is smooth, then G is locally convex at t.*

Proof. Immediate since $f_t(e) = e$. □

We are now in a position to state generalizations of two results derived in the real line case in [BOst6].

Proposition 9.12. *Let G be any locally convex group with a 2-Lipschitz norm. If f is $\frac{1}{2}$-convex and bounded below on a reflecting subset S of G, then f is locally bounded below on G.*

Proof. Suppose not. Let T be a reflecting subset of S. Let K be a lower bound on T. If f is not locally bounded from below, then at any point u in $\bar{T}$ there is a sequence $\{u_n\} \to u$ with $\{f(u_n)\} \to -\infty$. For some $w \in G$, we have $v_n^2 = wu_n \in T$, for infinitely many n. Then

$$K \leq f(v_n) \leq \frac{1}{2}f(w) + \frac{1}{2}f(u_n), \text{ or } 2K - f(w) \leq f(u_n),$$

i.e. $f(u_n)$ is bounded from below, a contradiction. □

Theorem 9.13 (Generalized Mehdi Theorem – cf. [Meh] Th. 3**).** *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed group, bounded above on an averaging subset S, is continuous on G.*

Proof. Let T be an averaging core of S. Suppose that f is not continuous, but is bounded above on T by K. Then f is not locally bounded above at some point of $u \in \bar{T}$. Then there is a null sequence $z_n \to e$ with $f(u_n) \to \infty$, where $u_n = uz_n$. Select $\{v_n\}$ and w in G so that, for infinitely many n, we have

$$u_n^2 = wv_n.$$

But for such n,we have

$$f(u_n) \leq \frac{1}{2}f(w) + \frac{1}{2}f(v_n) \leq \frac{1}{2}f(w) + \frac{1}{2}K,$$

contradicting the unboundedness of $f(u_n)$. □

The Generalized Mehdi Theorem, together with the Averaging Lemma, implies the classical result below and its generalizations.

Theorem 9.14 (**Császár-Ostrowski Theorem** [Csa], [Kucz] p. 210). *A convex function $f : \mathbb{R} \to \mathbb{R}$ bounded above on a set of positive measure/non-meagre set is continuous.*

Theorem 9.15 (Topological Császár-Ostrowski Theorem). *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed group, bounded above on a non-meagre subset, is continuous.*

Reference to the Generalized Borwein-Ditor Theorem yields the following.

Theorem 9.16 (Haar-measure Császár-Ostrowski Theorem). *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed group carrying a Radon measure, bounded above on a set of positive measure, is continuous.*

10 Automatic continuity: the Jones-Kominek Theorem

This section is dedicated to generalizations to normed groups and to a more general class of topological groups of the following result for the real line. Here we regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and so we say that T is a *spanning subset* of $\mathbb{R}$ if any real number is a finite rational combination of members of T. See below for the definition of an analytic set.

Theorem 10.1 (Theorems of Jones and Kominek). *Let f be additive on $\mathbb{R}$ and either have a continuous restriction, or a bounded restriction, $f|T$, where T is some analytic set spanning $\mathbb{R}$. Then f is continuous.*

The result follows from the Expansion Lemma and Darboux's Theorem (see below) that an additive function bounded on an interval is continuous. In fact the bounded case above (*Kominek's Theorem,* [Kom2]) implies the continuous case (*Jones's Theorem,* [Jones1], [Jones2]), as was shown in [BOst7]. [OC] develops limit theorems for sequences of functionals whose properties are given on various kinds of spanning sets including spanning in the sense of linear rational combinations.

Before stating the current generalizations we begin with some preliminaries on analytic subsets of a topological group.

We recall ([Jay-Rog], p. 11, or [Kech] Ch. III for the Polish space setting) that in a Hausdorff space X a K*-analytic set* is a set A that is the image under a compact-valued, upper semi-continuous map from $\mathbb{N}^{\mathbb{N}}$; if this mapping takes values that are singletons or empty, the set A is said to be *analytic*. In either case A is *Lindelöf*. (The topological notion of K-analyticity was introduced by Choquet, Frolik, Sion and Rogers under variant definitions, eventually found to be equivalent, as a consequence of a theorem of Jayne, see [Jay-Rog] Sect. 2.8 p. 37 for a discussion.) If the space X is a topological group, then the subgroup $\langle A\rangle$ spanned (generated) by an analytic subset A is also analytic and so Lindelöf (for which, see below); note the result due to Loy [Loy] and Christensen [Ch] that an analytic Baire group is Polish (cf. [HJ] Th. 2.3.6 p. 355). Note that a Lindelöf group need not be metric; see for example the construction due to Oleg Pavlov [Pav]. If additionally the group X is metric, then $\langle A\rangle$ is separable, and so in fact this K-analytic set is analytic (a continuous image of $\mathbb{N}^{\mathbb{N}}$ – see [Jay-Rog] Th. 5.5.1 (b), p. 110).

Definition. We say that a set S is *Souslin-$\mathcal{H}$* if it is of the form

$$S = \bigcup_{\alpha \in \omega^\omega} \cap_{n=1}^{\infty} H(\alpha|n),$$

with each $H(\alpha|n) \in \mathcal{H}$. We will often take $\mathcal{H}$ to be $\mathcal{F}(X)$, the family of closed subsets of the space X.

We recall that a set is *meagre* if it is a countable union of nowhere dense sets, a set is *Baire* if it is open modulo a meagre set, or equivalently if it is closed modulo a meagre set (cf. Engelking [Eng] especially p.198 Section 4.9 and Exercises 3.9.J, although we prefer 'meagre' to 'of first category').

Definition. Let G be any group. For any positive integer n and for any subset S let $S^{(n)}$, the *n-span* of S, denote the set of S-words of length n. Say that a subset H of G *spans* G *(*in the sense of group theory), or *generates the group* G, if for any $g \in G$, there are $h_1, ..., h_n$ in H such that

$$g = h_1^{\varepsilon_1} \cdot ... \cdot h_n^{\varepsilon_n}, \text{ with } \varepsilon_i = \pm 1.$$

(If H is *symmetric*, so that $h^{-1} \in H$ iff $h \in H$, there is no need for inverses.)
We begin with results concerning K-analytic groups.

Proposition 10.2. *The span of a K-analytic set is K-analytic; likewise for analytic sets.*

Proof. Since $f(v, w) = vw$ is continuous, $S^{(2)} = f(S \times S)$ is K-analytic by [Jay-Rog] Th 2.5.1 p. 23. Similarly all the sets $S^{(n)}$ are K-analytic. Hence the span, namely $\bigcup_{n \in \mathbb{N}} S^{(n)}$ is such ([Jay-Rog], Th. 2.5.4 p. 23). □

Theorem 10.3 (Intersection Theorem – [Jay-Rog] Th 2.5.3, p. 23). *The intersection of a K-analytic set with a Souslin-$\mathcal{F}(X)$ in a Hausdorff space X is K-analytic.*

Theorem 10.4 (Projection Theorem – [RW] and [Jay-Rog] Th 2.6.6, p. 30). *Let X and Y be topological spaces with Y a K-analytic set. Then the projection on X of a Souslin-$\mathcal{F}(X \times Y)$ is Souslin-$\mathcal{F}(X)$.*

Theorem 10.5 (Nikodym's Theorem – [Nik]; [Jay-Rog] p. 42) *The Baire sets of a space X are closed under the Souslin operation. Hence Souslin-$\mathcal{F}(X)$ sets are Baire.*

Definitions.

1. Say that a function $f : X \to Y$ between two topological spaces is $\mathcal{H}$*-Baire*, for $\mathcal{H}$ a class of sets in Y, if $f^{-1}(H)$ has the Baire property for each set H in $\mathcal{H}$. Thus f is $\mathcal{F}(Y)$-Baire if $f^{-1}(F)$ is Baire for all closed F in Y. Taking complements, since

$$f^{-1}(Y\backslash H) = X\backslash f^{-1}(H),$$

f is $\mathcal{F}(Y)$-Baire iff it is $\mathcal{G}(Y)$-Baire, when we will simply say that f is *Baire* ('f has the Baire property' is the alternative usage).

2. One must distinguish between functions that are $\mathcal{F}(Y)$-Baire and those that lie in the smallest family of functions closed under pointwise limits of sequences and containing the continuous functions (for a modern treatment see [Jay-Rog] Sect. 6). We follow tradition in calling these last *Baire-measurable.*

3. We will say that a function is *Baire-continuous* if it is continuous when restricted to some co-meagre set. In the real line case and with the density topology, this is Denjoy's approximate continuity ([LMZ], p.1); recall ([Kech], 17.47) that a set is (Lebesgue) measurable iff it has the Baire property under the density topology.

The connections between these concepts are given in the theorems below. See the cited papers for proofs.

Theorem 10.6 (Banach-Neeb Theorem – [Ban-T] Th. 4 pg. 35, and Vol I p. 206; [Ne]).

(i) *A Baire-measurable $f : X \to Y$ with X a Baire space and Y metric is Baire-continuous; and*

(ii) *a Borel-measurable $f : X \to Y$ with X, Y metric and Y separable is Baire-measurable.*

Remarks. In fact Banach shows that a Baire-measurable function is Baire-continuous on each perfect set ([Ban-T] Vol. II p. 206). Neeb assumes in addition that Y is arcwise connected, but as Pestov [Pes] remarks the arcwise connectedness may be dropped by referring to a result of Hartman and Mycielski [HM] that a separable metrizable group embeds as a subgroup of an arcwise connected separable metrizable group.

