Topological regular variation: II. The fundamental theorems.

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Abstract

This paper investigates fundamental theorems of regular variation (Uniform Convergence, Representation, and Characterization Theorems) some of which, in the classical setting of regular variation in $\mathbb{R}$, rely in an essential way on the additive semi-group of natural numbers $\mathbb{N}$ (e.g. de Bruijn's Representation Theorem for regularly varying functions). Other such results include Goldie's direct proof of the Uniform Convergence Theorem and Seneta's version of Kendall's theorem connecting sequential definitions of regular variation with their continuous counterparts (for which see [BOst15]). We show how to interpret these in the topological group setting established in [BOst13] as connecting $\mathbb{N}$-flow and $\mathbb{R}$-flow versions of regular variation, and in so doing generalize these theorems to $\mathbb{R}^d$. We also prove a flow version of the classical Characterization Theorem of regular variation.

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1 Introduction

In its classical setting regular variation is concerned with the asymptotic behaviour of $\sigma_h(t,x) := h(tx)h(x)^{-1}$ for $h : \mathbb{R}_+ \to \mathbb{R}_+$ with t fixed and x going to infinity. The foundation stone of the theory is the Uniform Convergence Theorem (UCT) which asserts that, for h Baire, if the limit $\partial_X h(t) := \lim_{x\to\infty} h(tx)/h(x)$ exists for all t, then convergence is uniform for t on compact sets. It is thus no surprise that there are as many eight or ten proofs (this count depends on what further assumptions are admitted), of which five are given in [BGT] Section 1.2 and a sixth referred to. Of these two (one due to Csiszár and Erdős, the other due to Elliott [E] Ch. 1) are known to give an extension of UCT to multivariate regular variation in $\mathbb{R}^d$, or, in the case of Elliott in $\mathbb{C}$. The ninth has a strong hypothesis on h (the continuity of $t \to h(tx)$) aimed at using the Baire Category Theorem, the tenth in similar spirit employs the Weil topology (but requires the strong assumption of T locally compact and H second countable). These last two were used by Bajšanski and Karamata [BajKar] (cf. [BGT] Appendix 1 and [Ba]) in a ground-breaking approach to provide a first proof of the UCT in their chosen general setting: a group theory formulation of regular variation wherein $h : G \to H$ with G, H groups and with x going to infinity along a filter in G (and t restricted to a co-meagre subgroup T of G). Recall for comparison the seven proofs establishing the functional equation of the Riemann zeta function given by Titchmarsh in Chapter 2 of [Ti].

The Csiszár and Erdős idea taken together with the (Baire) Category Embedding Theorem of [BOst11] has provided a first proof for the strongest form yet of the UCT in the topological-flow formulation of regular variation wherein, for groups X, H, T, the term tx results from a T-flow acting on X, and $h : X \to H$. Here there are actually two, dual UCTs, corresponding to a transposition of t and x afforded by the dual X-flow acting on T (so that respectively one of T or X is assumed to be a co-meagre group). In [Ost-knit] it is shown that the essential distinctions between flow and group formulations reside in the notions of divergence which the two theories admit.

The present paper explores, in the flow setting, the one and only direct proof of the UCT, due to Goldie, capable of topological generalization (the only other direct proof is Delange's, but lacks this capability, since it uses quantitative measure theory). We do so in two ways. With only the usual 'co-meagre group' hypothesis, the first direct step of his proof yields the flow version of the UCT (unfortunately, one needs a reductio ad absurdum to complete this step); this is the argument leading to what we observed in [BOst1] in the Euclidean setting was the Bounded Equivalence Principle. Just as there, so too here it yields the UCT. We are able to reproduce the completing second, direct step of Goldie in a locally compact, σ-compact topological group; specializing to the abelian, connected, locally connected case, this is very nearly the Euclidean setting (in view of the Pontryagin theory, for which see [Pont], [MZ], [We] and also [Ru] Ch. 2); when so specialized, this provides an eleventh proof!

Along the way, however, we have been able to clarify and unify other aspects of regular variation, namely the connections between discrete-time flow theory to the real-time flow theory and the limitations of the de Bruijn representation theory imposed by finite dimensions. It is useful to recall that one connection between this generalization of the representation theorem and the classic univariate paradigm is in the theory of domains of attraction ([BGT] Section 8.3.2 p. 345), connecting infinite divisibility to sequential regular variation via Kendall's Theorem (see [BGT] Section 1.9 p. 49, or [Ken], [KH, Th. 16, p.110.]). We show that this latter theorem generalizes to $\mathbb{R}^d$ (in fact, in much the same way as Goldie's direct proof of the UCT, [BGT] Section 1.2.2 'Second proof' – by reference to a slowly varying 'divergent net' on the space). Its significance lies in an immediate connection with recent work (see e.g. [HLMS]) wherein regular variation analysis is applied in two contexts: Euclidean (albeit with $\mathbb{R}^d$ replaced by its compactification $\overline{\mathbb{R}}^d$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$) and topological, with the function h replaced by a distribution function as below and the standard passage to the limit in the format

$$n\mathbb{P}(X/a_n \in .) \to^v \mu(.),$$

for some increasing sequence $a_n \uparrow \infty$ and with limits under the vague topology on the space of measures. (Recall the latter is defined in the context of the space of continuous functions with compact support, and their integrals.)