Theorem 10.7 (Baire Continuity Theorem). *A Baire function $f : X \to Y$ is Baire continuous in the following cases:*

(i) Baire condition (see e.g. [HJ] Th. 2.2.10 p. 346): *Y is a second-countable space*

(ii) Emeryk-Frankiewicz-Kulpa ([EFK]): *X is Čech-complete and Y has a base of cardinality not exceeding the continuum;*

(iii) Pol condition ([Pol]): *f is Borel, X is Borelian-K and Y is metrizable and of nonmeasurable cardinality;*

(iv) Hansell condition ([Han]): *f is σ-discrete and Y is metric.*

We will say that the pair (X, Y) *enables Baire continuity* if the spaces X, Y satisfy either of the two conditions (i) or (ii). In the applications below Y is usually the additive group of reals $\mathbb{R}$, so satisfies (i). Building on [EFK], Fremlin ([Frem] Section 10), characterizes a space X such that every Baire function $f : X \rightarrow Y$ is Baire-continuous for all metric Y in the language of 'measurable spaces with negligibles'; reference there is made to disjoint families of negligible sets all of whose subfamilies have a measurable union. For a discussion of discontinuous homomorphisms, especially counterexamples on $C(X)$ with X compact (e.g. employing Stone-Čech compactifications, $X = \beta\mathbb{N}\backslash\mathbb{N}$), see [Dal] Section 9.

Remarks. Hansell's condition, requiring the function f to be σ-discrete, is implied by f being analytic when X is absolutely analytic (i.e. Souslin-$\mathcal{F}(X)$ in any complete metric space X into which it embeds). Frankiewicz [Fr] considers implications of the Axiom of Constructibility.

The following result provides a criterion for verifying that f is Baire.

Theorem 10.8 (Souslin criterion). *Let X and Y be Hausdorff topological groups with Y a K-analytic set. If $f : X \rightarrow Y$ has Souslin-$\mathcal{F}(X \times Y)$ graph, then f is Baire.*

Proof. Let $G \subseteq X \times Y$ be the graph of f which is Souslin-$\mathcal{F}(X \times Y)$. For F closed in Y, we have

$$f^{-1}(F) = \mathrm{pr}_X[G \cap (X \times F)],$$

which, by the Intersection Theorem, is the projection of a *Souslin-$\mathcal{F}(X \times Y)$* set. By the Projection Theorem, $f^{-1}(F)$ is Souslin-$\mathcal{F}(X)$. Closed sets have the Baire property by definition, so by Nikodym's Theorem $f^{-1}(F)$ has the Baire property. $\square$

Before stating our next theorem we recall a classical result in the sharper form resulting from the enabling condition (ii) above.

Theorem 10.9 (Banach-Mehdi Theorem – [Ban-T] 1.3.4, p. 40, [Meh], [HJ] Th. 2.2.12 p. 348, or [BOst14])**.** *A Baire-continuous homomorphism $f : X \to Y$ between complete metric groups is continuous, when Y is separable, or has base of cardinality less than the continuum.*

The Souslin criterion and the next theorem together have as immediate corollary the classical Souslin-graph Theorem; in this connection recall (see the corollary of [HJ] Th. 2.3.6 p. 355) that a normed group which is Baire and analytic is Polish.

Theorem 10.10 (Baire Homomorphism Theorem). *Let X and Y be topological groups with Y a K-analytic group and X non-meagre. If $f : X \to Y$ is a Baire homomorphism, then f is continuous.*

Proof. Here we refer to the proof in [Jay-Rog] §2.10 of the Souslin-graph theorem; that proof may be construed as having two steps: one establishing the Souslin criterion (Th. 10.8 above), the other the Baire homomorphism theorem. □

Corollary 1 (Souslin-graph Theorem, Schwartz [Schw], cf. [Jay-Rog] p.50**)**. *Let X and Y be topological groups with Y a K-analytic group and X non-meagre. If $f : X \to Y$ is a homomorphism with Souslin-$\mathcal{F}(X \times Y)$ graph, then f is continuous.*

Proof. This follows from Theorems 10.8 and 10.10. □

Corollary 2 (Generalized Jones Theorem: Thinned Souslin-graph Theorem). *Let X and Y be topological groups with X non-meagre and Y a K-analytic set. Let S be a K-analytic set spanning X and $f : X \to Y$ a homomorphism with restriction to S continous on S. Then f is continuous.*

Proof. Since f is continuous on S, the graph $\{(x, y) \in S \times Y : y = f(x)\}$ is closed in $S \times Y$ and so is K-analytic by [Jay-Rog] Th. 2.5.3. Now $y = f(x)$ iff, for some $n \in \mathbb{N}$, there is $(y_1, ..., y_n) \in Y^n$ and $(s_1, ..., s_n) \in S^n$ such that $x = s_1 \cdot ... \cdot s_n$, $y = y_1 \cdot ... \cdot y_n$, and, for $i = 1, .., n$, $y_i = f(s_i)$. Thus $G := \{(x, y) : y = f(x)\}$ is K-analytic. Formally,

$$G = \mathrm{pr}_{X\times Y}\left[\bigcup_{n\in\mathbb{N}}\left[M_n \cap (X \times Y \times S^n \times Y^n) \cap \bigcap_{i\leq n} G_{i,n}\right]\right],$$

where

$$M_n := \{(x, y, s_1, ..., s_n, y_1, ..., y_n) : y = y_1 \cdot ... \cdot y_n \text{ and } x = s_1 \cdot ... \cdot s_n\},$$

and

$$G_{i,n} := \{(x, y, s_1, ..., s_n, y_1, ..., y_n) \in X \times Y \times X^n \times Y^n : y_i = f(s_i)\}, \text{for } i = 1, ..., n.$$

Here each set M_n is closed and each $G_{i,n}$ is K-analytic. Hence, by the Intersection and Projection Theorems, the graph G is K-analytic. By the Souslin-graph theorem f is thus continuous. □

This is a new proof of the Jones Theorem. We now consider results for the more special normed group context. Here again one should note the corollary of [HJ] Th. 2.3.6 p. 355 that a normed group which is Baire and analytic is Polish.

Our next result has a proof which is a minor adaptation of the proof in [BoDi]. We recall that a Hausdorff topological space is paracompact ([Eng] Ch. 5, or [Kel] Ch. 6, especially Problem Y) if every open cover has a locally finite open refinement and that (i) Lindelöf spaces and (ii) metrizable spaces are paracompact. Paracompact spaces are normal, hence topological groups need not be paracompact, as exemplified again by the example due to Oleg Pavlov [Pav] quoted earlier or by the example of van Douwen [vD] (see also [Com] Section 9.4 p. 1222); however, L. G. Brown [Br-2] shows that a locally complete group is paracompact (and this includes the locally compact case, cf. [Com] Th. 2.9 p. 1161). The assumption of paracompactness is thus natural.

Theorem 10. 11 (Second Generalized Measurable Kestelman-Borwein-Ditor Theorem – cf. Th. 6.6**).** *Let G be a paracompact topological group equipped with a locally-finite, inner regular Borel measure m (Radon measure) which is left-invariant, resp. right-invariant, (for example, G locally compact, equipped with a Haar measure).*

If A is a (Borel) measurable set with $0 < m(A) < \infty$ and $z_n \to e$, then, for m-almost all $a \in A$, there is an infinite set $\mathbb{M}_a$ such that the corresponding right-translates, resp. left-translates, of z_n are in A, i.e., in the first case

$$\{z_n a : n \in \mathbb{M}_a\} \subseteq A.$$

Proof. Without loss of generality we consider right-translation of the sequence $\{z_n\}$. Since G is paracompact, it suffices to prove the result for

A open and of finite measure. By inner-regularity A may be replaced by a σ-compact subset of equal measure. It thus suffices to prove the theorem for K compact with $m(K) > 0$ and $K \subseteq A$. Define a decreasing sequence of compact sets $T_k := \bigcup_{n \geq k} z_n^{-1} K$, and let $T = \bigcap_k T_k$. Thus $x \in T$ iff, for some infinite $\mathbb{M}_x$,

$$z_n x \in K \text{ for } m \in \mathbb{M}_x,$$

so that T is the set of 'translators' x for the sequence $\{z_n\}$. Since K is closed, for $x \in T$, we have $x = \lim_{n \in \mathbb{M}_x} z_n x \in K$; thus $T \subseteq K$. Hence, for each k,

$$m(T_k) \geq m(z_k^{-1} K) = m(K),$$

by left-invariance of the measure. But, for some n, $T_n \subseteq A$. (If $z_n^{-1} k_n \notin A$ on an infinite set $\mathbb{M}$ of n, then since $k_n \to k \in K$ we have $z_n^{-1} k_n \to k \in A$, but $k = \lim z_n^{-1} k_n \notin A$, a contradiction since A is open.) So, for some n, $m(T_n) < \infty$, and thus $m(T_k) \to m(T)$. Hence $m(K) \geq m(T) \geq m(K)$. So $m(K) = m(T)$ and thus almost all points of K are translators. □

Remark. It is quite consistent to have the measure left-invariant and the metric right-invariant.

Theorem 10.12 (Analytic Dichotomy Lemma on Spanning). *Let G be a connected, normed group. Suppose that an analytic set $T \subseteq G$ spans a set of positive measure or a non-meagre set. Then T spans G.*

Proof. In the category case, the result follows from the Banach-Kuratowski Dichotomy, Th. 5.8 ([Ban-G, Satz 1], [Kur-1, Ch. VI. 13. XII], [Kel, Ch. 6 Prob. P p. 211]) by considering S, the subgroup generated by T; since T is analytic, S is analytic and hence Baire, and, being non-meagre, is clopen and hence all of G, as the latter is a connected group.

In the measure case, by the Steinhaus Theorem, Th. 5.7 ([St], [BGT, Th. 1.1.1], [BOst3]), T^2 has non-empty interior, hence is non-meagre. The result now follows from the category case. □

Our next lemma follows directly from Choquet's Capacitability Theorem [Choq] (see especially [Del2, p. 186], and [Kech, Ch. III 30.C]). For completeness, we include the brief proof. Incidentally, the argument we employ goes back to Choquet's theorem, and indeed further, to [ROD] (see e.g. [Del1, p. 43]).

Theorem 10.13 (Compact Contraction Lemma). *In a normed group carrying a Radon measure, for T analytic, if $T \cdot T$ has positive Radon measure, then for some compact subset S of T, $S \cdot S$ has positive measure.*

Proof. We present a direct proof (see below for our original inspiration in Choquet's Theorem). As T^2 is analytic, we may write ([Jay-Rog]) $T^2 = h(H)$, for some continuous h and some $\mathcal{K}_{\sigma\delta}$ subset of the reals, e.g. the set H of the irrationals, so that $H = \bigcap_i \bigcup_j d(i,j)$, where $d(i,j)$ are compact and, without loss of generality, the unions are each increasing: $d(i,j) \subseteq d(i,j+1)$. The map $g(x,y) := xy$ is continuous and hence so is the composition $f = g \circ h$. Thus $T \cdot T = f(H)$ is analytic. Suppose that $T \cdot T$ is of positive measure. Hence, by the capacitability argument for analytic sets ([Choq], or [Si, Th.4.2 p. 774], or [Rog1, p. 90], there referred to as an 'Increasing sets lemma'), for some compact set A, the set $f(A)$ has positive measure. Indeed if $|f(H)| > \eta > 0$, then the set A may be taken in the form $\bigcap_i d(i,j_i)$, where the indices j_i are chosen inductively, by reference to the increasing union, so that $|f[H \cap \bigcap_{i<k} d(i,j_i)]| > \eta$, for each k. (Thus $A \subseteq H$ and $f(A) = \bigcap_i f[H \cap \bigcap_{i<k} d(i,j_i)]$ has positive measure, cf. [EKR].)

The conclusion follows as $S = h(A)$ is compact and $S \cdot S = g(S) = f(A)$. □

Note. The result may be deduced indirectly from the Choquet Capacitability Theorem by considering the capacity $I : G^2 \to \mathbb{R}$, defined by $I(X) = m(g(X))$, where, as before, $g(x,y) := xy$ is continuous and m denotes a Radon measure on G (on this point see [Del2, Section 1.1.1, p. 186]). Indeed, the set T^2 is analytic ([Rog2, Section 2.8, p. 37-41]), so $I(T^2) = \sup I(K^2)$, where the supremum ranges over compact subsets K of T. Actually, the Capacitability Theorem says only that $I(T^2) = \sup I(K_2)$, where the supremum ranges over compact subsets K_2 of T^2, but such a set may be embedded in K^2 where $K = \pi_1(K) \cup \pi_2(K)$, with π_i the projections onto the axes of the product space.

Corollary 10.14. *For T analytic and $\varepsilon_i \in \{\pm 1\}$, if $T^{\varepsilon_1} \cdot ... \cdot T^{\varepsilon_d}$ has positive measure (measure greater than η) or is non-meagre, then for some compact subset S of T, the compact set $K = S^{\varepsilon_1} \cdot ... \cdot S^{\varepsilon_d}$ has $K \cdot K$ of positive measure (measure greater than η).*

Proof. In the measure case the same approach may be used based now on the continuous function $g(x_1, ..., x_d) := x_1^{\varepsilon_1} \cdot ... \cdot x_d^{\varepsilon_d}$, ensuring that K

is of positive measure (measure greater than η). In the category case, if $T' = T^{\varepsilon_1} \cdot ... \cdot T^{\varepsilon_d}$ is non-meagre then, by the Steinhaus Theorem ([St], or [BGT, Cor. 1.1.3]), $T' \cdot T'$ has non-empty interior. The measure case may now be applied to T' in lieu of T. (Alternatively one may apply the Pettis-Piccard Theorem, Th. 5.3, as in the Analytic Dichotomy Lemma, Th. 10.12.) □

Theorem 10.15 (Compact Spanning Approximation). *For T analytic in X, if the span of T is non-null or is non-meagre, then there exists a compact subset of T which spans X.*

Proof. If T is non-null or non-meagre, then T spans all the reals (by the Analytic Dichotomy Lemma); then for some $\varepsilon_i \in \{\pm 1\}$, $T^{\varepsilon_1} \cdot ... \cdot T^{\varepsilon_d}$ has positive measure/ is non-meagre. Hence for some K compact $K^{\varepsilon_1} \cdot ... \cdot K^{\varepsilon_d}$ has positive measure/ is non-meagre. Hence K spans some and hence all reals. □

Theorem 10.16 (Analytic Covering Lemma – [Kucz, p. 227], cf. [Jones2, Th. 11]). *Given normed groups G and H, and T analytic in G, let $f : G \to H$ have continuous restriction $f|T$. Then T is covered by a countable family of bounded analytic sets on each of which f is bounded.*

Proof. For $k \in \omega$ define $T_k := \{x \in T : ||f(x)|| < k\} \cap B_k(e_G)$. Now $\{x \in T : ||f(x)|| < k\}$ is relatively open and so takes the form $T \cap U_k$ for some open subset U_k of G. The Intersection Theorem shows this to be analytic since U_k is an $\mathcal{F}_\sigma$ set and hence Souslin-$\mathcal{F}$. □

Theorem 10.17 (Expansion Lemma – [Jones2, Th. 4], [Kom2, Th. 2], and [Kucz, p. 215]). *Suppose that S is Souslin-$\mathcal{H}$, i.e. of the form*

$$S = \bigcup_{\alpha \in \omega^\omega} \cap_{n=1}^{\infty} H(\alpha|n),$$

with each $H(\alpha|n) \in \mathcal{H}$, for some family of analytic sets $\mathcal{H}$ on which f is bounded. If S spans the normed group G, then, for each n, there are sets $H_1, ..., H_k$ each of the form $H(\alpha|n)$, such that for some integers $r_1, ..., r_k$

$$T = H_1 \cdot ... \cdot H_k$$

has positive measure/ is non-meagre, and so $T \cdot T$ has non-empty interior.

Proof. For any $n \in \omega$ we have

$$S \subseteq \bigcup_{\alpha \in \omega^\omega} H(\alpha|n).$$

Enumerate the countable family $\{H(\alpha|n) : \alpha \in \omega^n\}$ as $\{T_h : h \in \omega\}$. Since S spans G, we have

$$G = \bigcup_{h \in \omega} \bigcup_{\mathbf{k} \in \mathbb{N}^h} (T_{k_1} \cdot \ldots \cdot T_{k_h}).$$

As each T_k is analytic, so too is the continuous image

$$T_{k_1} \cdot \ldots \cdot T_{k_h},$$

which is thus measurable. Hence, for some $h \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^h$ the set

$$T_{k_1} \cdot \ldots \cdot T_{k_h}$$

has positive measure/ is non-meagre. □

Definition. We say that S is a *pre-compact set* if its closure is compact. We will say that f is a *pre-compact function* if $f(S)$ is pre-compact for each pre-compact set S.

Theorem 10.18 (Jones-Kominek Analytic Automaticity Theorem for Metric Groups). *Let be G be either a non-meagre normed group, or a group supporting a Radon measure, and let H be K-analytic (hence Lindelöf, and so second countable in our metric setting). Let $h : G \to H$ be a homomorphism between metric groups and let T be an analytic set in G which finitely generates G.*

(i) (Jones condition) *If h is continuous on T, then h is continuous.*

(ii) (Kominek condition) *If h is pre-compact on T, then h is precompact.*

Proof. As in the Analytic Covering Lemma (Th. 10.16), write

$$T = \bigcup_{k \in \mathbb{N}} T_k.$$

(i) If h is not continuous, suppose that $x_n \to x_0$ but $h(x_n)$ does not converge to $h(x_0)$. Since

$$G = \bigcup_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} T_k^{(m)},$$

G is a union of analytic sets and hence analytic ([Jay-Rog] Th. 2.5.4 p. 23). Now, for some m, k the m-span $T_k^{(m)}$ is non-meagre, as is the m-span $S_k^{(m)}$ of some compact subset $S_k \subseteq T_k$. So for some shifted subsequence $tx_n \to tx_0$, where t and x_0 lie in $S_k^{(m)}$. Thus there is an infinite set $\mathbb{M}$ such that, for $n \in \mathbb{M}$,

$$tx_n = t_n^1 ... t_n^m \text{ with } t_n^i \in S_k.$$

W.l.o.g., as S_k is compact,

$$t_n^{(i)} \to t_0^{(i)} \in S_k \subset T,$$

and so

$$tx_n = t_n^1 ... t_n^m \to t_0^1 ... t_0^m = tx_0 \text{ with } t_0^i \in S_k \subset T.$$

Hence, as $t_n^i \to t_0^i \subset T$, we have, for $n \in \mathbb{M}$,

$$\begin{aligned} h(t)h(x_n) &= h(tx_n) = h(t_n^1 ... t_n^m) = h(t_n^1) ... h(t_n^m) \\ &\to h(t_0^1) ... h(t_0^m) = h(t_0^1 ... t_0^m) \\ &= h(tx_0) = h(t)h(x_0). \end{aligned}$$

Thus

$$h(x_n) \to h(x_0),$$

a contradiction.