2 Generalized Piccard-Pettis Theorem

We recall a number of definitions from [BOst13] to which we refer for justification and proof. Let X be a metric group with identity element e_X and with right-invariant metric d_X and associated group norm $||x|| := d_X(x, e_X)$. We denote by $Auth(X)$ the group of self-homeomorphisms of X under composition. $\mathcal{H}(X)$ denotes the subgroup

$$\{h \in Auth(X) : ||h|| < \infty\},$$

where, in turn,

$$||h|| := \sup d_X(h(x), x)),$$

denotes the group-normon $\mathcal{H}(X)$ which metrizes it by the right-invariant metric $d(g, h) = ||gh^{-1}||$.

Definition. (cf. [**?**]). Let $\{\psi_u : u \in I\}$ for I an open interval in $\mathbb{R}$ be a family of homeomorphisms in $\mathcal{H}(X)$. Let $u_0 \in I$. Say that ψ_u *converges to the identity as* $u \to u_0$ if

$$\lim_{u \to u_0} ||\psi_u|| = 0.$$

This property is preserved under topological conjugacy; more precisely we have the following result, whose proof is routine and hence omitted.

Lemma. *Let σ be a homeomorphism which is uniformly continuous, and write $u_0 = \sigma z_0$.*

If $\{\psi_z : z \in B_\varepsilon(z_0)\}$ converges to the identity as $z \to z_0$, then, as $u \to u_0$, so does the conjugate $\{\psi_u = \sigma\psi_z\sigma^{-1} : u \in B_\varepsilon(u_0), u = \sigma z\}$.

We recall that a subset A of a metric space is *Baire* if it has the Baire property, i.e., for an open set U and meagre sets M, N, we have $A = (M \cup U) \backslash N$. The result below generalizes the category version of the Steinhaus Theorem [St] of 1920, first stated explicitly by Piccard [P] in 1939, and restated in [Pet1] in 1950; in the current form it may be regarded as a 'localized-refinement' of [RaoRao].

Generalized Piccard-Pettis Theorem ([P], [Pet1],[Pet2], [BGT] Th. 1.1.1, [BOst3], [RaoRao], cf. [Kel] Ch. 6 Prb. P). *Let be X be a homogenous space. Suppose that ψ_u converges to the identity, as $u \to u_0$, and that A is Baire and non-meagre. Then, for some $\delta > 0$, we have*

$$A \cap \psi_u(A) \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta,$$

or, equivalently, for some $\delta > 0$

$$A \cap \psi_u^{-1}(A) \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta.$$

Proof. We may suppose that $A = V \backslash M$ with M meagre and V open. Hence, for any $v \in V \backslash M$, there is some $\varepsilon > 0$ with

$$B_\varepsilon(v) \subseteq U.$$

By definition of convergence, there is $\delta > 0$ such that, for u with $d(u, u_0) < \delta$, we have

$$d^*(\psi_u, id) < \varepsilon/2.$$

Hence, for any such u and any y in $B_{\varepsilon/2}(v)$, we have

$$d(\psi_u(y), y) < \varepsilon/2.$$

From this it follows that

$$W := \psi_u(B_{\varepsilon/2}(z_0)) \cap B_{\varepsilon/2}(z_0) \neq \varnothing,$$

and

$$W' := \psi_u^{-1}(B_{\varepsilon/2}(z_0)) \cap B_{\varepsilon/2}(z_0) \neq \varnothing.$$

For fixed u with $d(u, u_0) < \delta$, the set

$$M' := M \cup \psi_u(M) \cup \psi_u^{-1}(M)$$

is meagre. Let $w \in W \backslash M'$ (or $w \in W' \backslash M'$, as the case may be). Since $w \in B_\varepsilon(z_0) \backslash M \subseteq V \backslash M$, we have

$$w \in V \backslash M \subseteq A.$$

Similarly, $w \in \psi_u(B_\varepsilon(z_0)) \backslash \psi_u(M) \subseteq \psi_u(V) \backslash \psi_u(M)$. Hence

$$\psi_u^{-1}(w) \in V \backslash M \subseteq A.$$

In this case, as asserted,

$$A \cap \psi_u^{-1}(A) \neq \varnothing.$$

In the other case ($w \in W' \backslash M'$), one obtains similarly

$$\psi_u(w) \in V \backslash M \subseteq A.$$

Here too

$$A \cap \psi_u^{-1}(A) \neq \varnothing.$$

□

Remarks.