(ii) If $\{h(x_n)\}$ is not precompact with $\{x_n\}$ precompact, by the same argument, for some $S_k^{(n)}$ and some infinite set $\mathbb{M}$, we have $tx_n = t_n^1 ... t_n^m$ and $t_n^i \to t_0^i \subset T$, for $n \in \mathbb{M}$. Hence $h(tx_n) = h(t)h(x_n)$ is precompact and so $h(x_n)$ is precompact, a contradiction. □

The following result connects the preceeding theorem to Darboux's Theorem, that a locally bounded additive function on the reals is continuous ([Dar], or [AD]).

Definition. Say that a homomorphism between normed groups is $\mathbb{N}$-*homogeneous* if $||f(x^n)|| = n||f(x)||$, for any x and $n \in \mathbb{N}$. (cf. Section 2 where $\mathbb{N}$-*homogeneous* norms were considered, for which homomorphisms are automatically $\mathbb{N}$-*homogeneous*). Thus any homomorphism into the additive reals is $\mathbb{N}$-homogeneous. More generally, say that the norm is $\mathbb{N}$-*subhomogeneous* if there are constants κ_n with $\kappa_n \to \infty$ such that for all elements z of the group

$$\kappa_n ||z|| \leq ||z^n||,$$

or equivalently

$$||z^{1/n}|| \leq \frac{1}{\kappa_n}||z||.$$

Thus $z^{1/n} \to e$; a related condition was considered by McShane in [McSh] (cf. the Eberlein-McShane Theorem, Th. 9.1). In keeping with the convention of functional analysis (appropriately to our usage of norm) the next result refers to a locally bounded homomorphism as bounded.

Theorem 10.19 (Generalized Darboux Theorem – [Dar]). *A bounded homomorphism from a normed group to an* $\mathbb{N}$*-subhomogeneous normed group is continuous. In particular, a bounded, additive function on* $\mathbb{R}$ *is continuous.*

Proof. Suppose that $f : G \to H$ is a homomorphism to a normed $\mathbb{N}$-subhomogeneous group H; thus $||f(x^n)|| \geq \kappa_n||f(x)||$, for any $x \in G$ and $n \in \mathbb{N}$. Suppose that f is bounded by M and, for $||x|| < \eta$, we have

$$||f(x)|| < M.$$

Let $\varepsilon > 0$ be given. Choose N such that $\kappa_N > M/\varepsilon$, i.e. $M/\kappa_N < \varepsilon$. Now $x \to x^N$ is continuous, hence there is $\delta = \delta_N(\eta) > 0$ such that, for $||x|| < \delta$,

$$||x^N|| < \eta.$$

Consider x with $||x|| < \delta_N(\eta)$. Then $\kappa_N||f(x)|| \leq ||f(x)^N|| = ||f(x^N)|| < M$. So for x with $||x|| < \delta_N(\eta)$ we have

$$||f(x)|| < M/\kappa_N < \varepsilon,$$

proving continuity at e. $\square$

Compare [HJ] Th 2.4.9 p. 382.

The Main Theorem of [BOst7] may be given a combinatorial restatement in the group setting. We need some further definitions.

Definition. For G a metric group, let $\mathcal{C}(G) = \mathcal{C}(\mathbb{N}, G) := \{\mathbf{x} \in G^{\mathbb{N}} : \mathbf{x}$ is convergent$\}$. For $x \in \mathcal{C}(G)$ we write

$$L(\mathbf{x}) = \lim_n x_n.$$

We make $\mathcal{C}(G)$ into a group by setting

$$\mathbf{x} \cdot \mathbf{y} := \langle x_n y_n : n \in \mathbb{N}\rangle.$$

Thus $\mathbf{e} = \langle e_G \rangle$ and $\mathbf{x}^{-1} = \langle x_n^{-1} \rangle$. We identify G with the subgroup of constant sequences, that is

$$T = \{\langle g : n \in \mathbb{N} \rangle : g \in G\}.$$

The natural action of G or T on $\mathcal{C}(G)$ is then $t\mathbf{x} := \langle tx_n : n \in \mathbb{N} \rangle$. Thus $\langle g \rangle = g\mathbf{e}$, and then $t\mathbf{x} = t\mathbf{e} \cdot \mathbf{x}$.

Definition. For G a group, a set $\mathcal{G}$ of convergent sequences $\mathbf{u} = \langle u_n : n \in \mathbb{N} \rangle$ in $c(G)$ is a *G-ideal in the sequence space* $\mathcal{C}(G)$ if it is a subgroup closed under the mutiplicative action of G, and will be termed *complete* if it is closed under subsequence formation. That is, a complete G-ideal in $\mathcal{C}(G)$ satisfies

(i) $\mathbf{u} \in \mathcal{G}$ implies $t\mathbf{u} = \langle tu_n \rangle \in \mathcal{G}$, for each t in G,
(ii) $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ implies that $\mathbf{u}\mathbf{v}^{-1} \in \mathcal{G}$,
(iii) $\mathbf{u} \in \mathcal{G}$ implies that $\mathbf{u}_{\mathbb{M}} := \{u_m : m \in \mathbb{M}\} \in \mathcal{G}$ for every infinite $\mathbb{M}$.

If $\mathcal{G}$ isatisfies (i) and $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ implies only that $\mathbf{u}\mathbf{v} \in \mathcal{G}$, we say that $\mathcal{G}$ is a *G-subideal* in $\mathcal{C}(G)$.

Remarks.

0. In the notation of (iii) above, if $\mathcal{G}$ is merely an ideal then $\mathcal{G}^* = \{\mathbf{u}_{\mathbb{M}} :$ for $\mathbf{u} \in t$ and $\mathbb{M} \subset \mathbb{N}\}$ is a complete G-ideal; indeed $t\mathbf{u}_{\mathbb{M}} = (t\mathbf{u})_{\mathbb{M}}$ and $\mathbf{u}_{\mathbb{M}}\mathbf{v}_{\mathbb{M}}^{-1} = (\mathbf{u}\mathbf{v}^{-1})_{\mathbb{M}}$ and $\mathbf{u}_{\mathbb{M}\mathbb{M}'} = \mathbf{u}_{\mathbb{M}'}$ for $\mathbb{M}' \subset \mathbb{M}$.

1. We speak of a Euclidean sequential structure when G is the vector space $\mathbb{R}^d$ regarded as an additive group.

2. The conditions (i) and (ii) assert that $\mathcal{G}$ is similar in structure to a left-ideal, being closed under multiplication by G and a subgroup of $\mathcal{C}(G)$.

3. We refer only to the combinatorial properties of $\mathcal{C}(G)$; but one may give $\mathcal{C}(G)$ a pseudo-norm by setting

$$||x||_c := d^G(Lx, e) = ||Lx||, \text{ where } Lx := \lim x_n.$$

The corresponding pseudo-metric is

$$d(x, y) := \lim d^G(x_n, y_n) = d^G(Lx, Ly).$$

We may take equivalence of sequences with identical limit; then $\mathcal{C}(G)^\sim$ becomes a normed group. However, in our theorem below we do not wish to refer to such an equivalence.

Definitions. For a family $\mathcal{F}$ of functions from G to H, we denote by $\mathcal{F}(T)$ the family $\{f|T : f \in \mathcal{F}\}$ of functions in $\mathcal{F}$ restricted to $T \subseteq G$. Let us denote a convergent sequence with limit x_0, by $\{x_n\} \to x_0$. We say the property $\mathcal{Q}$ of functions (property being regarded set-theoretically, i.e. as a family of functions from G to H) is *sequential on* T if

$$f \in \mathcal{Q} \text{ iff } (\forall \{x_n : n > 0\} \subseteq T)[(\{x_n\} \to x_0) \implies f|\{x_n : n > 0\} \in \mathcal{Q}(\{x_n : n > 0\})].$$

If we further require the limit point to be enumerated in the sequence, we call $\mathcal{Q}$ *completely sequential on* T if

$$f \in \mathcal{Q} \text{ iff } (\forall \{x_n\} \subseteq T)[(\{x_n\} \to x_0) \implies f|\{x_n\} \in \mathcal{Q}(\{x_n\})].$$

Our interest rests on properties that are completely sequential; our theorem below contains a condition referring to completely sequential properties, that is, the condition is required to hold on convergent sequences with limit included (so on a compact set), rather than on arbitrary sequences.

Note that if $\mathcal{Q}$ is (completely) sequential then $f|\{x_n\} \in \mathcal{Q}(\{x_n\})$ iff $f|\{x_n : n \in \mathbb{M}\} \in \mathcal{Q}(\{x_n : n \in \mathbb{M}\})$, for every infinite $\mathbb{M}$.