1. In the theorem above it is possible to work with a weaker condition, namely local convergence at z_0, where one demands that for some neighbourhood $B_\eta(z_0)$ and some K

$$d(\psi_u(z), z) \leq K d(u, u_0), \text{ for } z \in B_\eta(z_0).$$

This implies that, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for $z \in B_\delta(z_0)$,

$$d(\psi_u(z), z) < \varepsilon, \text{ for } z \in B_\delta(z_0).$$

2. The Piccard-Pettis Theorem for topological groups (named by Kelley, [Kel] Ch. 6 Pblm P-(b), the Banach-Kuratowski-Pettis Theorem, say BKPT for short) asserts the category version of the Steinhaus Theorem [St] that, for A Baire and non-meagre, the set $A^{-1}A$ is a neighbourhood of the identity; our version of the Piccard theorem as stated implies this albeit only in the context of metric groups. Let d_X be a right-invariant metric on X and take $\psi_u(x) = ux$ and $u_0 = e$. Then ψ_u converges to the identity (see [BOst13] Section 3), and so the theorem implies that $B_\delta(e) \subseteq A^{-1} \cap A$ for some $\delta > 0$; indeed $a' \in A \cap \psi_u(A)$ for $u \in B_\delta(e)$ means that $a' \in A$ and, for some $a \in A$, also $ua = a'$ so that $u = a^{-1}a' \in A^{-1}A$. It is more correct to name the following important and immediate corollary, the BKPT, since it appears in this formulation in [Ban], [Kur1], derived by different means, and was used by Pettis in [Pet1] to deduce his Steinhaus-type theorem. A fundamental result for regular variation follows. Recall that a set A is *clopen* if A is both open and closed.

The Subgroup Dichotomy Theorem (Banach-Kuratowski-Pettis Theorem) ([Ban] Satz 1, [Kur1] Ch. VI. 13. XII; cf. [Kel] Ch. 6 Pblm P; cf. [BGT] Cor. 1.1.4 and also [BCS] and [1] for the measure variant).

Let X be a topological group which is non-meagre and let A any Baire subgroup. Then A is either meagre or clopen in X.

The antecedent Kuratowski Theorem ([Kur1], Ch. I Para. 13.XI) and the category analogue of the Hewitt-Savage zero-one law of [RaoRao] are related.

The Characterization Theorem ([BGT] Thm. 1.4.1). *Let X and H be normed groups, $T \subseteq \mathcal{H}(X)$ be a connected non-meagre subgroup acting on the group X, $h : X \to H$ be Baire. If the limit*

$$\partial_X(t) := \lim_{||x|| \to \infty} h(tx)h(x)^{-1}$$

exists on a non-meagre subset of T, then $\partial_X(t)$ exists on all of T and is a continuous homomorphism from T to H.

Proof. The set

$$S := \{t \in T : \lim_{||x|| \to \infty} h(tx)h(x)^{-1}\text{exists}\}$$

is a non-meagre subgroup of T. Hence S is non-empty and clopen; but T is a connected group, so $S = T$. The final assertion follows from the Continuous Homomorphism Theorem of [BOst15]. □

Remark. With $X = \mathbb{R}$ and T the group of isometries, this theorem implies the classical caracterization theorem of regular variation. The implication follows from the result of van Dantzig and van der Waerden that for X a connected, locally compact metric space the isometries under the compact-open topology form a locally compact group. For a proof see [KoNo] Th. 4.7 (cf [BH] Ch. I.Prop 8.7 for the compact case). More is known – see the generalization by Strantzalos [Str] and [Itz] for an analogous result for locally compact uniform spaces.

As a second corollary, we have a far-reaching generalization.

Corollary (cf. [1],[BCS] for the measure variant). *Let X be a topological group which is non-meagre. For A, B non-meagre and $\alpha \in A$, $\beta \in B$, put $\tau(x) := \beta\alpha^{-1}x$. For ψ_u convergent to the identity as $u \to u_0$, there is $\delta > 0$ such that*

$$B \cap \tau\psi_u(A) \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta.$$

In particular,

$$\beta\alpha^{-1}B_\delta(e) \subseteq BA^{-1},$$

and hence, for any non-meagre set A, the set $A^2 = AA$ contains an open set.

Proof. W.l.o.g. $A' := \tau^{-1}(B) \subseteq A$ and hence there is $\delta > 0$ such that

$$A' \cap \psi_u(A') \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta.$$

Hence

$$B \cap \tau\psi_u\tau^{-1}(B) = \tau(A') \cap \tau\psi_u(A') \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta,$$

So

$$B \cap \tau\psi_u\tau^{-1}(B) = B \cap \tau\psi_u(A) \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta.$$

Since inversion is a homeomorphism, we may replace A by A^{-1}; then taking $B = A$ we obtain the final assertion. □

The final result of this section concerning subsemigroups is of interest to regular variation in the Euclidean setting; for the role of subsemigroup arguments, see for instance [BGT] Thms 2.0.1 p. 61 and 3.2.5 p. 141. Clearly it is applicable to an ordered group setting such as developed in [Ru] Ch. 8.

The Subsemigroup Theorem (cf. [BCS], [1]; cf. [BGT] Cor. 1.1.4). *Let X be a topological group which is non-meagre and let A any Baire subsemigroup. Then A contains an open set. In particular, if $A = \mathbb{R}^d$, then A contains an open sector.*

3 Bounded Equivalence Principle

We may now prove in a general context the following Bounded Equivalence Principle (BEP) whose real-line version is implicit but embedded in the course of the 'Second direct proof of the UCT' in [BGT] (p. 7-8, and due to Goldie). It was first isolated in [BOst1]. We give its proof below. Then, for convenience, we reproduce the indirect proof of the general UCT which the BEP also facilitates; a direct proof of the UCT can be deduced from the Principle albeit in the narrower setting of a locally compact, σ-compact group. That proof is offered in the next section. We recall, for X a metric space with a *distinguished point* z_0 and metric d_X, that a self-homeomorphisms φ of X is *bounded* if

$$||\varphi|| := \sup_{x \in X} d_X(\varphi(x), x) < \infty,$$

a sequnece $\{\varphi_n\}$ of self-homeomorphisms is divergent if $||\varphi_n|| \to \infty$, and correspondingly a function $h : X \to \mathbb{R}$ is $\{\varphi_n\}$-slowly varying if, for each t,

$$h(\varphi_n(t)) - h(\varphi_n(z_0)) \to 0.$$

Bounded Equivalence Principle (BEP). *Suppose the following:*

(i) *X is a Baire space,*

(ii) *X is uniformly homogeneous, i.e. for any pair of points z, u there is a uniformly continuous homeomorphism σ such that $\sigma(z) = u$;*

(iii) *For some $\delta > 0$, there is a family of homeomorphisms $\{\psi_z : z \in B_\delta(z_0)\}$ converging to the identity as $z \to z_0$, such that $\psi_z(z_0) = z$.*

Then for $h : X \to \mathbb{R}$ Baire slowly varying, the following are equivalent:

(a) $h_n(t) := h(\varphi_n(t)) - h(\varphi_n(z_0)) \to 0$, *uniformly in t on compact sets for any divergent φ_n,*

(b) $\lim_{n\to\infty} |h(\varphi_n(u_n)) - h(\varphi_n(z_0))| = 0$, *whenever $\mathbf{u} = \{u_n\}$ is a sequence converging to z_0, and φ_n is divergent,*

(c) $\lim_{n\to\infty} |h(\varphi_n(u_n)) - h(\varphi_n(z_0))| = 0$, *whenever $\mathbf{u} = \{u_n\}$ is a bounded sequence, and φ_n is divergent.*

Indirect Proof. Since (c) includes (a), it is enough to prove:

(I) that (a) implies (b) and

(II) that (b) implies (c).

(I) For brevity we write $t = z_0$. Choose ψ_k converging to the identity with $\psi_k(z_0) = z_k$. For any $\varepsilon > 0$, put

$$A_{n,t} := \bigcap_{k\geq n} \{y : |h(\varphi_k(y)) - h(\varphi_k(z_0))| < \varepsilon \text{ and } |h(\varphi_k\psi_k(y)) - h(\varphi_k\psi_k(z_0))| < \varepsilon\}.$$

Then each set $A_{n,t}$ is Baire and

$$X = \bigcup_n A_{n,t},$$

since for any fixed y we have both $h(\varphi_k(y)) - h(\varphi_k(z_0)) \to 0$ and $h(\varphi_k\psi_k(y)) - h(\varphi_k\psi_k(z_0)) \to 0$ (as h is slowly varying). Hence, for some $N = N(t)$, the set $A = A_{N(t),t}$ is non-meagre. By the Generalized Piccard-Pettis Theorem, there is $\delta > 0$ such that, for each $n > N(t)$ and $u = u_n$, we have

$$A \cap \psi_u(A) \neq \varnothing, \text{ for all } u \text{ with } d(u,t) < \delta.$$