Definition. Let $h : G \to H$, with G, H metric groups. Say that a sequence $\mathbf{u} = \{u_n\}$ is $\mathcal{Q}$-*good* for h if

$$h|\{u_n\} \in \mathcal{Q}|\{u_n\},$$

and put

$$\mathcal{G}_{h\mathcal{Q}} = \{\mathbf{u} : h|\{u_n\} \in \mathcal{Q}|\{u_n\}\}.$$

If $\mathcal{Q}$ is completely sequential, then $\mathbf{u}$ is $\mathcal{Q}$-good for h iff every subsequence of $\mathbf{u}$ is $\mathcal{Q}$-good for h, so that $\mathcal{G}_{h\mathcal{Q}}$ is a G-ideal iff it is a complete G-ideal. One then has:

Lemma 10.20. *If $\mathcal{Q}$ is completely sequential and $\mathcal{F}$ preserves $\mathcal{Q}$ under shift and multiplication and division on compacts, then $\mathcal{G}_{h\mathcal{Q}}$ for $h \in \mathcal{F}$ is a G-ideal.*

Theorem 10.21 (Analytic Automaticity Theorem - combinatorial form). *Suppose that functions of $\mathcal{F}$ having $\mathcal{Q}$ on G have $\mathcal{P}$ on G, where $\mathcal{Q}$ is a property of functions from G to H that is completely sequential on G.*

Suppose that, for all $h \in \mathcal{F}$, $\mathcal{G}_{h\mathcal{Q}}$, the family of $\mathcal{Q}$-good sequences is a G-ideal. Then, for any analytic set T spanning G, functions of $\mathcal{F}$ having $\mathcal{Q}$ on T have $\mathcal{P}$ on G.

This theorem is applied with $G=\mathbb{R}^d$ and $H=\mathbb{R}$ in [BOst6] to subadditive functions, convex functions, and to regularly varying functions (defined on $\mathbb{R}^d$) to derive automatic properties such as automatic continuity, automatic local boundedness and automatic uniform boundedness.

11 Duality in normed groups

In this section – to distinguish two contexts – we use the generic notation of S for a group with metric d^S; recall from Section 3 that $Auth(S)$ denotes the auto-homeomorphisms of S; $\mathcal{H}(S)$ denotes the bounded elements of $Auth(S)$. We write $\mathcal{A}\subseteq\mathcal{H}(S)$ for a subgroup of auto-homeomorphisms of S. We work in the category of normed groups. However, by specializing to $\mathcal{A}=\mathcal{H}_u(S)$, the homeomorphisms that are bi-uniformly continuous (relative to d^S), we can regard the development as also taking place inside the category of topological groups, by Th. 3.11. We assume that $\mathcal{A}$ is metrized by the supremum metric

$$d^T(t_1,t_2)=\sup_{s\in S} d^X(t_1(s),t_2(s)).$$

Note that $e_{\mathcal{A}}=id_S$. The purpose of this notation is to embrace the two cases:
(i) $S=X$ and $\mathcal{A}=\mathcal{H}_u(X)$, and
(ii) $S=\mathcal{H}_u(X)$ and $\mathcal{A}=\mathcal{H}_u(\mathcal{H}_u(X))$.

In what follows, we regard the group $\mathcal{H}_u(X)$ as the topological (uniform) dual of X and verify that (X,d^X) is embedded in the second dual $\mathcal{H}_u(\mathcal{H}_u(X))$. As an application one may use this duality to clarify, in the context of a non-autonomous differential equation with initial conditions, the link between its solutions trajectories and flows of its varying 'coefficient matrix'. See [Se1] and [Se2], which derive the close relationship for a general non-autonomous differential equation $u'=f(u,t)$ with $u(0)=x\in X$, between its trajectories in X and local flows in the function space Φ of translates f_t of f (where $f_t(x,s)=f(x,t+s)$).

One may alternatively capture the topological duality as algebraic complementarity – see [Ost-knit] for details. A summary will suffice here. One first considers the commutative diagram below where initially the maps are only *homeomorphisms* (herein $T\subseteq\mathcal{H}_u(X)$ and $\Phi^T(t,x)=(t,tx)$ and $\Phi^X(x,t)=(t,xt)$ are embeddings). Then one extends the diagram to a diagram of *isomorphisms,* a change facilitated by forming the direct product group $G:=T\times X$. Thus $G=T_GX_G$ where T_G and X_G are normal subgroups, commuting elementwise, and isomorphic respectively to T and X; moreover, the subgroup T_G, acting multiplicatively on X_G, represents the T-flow on X and simultaneously the multiplicative action of X_G on G represents the X-flow on $T_X = \{t_x : t \in T, x \in X\}$, the group of right-translates of T, where $t_x(u) = \theta_x(t)(u) = t(ux)$. If G has an invariant metric d^G, and T_G and X_G are now regarded as groups of translations on G, then they may be metrized by the supremum metric $\hat{d}^G$, whereupon each is isometric to itself as subgroup of G. Our approach here suffers a loss of elegance, by dispensing with G, but gains analytically by working directly with d^X and $\hat{d}^X$.

$$\begin{array}{ccc} (t,x) & \stackrel{\Phi^T}{\longleftrightarrow} & (t,tx) \\ \updownarrow & & \updownarrow \\ (x,t) & \stackrel{\Phi^X}{\longleftrightarrow} & (t,xt) \end{array}$$

Here the two vertical maps may, and will, be used as identifications, since $(t,tx) \leftrightarrows (t,x) \leftrightarrows (t,xt)$ are bijections (more in fact is true, see [Ost-knit]).

Definitions. Let X be a topological group with right-invariant metric d^X. We define for $x \in X$ a map $\xi_x : \mathcal{H}(X) \to \mathcal{H}(X)$ by putting

$$\xi_x(s)(z) = s(\lambda_x^{-1}(z)) = s(x^{-1}z), \text{ for } s \in \mathcal{H}_u(X), z \in X.$$

We set

$$\Xi := \{\xi_x : x \in X\}.$$

By restriction we may also write $\xi_x : \mathcal{H}_u(X) \to \mathcal{H}_u(X)$.

Proposition 11.1. *Under composition Ξ is a group of isometries of $\mathcal{H}_u(X)$ isomorphic to X.*

Proof. The identity is given by $e_\Xi = \xi_e$, where $e = e_X$. Note that

$$\xi_x(e_S)(e_X) = x^{-1},$$

so the mapping $x \to \xi_x$ from X to Ξ is bijective. Also, for $s \in \mathcal{H}(X)$,

$$\begin{aligned}(\xi_x \circ \xi_y(s))(z) &= \xi_x(\xi_y(s))(z) = (\xi_y(s))(x^{-1}z) \\ &= s(y^{-1}x^{-1}z) = s((xy)^{-1}z) = \xi_{xy}(s)(z),\end{aligned}$$

so ξ is an isomorphism from X to Ξ and so $\xi_x^{-1} = \xi_{x^{-1}}$.

For x fixed and $s \in \mathcal{H}_u(X)$, note that by Lemma 3.8 and Cor. 3.6 the map $z \to s(x^{-1}z)$ is in $\mathcal{H}_u(X)$. Furthermore

$$d_{\mathcal{H}}(\xi_x(s), \xi_x(t)) = \sup_z d^X(s(x^{-1}z), t(x^{-1}z)) = \sup_y d^X(s(y), t(y)) = d_{\mathcal{H}}(s, t),$$

so ξ_x is an isometry, and hence is continuous. ξ_x is indeed an auto-homeomorphism of $\mathcal{H}_u(X)$, as $\xi_{x^{-1}}$ is the continuous inverse of ξ_x. □

Remark. The definition above lifts the isomorphism $\lambda : X \to Tr_L(X)$ to $\mathcal{H}_u(X)$. If $T \subseteq \mathcal{H}_u(X)$ is λ-invariant, we may of course restrict λ to operate on T. Indeed, if $T = Tr_L(X)$, we then have $\xi_x(\lambda_y)(z) = \lambda_y \lambda_x^{-1}(z)$, so $\xi_x(\lambda_y) = \lambda_{yx^{-1}}$.

In general it will not be the case that $\xi_x \in \mathcal{H}_u(\mathcal{H}_u(X))$, unless d^X is bounded. Recall that

$$||x||_\infty := \sup_{s \in \mathcal{H}(X)} ||x||_s = \sup_{s \in \mathcal{H}(X)} d_s^X(x, e) = \sup_{s \in \mathcal{H}(X)} d^X(s(x), s(e)).$$

By contrast we have

$$||f||_\infty = \sup_z \sup_g d_g^X(f(z), z).$$

However, for $f(z) = \lambda_x(z) := xz$, putting $s = g \circ \rho_z$ brings the the two formulas into alignment, as

$$||\lambda_x||_\infty = \sup_z \sup_g d^X(g(xz), g(z)) = \sup_z \sup_g d^X(g(\rho_z(x)), g(\rho_z(e))).$$

This motivates the following result.