In particular, there is $w_n \in A$ and $a_n \in A$ such that

$$w_n = \psi_u(a_n).$$

So

$$h(\varphi_k(w_n)) = h(\varphi_k(\psi_u(a_n)).$$

For such n, we have

$$|h(\varphi_k(w_n)) - h(\varphi_k(z_0))| < \varepsilon,$$

as $w_n \in A$ (using the first condition), and also

$$|h(\varphi_k\psi_k(a_n)) - h(\varphi_k\psi_k(z_0))| < \varepsilon,$$

as $a_n \in A$ (using the second condition). Hence

$$\begin{aligned} |h(\varphi_k\psi_k(z_0)) - h(\varphi_k(z_0))| &\leq |h(\varphi_k(w_n)) - h(\varphi_k(z_0))| \\ &\quad +|h(\varphi_k(\psi_k(a_n))) - h(\varphi_k\psi_k(z_0))| \\ &< 2\varepsilon. \end{aligned}$$

Since $\psi_k(z_0) = z_k$ and $\varepsilon > 0$ is arbitrary, we deduce that $\lim_{n\to\infty} |h(\varphi_n(u_n)) - h(\varphi_n(z_0))| = 0$.

(II) Let T be the closure of $\{u_n\}$. Let $t \in T$. We show how to reduce this situation to that in (i) by using a uniformly continuous shift . Here we take a uniform shift σ such that $\sigma z_0 = t$ and use a homeomorphism to repeat the argument. In the expression for $A_{n,t}$, we need to replace the convergence to the identity ψ_z, with z near z_0, by convergence to the identity ψ_u, with u near t. These movements are generated using the conjugate homeomorphism $\psi_u = \sigma\psi_z\sigma^{-1}$ with $u = \sigma z$. By the Lemma of Section 2 the conjugate homeomorphisms also converge to the identity.

As before, for any $\varepsilon > 0$, put

$$A_{n,t} := \bigcap_{k\geq n}\{y : |h(\varphi_k(y)) - h(\varphi_k(z_0))| < \varepsilon \text{ and } |h(\varphi_k\psi_{u(k)}(y)) - h(\varphi_k\psi_{u(k)}(z_0))| < \varepsilon\}.$$

Then each set $A_{n,t}$ is Baire and

$$X = \bigcup_n A_{n,t},$$

since for any fixed y we have both $h(\varphi_k(y)) - h(\varphi_k(z_0)) \to 0$ and $h(\varphi_k\psi_k(y)) - h(\varphi_k\psi_k(z_0)) \to 0$ (as h is slowly varying). Hence, for some $N = N(t)$, the set $A = A_{N(t),t}$ is non-meagre. By the Generalized Piccard-Pettis Theorem, there is $\delta > 0$ such that, for each $n > N(t)$ and $u = u_n$, we have

$$A \cap \psi_u(A) \neq \varnothing, \text{ for all } u \text{ with } d(u,t) < \delta.$$

In particular, there is $w_n \in A$ and $a_n \in A$ such that

$$w_n = \psi_u(a_n).$$

So

$$h(\varphi_k(w_n)) = h(\varphi_k(\psi_u(a_n)).$$

For such n, we have

$$|h(\varphi_k(w_n)) - h(\varphi_k(z_0))| < \varepsilon,$$

as $w_n \in A$ (using the first condition), and also

$$|h(\varphi_k\psi_{u(k)}(a_n)) - h(\varphi_k\psi_{u(k)}(z_0))| < \varepsilon,$$

as $a_n \in A$ (using the second condition). Hence

$$\begin{aligned} |h(\varphi_k\psi_{u(k)}(z_0)) - h(\varphi_k(z_0))| &\leq |h(\varphi_k(w_n)) - h(\varphi_k(z_0))| \\ &\quad + |h(\varphi_k(\psi_{u(k)}(a_n))) - h(\varphi_k\psi_{u(k)}(z_0))| \\ &< 2\varepsilon. \end{aligned}$$

i.e., since $\psi_{u(k)}(z_0) = u_k$,

$$|h(\varphi_k(u_k)) - h(\varphi_k(t))| \leq 2\varepsilon.$$

Since $\psi_{u(k)}(z_0) = u_k$ and $\varepsilon > 0$ is arbitrary, we deduce that $\lim_{n\to\infty} |h(\varphi_n(u_n)) - h(\varphi_n(z_0))| = 0$. □

The argument above and the compactness argument of [BGT] p. 8 may now be repeated in the general context to yield another indirect proof of the UCT. As promised, we reproduce the argument here for convenience.