Proposition 11.2. *The subgroup $H_X := \{x \in X : ||x||_\infty < \infty\}$ equipped with the norm $||x||_\infty$ embeds isometrically under ξ into $\mathcal{H}_u(\mathcal{H}_u(X))$ as*

$$\Xi_H := \{\xi_x : x \in H_X\}.$$

Proof. Writing $y = x^{-1}z$ or $z = xy$, we have

$$\begin{aligned} d_{\mathcal{H}}(\xi_x(s), s) &= \sup_{z \in X} d^X(s(x^{-1}z), s(z)) = \sup_{y \in X} d^X(s(y), s(xy)) \\ &= \sup_{y \in X} d_s^X(\rho_y e, \rho_y x) = \sup_y d_{s\text{-}y}^X(e, x). \end{aligned}$$

Hence

$$||\xi_x||_{\mathcal{H}} = \sup_{s\in\mathcal{H}(X)} d_{\mathcal{H}}(\xi_x(s), s) = ||\lambda_x||_\infty = \sup_{s\in\mathcal{H}(X)} \sup_{y\in X} d_s^X(y, xy) = ||x||_\infty.$$

Thus for $x \in H_X$ the map ξ_x is bounded over $\mathcal{H}_u(X)$ and hence is in $\mathcal{H}_u(\mathcal{H}_u(X))$. $\square$

The next result adapts ideas of Section 3 on the Lipschitz property in $\mathcal{H}_u$ (Th. 3.20) to the context of ξ_x and refers to the inverse modulus of continuity $\delta(s)$ which we recall:

$$\delta(g) = \delta_1(g) := \sup\{\delta > 0 : d^X(g(z), g(z')) \le 1, \text{ for all } d^X(z, z') \le \delta\}.$$

Proposition 11.3 (Further Lipschitz properties of $\mathcal{H}_u$). *Suppose that the normed group X has a vanishingly small global word-net. Then for $x, z \in X$ and $s \in \mathcal{H}_u(X)$ the s-z-shifted norm (recalled below) satisfies*

$$||x||_{s\text{-}z} := d_{s\text{-}z}^X(x, e) = d^X(s(z), s(xz)) \le 2||x||/\delta(s).$$

Hence

$$||\xi_e||_{\mathcal{H}(\mathcal{H}_u(X))} = \sup_{s\in\mathcal{H}_u(X)} \sup_{z\in X} ||e||_{s\text{-}z} = 0,$$

and so $\xi_e \in \mathcal{H}(\mathcal{H}_u(X))$. Furthermore, if $\{\delta(s) : s \in \mathcal{H}_u(X)\}$ is bounded away from 0, then

$$\begin{aligned} ||\xi_x||_{\mathcal{H}(\mathcal{H}_u(X))} &= \sup_{s\in\mathcal{H}_u(X)} d_{\mathcal{H}(X)}(\xi_x(s), s) = \sup_{s\in\mathcal{H}_u(X)} \sup_{z\in X} d^X(s(x^{-1}z), s(z)) \\ &\le 2||x||/\inf\{\delta(s) : s \in \mathcal{H}_u(X)\}, \end{aligned}$$

and so $\xi_x \in \mathcal{H}(\mathcal{H}_u(X))$.

In particular this is so if in addition X is compact.

Proof. Writing $y = x^{-1}z$ or $z = xy$, we have

$$d_{\mathcal{H}}(\xi_x(s), s) = \sup_{z\in X} d^X(s(x^{-1}z), s(z)) = \sup_{y\in X} d^X(s(y), s(xy)).$$

Fix s. Since s is uniformly continuous, $\delta = \delta(s)$ is well-defined and

$$d(s(z'), s(z)) \le 1,$$

for z, z' such that $d(z, z') < \delta$. In the definition of the word-net take $\varepsilon < 1$. Now suppose that $w(x) = w_1...w_{n(x)}$ with $||z_i|| = \frac{1}{2}\delta(1 + \varepsilon_i)$ and $|\varepsilon_i| < \varepsilon$, where $n(x) = n(x, \delta)$ satisfies

$$1 - \varepsilon \leq \frac{n(x)\delta}{||x||} \leq 1 + \varepsilon.$$

Put $z_0 = z$, for $0 < i < n(x)$

$$z_{i+1} = z_i w_i,$$

and $z_{n(x)+1} = x$; the latter is within δ of x. As

$$d(z_i, z_{i+1}) = d(e, w_i) = ||w_i|| < \delta,$$

we have

$$d(s(z_i), s(z_{i+1})) \leq 1.$$

Hence

$$d(s(z), s(xz)) \leq n(x) + 1 < 2||x||/\delta.$$

The final assertion follows from the subadditivity of the Lipschitz norm (cf. Theorem 3.25). □

If $\{\delta(s) : s \in \mathcal{H}_u(X)\}$ is unbounded (i.e. the inverse modulus of continuity is unbounded), we cannot develop a duality theory. However, a comparison with the normed vector space context and the metrization of the translations $x \to t(z + x)$ of a linear map $t(z)$ suggests that, in order to metrize Ξ by reference to $\xi_x(t)$, we need to take account of $||t||$. Thus a natural metric here is, for any $\varepsilon \geq 0$, the magnification metric

$$d_T^\varepsilon(\xi_x, \xi_y) := \sup_{||t|| \leq \varepsilon} d^T(\xi_x(t), \xi_y(t)). \tag{5}$$

By Proposition 2.14 this is a metric; indeed with $t = e_{\mathcal{H}(X)} = id_X$ we have $||t|| = 0$ and, since d^X is assumed right-invariant, for $x \neq y$, we have with $z_{xy} = e$ that $d^X(x^{-1}z, y^{-1}z) = d^X(x^{-1}, y^{-1}) > 0$. The presence of the case $\varepsilon = 0$ is not fortuitous; see [Ost-knit] for an explanation via an isomorphism theorem. We trace the dependence on $||t||$ in Proposition 11.5 below. We refer to Gromov's notion [Gr1], [Gr2] of quasi-isometry under π, in which π is a mapping between spaces. In a first application we take π to be a self-homeomorphism, in particular a left-translation; in the second $\pi(x) = \xi_x(t)$

with t fixed is an evaluation map appropriate to a dual embedding. We begin with a theorem promised in Section 3.

Theorem 11.4 (Uniformity Theorem for Conjugation). *Let $\Gamma : G^2 \to G$ be the conjugation $\Gamma(g,x) := g^{-1}xg$.*
Under a bi-invariant Klee metric, for all a, b, g, h ,

$$d^G(a,b) - 2d^G(g,h) \leq d^G(gag^{-1}, hbh^{-1}) \leq 2d^G(g,h) + d^G(a,b),$$

and hence conjugation is uniformly continuous.

Proof. Referring to the Klee property, via the cyclic property we have

$$\begin{aligned} d^G(gag^{-1}, hbh^{-1}) &= ||gag^{-1}hb^{-1}h^{-1}|| = ||h^{-1}gag^{-1}hb^{-1}|| \\ &\leq ||h^{-1}g|| + ||ag^{-1}hb^{-1}|| \\ &\leq ||h^{-1}g|| + ||ab^{-1}|| + ||g^{-1}h||, \end{aligned}$$

for all a, b, yielding the right-hand side inequality. Then substitute $g^{-1}ag$ for a etc., g^{-1} for g etc., to obtain

$$d^G(a,b) \leq 2d^G(g^{-1}, h^{-1}) + d^G(gag^{-1}, hbh^{-1}).$$

This yields the left-hand side inequality, as d^G is bi-invariant and so

$$d^G(g^{-1}, h^{-1}) = \tilde{d}^G(g,h) = d^G(g,h). \qquad \square$$

Proposition 11.5 (Permutation metric). *For $\pi \in \mathcal{H}(X)$, let $d_\pi(x,y) := d^X(\pi(x), \pi(y))$. Then d_π is a metric, and*

$$d^X(x,y) - 2||\pi|| \leq d_\pi(x,y) \leq d^X(x,y) + 2||\pi||.$$

In particular, if d^X is right-invariant and $\pi(x)$ is the left-translation $\lambda_z(x) = zx$, then

$$d^X(x,y) - 2||z|| \leq d_z^X(x,y) = d^X(zx, zy) \leq d^X(x,y) + 2||z||.$$

Proof. By the triangle inequality,

$$d^X(\pi(x), \pi(y)) \leq d^X(\pi(x), x) + d^X(x,y) + d^X(y, \pi(y)) \leq 2||\pi|| + d^X(x,y).$$

Likewise,

$$\begin{aligned} d^X(x,y) &\le d^X(x,\pi(x)) + d^X(\pi(x),\pi(y)) + d^X(\pi(y),y) \\ &\le 2||\pi|| + d^X(\pi(x),\pi(y)). \end{aligned}$$

If $\pi(x) := zx$, then $||\pi|| = \sup d(zx,x) = ||z||$ and the result follows. □

Recall from Proposition 2.2 that for d a metric on a group X, we write $\tilde{d}(x,y) = d(x^{-1},y^{-1})$ for the *(inversion) conjugate metric*. The conjugate metric $\tilde{d}$ is left-invariant iff the metric d is right-invariant. Under such circumstances both metrics induce the same norm (since $d(e,x) = d(x^{-1},e)$, as we have seen above). In what follows note that $\xi_x^{-1} = \xi_{x^{-1}}$.

Theorem 11.6 (**Quasi-isometric duality**). *If the metric d^X on X is right-invariant and $t \in T \subset \mathcal{H}(X)$ is a subgroup, then*

$$\tilde{d}^X(x,y) - 2||t||_{\mathcal{H}(X)} \le d^T(\xi_x(t),\xi_y(t)) \le \tilde{d}^X(x,y) + 2||t||_{\mathcal{H}(X)},$$

and hence, for each $\varepsilon \ge 0$, the magnification metric (5) *satisfies*

$$\tilde{d}^X(x,y) - 2\varepsilon \le d_T^\varepsilon(\xi_x,\xi_y) \le \tilde{d}^X(x,y) + 2\varepsilon.$$

Equivalently, in terms of conjugate metrics,

$$d^X(x,y) - 2\varepsilon \le \tilde{d}_T^\varepsilon(\xi_x,\xi_y) \le d^X(x,y) + 2\varepsilon.$$

Hence,

$$||x|| - 2\varepsilon \le ||\xi_x||_\varepsilon \le ||x|| + 2\varepsilon,$$

and so $||x_n|| \to \infty$ *iff* $d^T(\xi_{x(n)}(t),\xi_e(t)) \to \infty$.

Proof. We follow a similar argument to that for the permutation metric. By right-invariance,

$$\begin{aligned} d^X(t(x^{-1}z),t(y^{-1}z)) &\le d^X(t(x^{-1}z),x^{-1}z) + d^X(x^{-1}z,y^{-1}z) + d^X(y^{-1}z,t(y^{-1}z)) \\ &\le 2||t|| + d^X(x^{-1},y^{-1}), \end{aligned}$$

so

$$d^T(\xi_x(t),\xi_y(t)) = \sup_z d^X(t(x^{-1}z),t(y^{-1}z)) \le 2||t|| + d^X(x,e_X).$$

Now, again by right-invariance,

$$d^X(x^{-1}, y^{-1}) \leq d(x^{-1}, t(x^{-1})) + d(t(x^{-1}), t(y^{-1})) + d(t(y^{-1}), y^{-1}).$$

But

$$d(t(x^{-1}), t(y^{-1})) \leq \sup_z d^X(t(x^{-1}z), t(y^{-1}z)),$$

so

$$d^X(x^{-1}, y^{-1}) \leq 2||t|| + \sup_z d^X(t(x^{-1}z), t(y^{-1}z)) = 2||t|| + d^T(\xi_x(t), \xi_y(t)),$$

as required. □

We thus obtain the following result.

Theorem 11.7 (Topological Quasi-Duality Theorem).
For X a normed group, the second dual Ξ is a normed group isometric to X which, for any $\varepsilon \geq 0$, is ε-quasi-isometric to X in relation to $\tilde{d}_T^\varepsilon(\xi_x, \xi_y)$ and the $|| \cdot ||^\varepsilon$ norm. Here $T = \mathcal{H}_u(X)$.

Proof. We metrize Ξ by setting $d_\Xi(\xi_x, \xi_y) = d^X(x, y)$. This makes Ξ an isometric copy of X and an ε-quasi-isometric copy in relation to the conjugate metric $\tilde{d}_T^\varepsilon(\xi_x, \xi_y)$ which is given, for any $\varepsilon \geq 0$, by

$$\tilde{d}_T^\varepsilon(\xi_x, \xi_y) := \sup_{||t|| \leq \varepsilon} d^T(\xi_x^{-1}(t), \xi_y^{-1}(t)).$$

In particular for $\varepsilon = 0$ we have

$$d^T(\xi_x^{-1}(e), \xi_y^{-1}(e)) = \sup_z d^X(xz, yz) = d(x, y).$$

Assuming d^X is right-invariant, d_Ξ is right-invariant, since

$$d_\Xi(\xi_x\xi_z, \xi_y\xi_z) = d_\Xi(\xi_{xz}, \xi_{yz}) = d^X(xz, yz) = d^X(x, y). \qquad \square$$

Remark. Alternatively, working in $Tr_L(X)$ rather than in $\mathcal{H}_u(X)$ and with d_R^X again right-invariant, and noting that $\xi_x(\lambda_y)(z) = \lambda_y\lambda_x^{-1}(z)) = \lambda_{yx^{-1}}(z)$, we have

$$\sup_w d_\mathcal{H}(\xi_x(\lambda_w), \xi_e(\lambda_w)) = \sup_v d_v^X(e, x) = ||x||_\infty^X,$$

possibly infinite. Indeed

$$\begin{aligned}\sup_w d_{\mathcal{H}}(\xi_x(\lambda_w), \xi_y(\lambda_w)) &= \sup_w \sup_z d_R^X(\xi_x(\lambda_w)(z), \xi_y(\lambda_w)(z)) \\ &= \sup_w \sup_z d_R^X(wx^{-1}z, wy^{-1}z) = \sup_w d_R^X(vxx^{-1}, vxy^{-1}) \\ &= \sup_v d_R^X(vy, vx) = \sup_v d_v^X(y, x).\end{aligned}$$

(Here we have written $w = vx$.)

The refinement metric $\sup_v d^X(vy, vx)$ is left-invariant on the bounded elements (bounded under the related norm; cf. Proposition 2.12). Of course, if d^X were bi-invariant (both right- and left-invariant), we would have

$$\sup_w d_H(\xi_x(\lambda_w), \xi_y(\lambda_w)) = d^X(x, y).$$

12 Divergence in the bounded subgroup

In earlier sections we made on occasion the assumption of a bounded norm. Here we are interested in norms that are unbounded.

For S a space and $\mathcal{A}$ a subgroup of $Auth(S)$ equipped with the supremum norm, suppose $\varphi : \mathcal{A} \times S \to S$ is a continuous flow (see Lemma 3.6, for an instance). We will write $\alpha(s) := \varphi^\alpha(s) = \varphi(\alpha, s)$. This is consistent with $\mathcal{A}$ being a subgroup of $Auth(S)$. As explained at the outset of Section 11, we have in mind two pairs $(\mathcal{A}, S)$, as follows.

Example 1. Take $S = X$ to be a normed topological group and $\mathcal{A} = T \subseteq \mathcal{H}(X)$ to be a subgroup of automorphisms of X such that T is a topological group with supremum metric

$$d^T(t_1, t_2) = \sup_x d^X(t_1(x), t_2(x)),$$

e.g. $T = \mathcal{H}_u(X)$. Note that here $e_T = id_X$.

Example 2. $(\mathcal{A}, S) = (\Xi, T) = (X, T)$. Here X is identified with its second dual Ξ (of the preceding section).

Given a flow $\varphi(t, x)$ on $T \times X$, with T closed under translation, the action defined by

$$\varphi(\xi_x, t) := \xi_{x^{-1}}(t)$$

is continuous, hence a flow on $\Xi \times T$, which is identified with $X \times T$. Note that $\xi_{x^{-1}}(t)(e_X) = t(x)$, i.e. projection onto the e_X coordinate retrieves the T-flow φ. Here, for $\xi = \xi_{x^{-1}}$, writing $x(t)$ for the translate of t, we have

$$\xi(t) := \varphi^{\xi}(t) = \varphi(\xi, t) = x(t),$$

so that φ may be regarded as a X-flow on T.

We now formalize the notion of a sequence converging to the identity and divergent sequence. These are critical to the definition of regular variation [BOst13].

Definition. Let $\psi_n : X \to X$ be auto-homeomorphisms.
We say that a sequence ψ_n in $\mathcal{H}(X)$ *converges to the identity* if

$$||\psi_n|| = \hat{d}(\psi_n, id) := \sup_{t \in X} d(\psi_n(t), t) \to 0.$$

Thus, for all t, we have $z_n(t) := d(\psi_n(t), t) \leq ||\psi_n||$ and $z_n(t) \to 0$. Thus the sequence $||\psi_n||$ is bounded.

Illustrative examples. In $\mathbb{R}$ we may consider $\psi_n(t) = t + z_n$ with $z_n \to 0$. In a more general context, we note that a natural example of a convergent sequence of homeomorphisms is provided by a flow parametrized by *discrete time* (thus also termed a 'chain') towards a sink. If $\psi : \mathbb{N} \times X \to X$ is a flow and $\psi_n(x) = \psi(n, x)$, then, for each t, the orbit $\{\psi_n(t) : n = 1, 2, ...\}$ is the image of the real null sequence $\{z_n(t) : n = 1, 2, ...\}$.

Proposition 12.1. (i) *For a sequence ψ_n in $\mathcal{H}(X)$, ψ_n converges to the identity iff ψ_n^{-1} converges to the identity.*

(ii) Suppose X has abelian norm. For $h \in \mathcal{H}(X)$, if ψ_n converges to the identity then so does $h^{-1}\psi_n h$.

Proof. Only (ii) requires proof, and that follows from $||h^{-1}\psi_n h|| = ||hh^{-1}\psi_n|| = ||\psi_n||$, by the cyclic property. □

Definitions. 1. Again let $\varphi_n : X \to X$ be auto-homeomorphisms. We say that the sequence φ_n in $\mathcal{G}$ *diverges uniformly* if for any $M > 0$ we have, for all large enough n, that

$$d(\varphi_n(t), t) \geq M, \text{ for all } t.$$

Equivalently, putting

$$d_*(h,h') = \inf_{x\in X} d(h(x),h'(x)),$$

$$d_*(\varphi_n, id) \to \infty.$$

2. More generally, let $\mathcal{A} \subseteq \mathcal{H}(S)$ with $\mathcal{A}$ a metrizable topological group. We say that α_n is a *pointwise divergent sequence in* $\mathcal{A}$ if, for each $s \in S$,

$$d_S(\alpha_n(s), s) \to \infty,$$

equivalently, $\alpha_n(s)$ does not contain a bounded subsequence.

3. We say that α_n is a *uniformly divergent sequence in* $\mathcal{A}$ if

$$||\alpha_n||_{\mathcal{A}} := d_{\mathcal{A}}(e_{\mathcal{A}}, \alpha_n) \to \infty,$$

equivalently, α_n does not contain a bounded subsequence.

Examples. In $\mathbb{R}$ we may consider $\varphi_n(t) = t + x_n$ where $x_n \to \infty$. In a more general context, a natural example of a uniformly divergent sequence of homeomorphisms is again provided by a flow parametrized by discrete time from a source to infinity. If $\varphi : \mathbb{N} \times X \to X$ is a flow and $\varphi_n(x) = \varphi(n,x)$, then, for each x, the orbit $\{\varphi_n(x) : n = 1,2,...\}$ is the image of the divergent real sequence $\{y_n(x) : n = 1,2,...\}$, where $y_n(x) := d(\varphi_n(x), x) \geq d_*(\varphi_n, id)$.

Remark. Our aim is to offer analogues of the topological vector space characterization of boundedness: for a bounded sequence of vectors $\{x_n\}$ and scalars $\alpha_n \to 0$ ([Ru-FA2] cf. Th. 1.30), $\alpha_n x_n \to 0$. But here $\alpha_n x_n$ is interpreted in the spirit of duality as $\alpha_n(x_n)$ with the homeomorphisms α_n converging to the identity.