Proof of the UCT via the BEP

Let φ_n be divergent. Let h be φ-regularly varying. By (c) of BEP we must show that

$$\lim_{n\to\infty} |h(\varphi_n(u_n)) - h(\varphi_n(z_0))| = 0,$$

whenever $u = \{u_n\}$ is a bounded sequence. Suppose otherwise. Then there is a bounded sequence of points u_n, say in the compact set K, and $\varepsilon > 0$ such that

$$|h(\varphi_n(u_n)) - h(\varphi_n(e))| \geq 3\varepsilon.$$

We deduce a contradiction. For each $t \in K$, arguing as in the proof of the BEP above, we may select $\delta = \delta(t) > 0$ such that, for u_k with $d(u_k, t) < \delta(t)$,

$$|h(\varphi_k(u_k)) - h(\varphi_k(t))| \leq 2\varepsilon. \tag{1}$$

Now $\{B_{\delta(t)}(t) : t \in K\}$ covers the compact set K. So we may choose a finite subset F of K such that $\{B_{\delta(t)}(t) : t \in F\}$ covers K. Thus $\delta := \min\{\delta(t) : t \in F\} > 0$. For $t \in F$, since $h_n(t) \to 0$, there is $N(t)$ such that for $n > N(t)$

$$|h(\varphi_n(u)) - h(\varphi_n(e))| < \varepsilon. \tag{2}$$

Put $N := \max\{N(t) : t \in F\}$. Consider any $n \geq N$. The point u_n lies in $B_{\delta(t)}(t)$ for some $t \in F$, so by (1)

$$|h(\varphi_n(u_n)) - h(\varphi_n(t))| \leq 2\varepsilon.$$

Combining with (2), we obtain

$$|h(\varphi_n(u_n)) - h(\varphi_n(e))| < 3\varepsilon,$$

a contradiction. □

4 Generalized Goldie Theorem (Direct Proof of UCT)

We work here with a metric group X. The Birkhoff-Kakutani Theorem ([Bir], [Kak]) asserts that a first-countable Hausdorff group has a right-invariant metric (see [Klee], [Bour] Part 2, Section 3.1, and [ArMa], compare [**?**] Exercise 8.1.G and Th. 8.1.21). We thus assume that X has a right-invariant metric d_X and as usual we denote the corresponding group-norm by $||x||_X := d_X(e, x)$, so that $d_X(x, y) = ||xy^{-1}||$ (see [**?**] for an exposition of the relation between metric groups and groups carrying a group-norm). The conjugate left-invariant metric, given by

$$\tilde{d}_X(x, y) = d_X(x^{-1}, y^{-1}),$$

makes an appearance in the definition below and consequently is critical to the immediately following Theorem. It plays no further explicit role. We note that $\tilde{d}_X(e,x) = d_X(e,x^{-1}) = d_X(x,e) = ||x||$, so the norm may refer ambiguously to either metric, however $\tilde{d}_X(x,y) = ||y^{-1}x||$.

Definitions.
1. For $\varepsilon > 0$, we say that $\{x_n\}$ is a *divergent ε-net* of the space X, if
(i) $||x_n|| \to \infty$,
(ii) for each $x \in X$ there is $n = n(x)$ such that $||x_n^{-1}x|| < \varepsilon$, i.e. $d_X(x^{-1}, x_n^{-1}) < \varepsilon$.
2. The *ε-swelling* is defined by $B_\varepsilon(K) := \{z : d_X(z,k) < \varepsilon$ for some $k \in K\}$.

We begin with a routine result, whose proof we include for completness, as it is short.

Lemma. *(i) If the closed ε-balls are compact in the locally compact group X and K is compact, then $B_{\varepsilon/2}(K)$ is pre-compact.*
(ii) $B_\varepsilon(K) = \{wk : k \in K, ||w|| < \varepsilon\}$.

Proof. (i) If $x_n \in B_{\varepsilon/2}(K)$, then we may choose $k_n \in K$ with $d(k_n, x_n) < \varepsilon/2$. W.l.o.g. k_n converges to k. Thus there exists N such that, for $n > N$, we have $d(k_n, k) < \varepsilon/2$. Then, for such n, we have $d(x_n, k) < \varepsilon$. Hence the sequence x_n lies in the compact closed ε-ball centred at k and so has a convergent subsequence.
(ii) If $||w|| < \varepsilon$, then $d_X(wk, k) = d_X(w, e) = ||w|| < \varepsilon$, so $wk \in B_\varepsilon(K)$. Conversely, if $\varepsilon > d_X(z,k) = d_X(zk^{-1}, e)$, then, putting $w = zk^{-1}$, we have $z = wk \in B_\varepsilon(K)$. $\square$

Theorem. *Let X be a locally-compact σ-compact group with unbounded metric. Then X possesses a divergent ε-net for all small enough $\varepsilon > 0$. In particular, this is so for $\mathbb{R}^d$*