Theoretical examples motivated by duality.

1. Evidently, if $S = X$, the pointwise definition reduces to functional divergence in $\mathcal{H}(X)$ defined pointwise:

$$d^X(\alpha_n(x), x) \to \infty.$$

The uniform version corresponds to divergence in the supremum metric in $\mathcal{H}(X)$.

2. If $S = T$ and $\mathcal{A} = X = \Xi$, we have, by the Quasi-isometric Duality Theorem (Th. 11.7), that

$$d^T(\xi_{x(n)}(t), \xi_e(t)) \to \infty$$

iff

$$d^X(x_n, e_X) \to \infty,$$

and the assertion is ordinary divergence in X. Since

$$d_\Xi(\xi_{x(n)}, \xi_e) = d^X(x_n, e_X),$$

the uniform version also asserts that

$$d^X(x_n, e_X) \to \infty.$$

Recall that $\xi_x(s)(z) = s(\lambda_x^{-1}(z)) = s(x^{-1}z)$, so the interpretation of Ξ as having the action of X on T was determined by

$$\varphi(\xi_x, t) = \xi_{x^{-1}}(t)(e) = t(x).$$

One may write

$$\xi_{x(n)}(t) = t(x_n).$$

When interpreting $\xi_{x(n)}$ as x_n in X acting on t, note that

$$d_X(x_n, e_X) \leq d^X(x_n, t(x_n)) + d^X(t(x_n), e_X) \leq ||t|| + d^X(t(x_n), e_X),$$

so, as expected, the divergence of x_n implies the divergence of $t(x_n)$.

The next definition extends our earlier one from sequential to continuous limits.

Definition. Let $\{\psi_u : u \in I\}$ for I an open interval be a family of homeomorphisms (cf. [Mon2]). Let $u_0 \in I$. Say that ψ_u *converges to the identity as* $u \to u_0$ if

$$\lim_{u \to u_0} ||\psi_u|| = 0.$$

This property is preserved under topological conjugacy; more precisely we have the following result, whose proof is routine and hence omitted.

Lemma 12.2. *Let $\sigma \in \mathcal{H}_{unif}(X)$ be a homeomorphism which is uniformly continuous with respect to d^X, and write $u_0 = \sigma z_0$.*

If $\{\psi_z : z \in B_\varepsilon(z_0)\}$ converges to the identity as $z \to z_0$, then as $u \to u_0$ so does the conjugate $\{\psi_u = \sigma\psi_z\sigma^{-1} : u \in B_\varepsilon(u_0), u = \sigma z\}$.

Lemma 12.3 *Suppose that the homeomorphisms $\{\varphi_n\}$ are uniformly divergent, $\{\psi_n\}$ are convergent and σ is bounded, i.e. is in $\mathcal{H}(X)$. Then $\{\varphi_n\sigma\}$ is uniformly divergent and likewise $\{\sigma\varphi_n\}$. In particular $\{\varphi_n\psi_n\}$ is uniformly divergent, and likewise $\{\varphi_n\sigma\psi_n\}$, for any bounded homeomorphism $\sigma \in \mathcal{H}(X)$.*

Proof. Consider $s := ||\sigma|| = \sup d(\sigma(x), x) > 0$. For any M, from some n onwards we have

$$d_*(\varphi_n, id) = \inf_{x \in X} d(\varphi_n(x), x) > M,$$

i.e.

$$d(\varphi_n(x), x) > M,$$

for all x. For such n, we have $d_*(\varphi_n\sigma, id) > M - s$, i.e. for all t we have

$$d(\varphi_n(\sigma(t)), t)) > M - s.$$

Indeed, otherwise at some t this last inequality is reversed, and then

$$\begin{aligned} d(\varphi_n(\sigma(t)), \sigma(t)) &\leq d(\varphi_n(\sigma(t)), t) + d(\sigma(t), t) \\ &\leq M - s + s = M. \end{aligned}$$

But this contradicts our assumption on φ_n with $x = \sigma(t)$. Hence $d_*(\varphi_n\sigma, id) > M - s$ for all large enough n.

The other cases follow by the same argument, with the interpretation that now $s > 0$ is arbitrary; then we have for all large enough n that $d(\psi_n(x), x) < s$, for all x. □

Remark. Lemma 12.3 says that the filter of sets (countably) generated from the sets

$$\{\varphi | \varphi : X \to X \text{ is a homeomorphism and } ||\varphi|| \geq n\}$$

is closed under composition with elements of $\mathcal{H}(X)$.

We now return to the notion of divergence.

Definition. We say that pointwise (resp. uniform) divergence is *unconditional* in $\mathcal{A}$ if, for any (pointwise/uniform) divergent sequence α_n,
(i) for any bounded σ, the sequence $\sigma\alpha_n$ is (pointwise/uniform) divergent;

and,
(ii) for any ψ_n convergent to the identity, $\psi_n\alpha_n$ is (pointwise/uniform) divergent.

Remarks. In clause (ii) each of the functions ψ_n has a bound depending on n. The two clauses could be combined into one by requiring that if the bounded functions ψ_n converge to ψ_0 in the supremum norm, then $\psi_n\alpha_n$ is (pointwise/uniform) divergent.

By Lemma 12.3 uniform divergence in $\mathcal{H}(X)$ is unconditional. We move to other forms of this result.

Proposition 12.4. *If the metric on $\mathcal{A}$ is left- or right-invariant, then uniform divergence is unconditional in $\mathcal{A}$.*

Proof. If the metric $d = d_{\mathcal{A}}$ is left-invariant, then observe that if β_n is a bounded sequence, then so is $\sigma\beta_n$, since

$$d(e, \sigma\beta_n) = d(\sigma^{-1}, \beta_n) \leq d(\sigma^{-1}, e) + d(e, \beta_n).$$

Since $||\beta_n^{-1}|| = ||\beta_n||$, the same is true for right-invariance. Further, if ψ_n is convergent to the identity, then also $\psi_n\beta_n$ is a bounded sequence, since

$$d(e, \psi_n\beta_n) = d(\psi_n^{-1}, \beta_n) \leq d(\psi_n^{-1}, e) + d(e, \beta_n).$$

Here we note that, if ψ_n is convergent to the identity, then so is ψ_n^{-1} by continuity of inversion (or by metric invariance). The same is again true for right-invariance. □

The case where the subgroup $\mathcal{A}$ of auto-homeomorphisms is the translations Ξ, though immediate, is worth noting.

Theorem 12.5 (The case $\mathcal{A} = \Xi$). *If the metric on the group X is left- or right-invariant, then uniform divergence is unconditional in $\mathcal{A} = \Xi$.*

Proof. We have already noted that Ξ is isometrically isomorphic to X. □

Remarks.

1. If the metric is bounded, there may not be any divergent sequences.
2. We already know from Lemma 12.3 that uniform divergence in $\mathcal{A} = \mathcal{H}(X)$ is unconditional.

3. The unconditionality condition (i) corresponds directly to the technical condition placed in [BajKar] on their filter $\mathcal{F}$. In our metric setting, we thus employ a stronger notion of limit to infinity than they do. The filter implied by the pointwise setting is generated by sets of the form

$$\bigcap_{i\in I}\{\alpha : d^X(\alpha_n(x_i), x_i) > M \text{ ultimately}\} \text{ with } I \text{ finite.}$$

However, whilst this is not a countably generated filter, its projection on the x-coordinate:

$$\{\alpha : d^X(\alpha_n(x), x) > M \text{ ultimately}\},$$

is.

4. When the group is locally compact, 'bounded' may be defined as 'pre-compact', and so 'divergent' becomes 'unbounded'. Here divergence is unconditional (because continuity preserves compactness).

The supremum metric need not be left-invariant; nevertheless we still do have unconditional divergence.

Theorem 12.6. *For $\mathcal{A} \subseteq \mathcal{H}(S)$, pointwise divergence in $\mathcal{A}$ is unconditional.*

Proof. For fixed $s \in S$, $\sigma \in \mathcal{H}(S)$ and $d^X(\alpha_n(s), s))$ unbounded, suppose that $d^X(\sigma\alpha_n(s), s))$ is bounded by K. Then

$$\begin{aligned} d_S(\alpha_n(s), s)) &\leq d_S(\alpha_n(s), \sigma(\alpha_n(s))) + d_S(\sigma(\alpha_n(s)), s) \\ &\leq ||\sigma||_{\mathcal{H}(S)} + K, \end{aligned}$$

contradicting that $d_S(\alpha_n(s), s))$ is unbounded. Similarly, for ψ_n converging to the identity, if $d_S(\psi_n(\alpha_n(x)), x)$ is bounded by L, then

$$\begin{aligned} d_S(\alpha_n(s), s)) &\leq d_S(\alpha_n(s), \psi_n(\alpha_n(s))) + d_S(\psi_n(\alpha_n(s)), s) \\ &\leq ||\psi_n||_{\mathcal{H}(S)} + L, \end{aligned}$$

contradicting that $d_S(\alpha_n(s), s))$ is unbounded. □

Corollary 1. *Pointwise divergence in $\mathcal{A} \subseteq \mathcal{H}(X)$ is unconditional.*

Corollary 2. *Pointwise divergence in $\mathcal{A} = \Xi$ is unconditional.*

Proof. In Theorem 12.6, take $\alpha_n = \xi_{x(n)}$. Then unboundedness of $d^T(\xi_{x(n)}(t), t)$ implies unboundedness of $d^T(\sigma\xi_{x(n)}(t), t)$ and of $d^T(\psi_n\xi_{x(n)}(t)), t)$. □

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