Proof. Here exceptionally we work with the metric $\tilde{d}_X$. For some small enough $\eta > 0$, the closed η-ball is compact. Write $X = \bigcup K_n$ with K_n increasing and compact. The mapping $x \to ||x||$ is continuous and hence bounded on each K_n. Hence $M_n := \sup\{||x|| : x \in K_n\}$ defines an increasing sequence. For $\varepsilon < \eta/2$, select a finite number of ε-balls under the $\tilde{d}_X$ metric covering the annular set $A_{n+1} := K_{n+1} \backslash B_\varepsilon(K_n)$ whose centres thus form an ε-net. Enumerate these centres listing the nets of each annulus consecutively, and we are done. Evidently, if $X = \mathbb{R}^d$, then it is enough to decompose X into concentric annuli centered at the origin. In each annulus choose a finite ε-net. Let x_n be an enumeration of these nets listing the nets of each consecutive annulus. $\square$

Generalized Goldie Theorem. *Let X be a locally compact group with right-invariant metric, and suppose that, for some $\varepsilon > 0$, X has the closed 2ε-ball $\bar{B}_{2\varepsilon}(e_X)$ compact, and possesses a divergent ε-net. Then the UCT for X has a direct proof in the case of $\varphi_n(t) = s_n t$, where s_n is a divergent ε-net.*

Proof. Let s_n be a divergent ε-net. Let $\varphi_n(t) = s_n t$. Thus $||\varphi_n|| = \sup_t d(s_n t, t) = d(s_n, e) = ||s_n|| \to \infty$. Let $h : X \to \mathbb{R}$ be φ-slowly varying. Since X is Baire, by the Bounded Equivalence Theorem, we have for any precompact sequence u_n that

$$h(\varphi_n(u_n)) - h(\varphi_n(e)) \to 0.$$

Let K be compact. We now claim that

$$\sup\{u \in K : |h(\varphi_n(u)) - h(\varphi_n(e))|\} < \infty.$$

Indeed, for each integer n may select $u_n \in K$ either with the property that

$$|h(\varphi_n(u_n)) - h(\varphi_n(e))| \geq n,$$

when possible, or, when not (so that $\sup\{u \in K : |h(\varphi_n(u)) - h(\varphi_n(e))|\} < n$) with

$$|h(\varphi_n(u_n)) - h(\varphi_n(e))| \geq \sup\{u \in K : |h(\varphi_n(u)) - h(\varphi_n(e))|\} - \frac{1}{n}.$$

Since $h(\varphi_n(u_n)) - h(\varphi_n(e)) \to 0$, the latter alternative is impossible for large enough n. It follows that, for large n,

$$\sup\{u \in K : |h(\varphi_n(u)) - h(\varphi_n(0))|\} \leq |h(\varphi_n(u_n)) - h(\varphi_n(e))|,$$

and so $h(\varphi_n(u)) - h(\varphi_n(e)) \to 0$, as $n \to \infty$, uniformly on K.

Since s_n is an ε-net, for each x, we may find $n(x)$ such that $||s_{n(x)}^{-1} x|| < \varepsilon$. Put $u_n = s_{n(x)}^{-1} x$, then $||u_n|| < \varepsilon$. Thus $x = s_n u_n$.

Now if $u \in K$, then $xu = s_{n(x)}v_n$, where $v_n = v_n(u) = u_n u$. Thus by the Lemma $v_n \in B_\varepsilon(K)$ and $B_\varepsilon(K)$ is pre-compact. Thus

$$\begin{aligned}
& \sup_{u \in K} |h(xu) - h(x)| \\
\leq \;& \sup_{u \in K} \left[|h(xu) - h(s_{n(x)})| + |h(s_{n(x)}) - h(x)| \right] \\
= \;& \sup_{u \in K} \left[|h(s_{n(x)}v_n) - h(s_{n(x)})| + |h(s_{n(x)}u_n) - h(s_{n(x)})| \right] \\
= \;& \sup_{u \in K} \left[|h(\varphi_{n(x)}(v_n) - h(\varphi_{n(x)}(e))| + |h(\varphi_{n(x)}(u_n)) - h(\varphi_{n(x)}(e)| \right] \\
\leq \;& \sup_{v \in B_\varepsilon(K)} |h(\varphi_{n(x)}(v) - h(\varphi_{n(x)}(e))| + \sup_{u \in B_\varepsilon(e)} |h(\varphi_{n(x)}(u)) - h(\varphi_{n(x)}(e)| \\
\to \;& 0,
\end{aligned}$$

as $n(x) \to \infty$. $\square$

5 De Bruijn-Karamata Representation theorem

Using the UCT Karamata characterized slowly varying functions in an integral form (see below); his approach to the integral necessitated a restriction of the slowly varying functions to measurable ones. De Bruijn's later alternative proof in [deB] (1959) carries the theorem over to a wider context. Classically it was applied to Baire functions ([BGT] Section 1.3, p. 15) and it gives rise to the Smooth Variation Theorem ([BGT] Section 1.8, Th 1.8.2 p.45). More recently it has been applied to the natural classical context of regular variation to the functions of class $\mathbf{\Delta}^1_2$ (those with graph ambiguously analytic and co-analytic in the sense of classical descriptive set theory) – indeed to any class of functions closed under additive shift. Here we show how to extend de Bruijn's theorem to Euclidean spaces. We leave unanswered the question of whether the theorem extends to locally compact σ-compact groups for lack of an appropriate theory of Stieltjes integration.

For $x \in \mathbb{R}^d$ let

$$||x|| = \max\{|x|_1, ..., |x_d|\}.$$

We let $B(x) = \{y : 0 \leq y_i \leq x_i \text{ for } i = 1, ..., d.\}$ and denote Lebesgue measure by λ_d.

Representation theorem (cf. de Bruijn [deB], [BGT] Section 1.3, p. 15)

For $h : \mathbb{R}^d \to \mathbb{R}$, h is Baire slowly varying iff

$$h(x) = c(x) + \int_{B(x)} \eta(z) d\lambda_d(z),$$

for some Baire $c(.)$ and C^∞ function $\eta(.)$ such that.

$$c(x) \to h(e_X),\ \eta(x) \to 0,\ \text{as } ||x|| \to \infty.$$

Proof. For $x \in \mathbb{R}^d$, let

$$[x] := ([x_1], ..., [x_d]) \in \mathbb{Z}^d.$$

Define the square $S(z) := \{x : z_i - 1 \le x_i < z_i \text{ for } i = 1, ..., d\}$ and the annulus

$$A(z) := \{x : x_i < z_i \text{ for } i = 1, ..., d\} \backslash \{x : x_i \le z_i - 1 \text{ for } i = 1, ..., d\}.$$

For $x \in S(z)$, we have $[x] = z - 1_d$, where $1_d = (1, ..., 1)$. Let $p(x, z)$ be a C^∞ density on the unit square $S(z)$. We may indeed take $p(x, z) = p(x - [x])$, where $p(x) = p(x, 1_d)$.

For $z \in \mathbb{Z}^d$, put $\sigma(z) = z_1 + ... + z_d$. For $z \in \mathbb{Z}^d$ and $x \in S(z)$, put

$$\Delta h(x) := \sum_{\varepsilon \in \{-1,0\}^d} (-1)^{\sigma(\varepsilon)} h(z + \varepsilon), \qquad \eta(x) := p(x, z) \Delta h(x).$$

Thus, for $d = 2$, we have

$$\Delta h((i, j)) = \{h(i, j) - h(i - 1, j) + h(i - 1, j - 1) - h(i, j - 1)\}.$$

Put

$$h_1(x) = h(0) + \int_{B(x)} \Delta h(y) \cdot p(y - [y]) d\lambda_d(y).$$

Then, for $z \in \mathbb{Z}^d$,

$$h_1(z) = h(z).$$

For $\delta > 0$, suppose that, for $z \in \mathbb{Z}^d$ and $x \in A(z)$, we have

$$|\Delta h(x)| \le \delta.$$

For h slowly varying, this will be the case for $||z||$ large enough, since the unit ball is compact. Hence, for $x \in S(z)$,

$$h_1(x) = h([x]) + \int_{A(z)} \Delta h(y) \cdot p(y - [y]) d\lambda_d(y).$$

Thus

$$h(x) - h_1(x) = h(x) - h([x]) - \int_{A(z)} \Delta h(y) \cdot p(y - [y]) d\lambda_d(y),$$

and, with our assumptions above,

$$\begin{aligned} | \int_{A(z)} \Delta h(y) \cdot p(y - [y]) d\mu_d(y)| &\leq \int_{A(z)} |\Delta h(y)| \cdot p(y - [y]) d\lambda_d(y) \\ &\leq \delta \int_{A(z)} p(y - [y]) d\lambda_d(y) = \delta. \end{aligned}$$

Hence

$$h(x) - h_1(x) \to 0, \text{ as } x \to \infty.$$

Thus

$$h(x) = c(x) + \int_{B(x)} \eta(z) d\mu_d(z),$$

where

$$c(x) = [h(0) + h(x) - h_1(x)] \to h(0).$$

Indeed

$$\begin{aligned} h(x) &= h_1(x) + h(x) - h_1(x) \\ &= [h(x_0) + h(x) - h_1(x)] + \int_{B(x)} \eta(z) d\lambda_d(z). \end{aligned}$$

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