Topological regular variation: III: Regular variation.

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Abstract

This paper extends the topological theory of regular variation of the slowly varying case of [BOst13] to the regularly varying functions between metric groups, viewed as normed groups (see also [BOst14]). This employs the language of topological dynamics, especially flows and cocycles. In particular we show that regularly varying functions obey the chain rule and in the non-commutative context we characterize pairs of regularly varying functions whose product is regularly varying. The latter requires the use of a 'differential modulus' akin to the modulus of Haar integration.

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1 Introduction

In [BOst13] and [BOst14] we developed the basic theory of regular variation up to the Uniform Convergence Theorem (UCT) for functions h defined on a metric (topological) group X with values in a metric group H. We employed the language of topological dynamics (for which see [Ell1]), specifically T-flows on X, for T a group, that is continuous maps $\varphi : T \times X \to X$ satisfying

$$\varphi(st, x) = \varphi(s, \varphi(t, x)), \text{ with } \varphi(e_T, x) = x,$$

where e_T is the identity element of T. With tx denoting $t(x) := \varphi(t, x)$, this enabled us to define the dual cocycles (for which see [Ell2]) of regular variation as

$$\sigma_h(t, x) := h(tx)h(te_X)^{-1} \text{ and } \tilde{\sigma}_h(x, t) := h(tx)h(x)^{-1},$$

leading to the formulas

$$\partial_T h(x) = \lim_{x\to\infty} \tilde{\sigma}_h(x, t) \text{ and } \partial_X h(t) = \lim_{x\to\infty} \sigma_h(x, t), \text{ i.e., as } d(x, e_X) \to \infty.$$

Here we assume the limits are defined and exist. (The Characterization Theorem of [BOst14] asserts that it suffices for the limits to exist on a non-meagre set.) When either limit is identically the identity element, respectively of X or T, the function h is said to be *slowly varying*; two corresponding theorems assert uniform convergence on compacts. When $X = \mathbb{R}$ and $T = \mathbb{R}^*_+$ (the multiplicative group of strictly positive reals), these formulas yield one and the same classical definition of regular variation, for which see [BGT].

Here we extend the theory to regularly varying functions and consider the their 'calculus': matters such as factorization of a regularly varying function into a multiplicative function and a slowly varying one, and circumstances under which products of regularly varying functions are regularly varying. These matters are straightforward in an abelian-group setting. Here we find that there is a satisfactory non-commutative theory, provided the metric is appropriately invariant, although on occasion a Haar-like modulus function is required (cf. [Na]).

We recall a number of definitions from [BOst12], to which we refer for justification and proof in the absence of other citations. Let X be a metric group with identity element e_X and with a metric d_X, which we assume is right-invariant (the Birkhoff-Kakutani Metrization Theorem secures this property, cf. [Bir], [Kak]). It is helpful to refer to the associated *group-norm* $||x|| := d_X(x, e_X)$, an equivalent way of describing the right-invariant metric structure, where a group-norm $||\cdot|| : X \to \mathbb{R}_+$ is defined by the following three properties:

(i) Subadditivity (Triangle inequality): $||xy|| \leq ||x|| + ||y||$;

(ii) Positivity: $||x|| > 0$ for $x \neq e$;

(iii) Inversion (Symmetry): $||x^{-1}|| = ||x||$.

We can thus be guided by the normed vector-space calculus paradigm. We denote by $Auth(X)$ the group of self-homeomorphisms of X under composition. $\mathcal{H}(X)$ denotes the subgroup

$$\{h \in Auth(X) : ||h|| < \infty\},$$

where, in turn,

$$||h|| := d_X^*(h, e_{\mathcal{H}(X)}) = \sup_{x \in X} d_X(h(x), x))$$

denotes the group-norm on $\mathcal{H}(X)$, which metrizes it by the right-invariant metric $d(g, h) = ||gh^{-1}||$.

2 Topological regular variation : Fréchet case

Definitions. Let X be a metric space with a distinguished point z_0. This will usually be e_X, but on occassion other choices are convenient. Let $\mathcal{G}$ be some fixed 'ground group' of homeomorphisms of X into itself acting transitively on X. Thus X is a homogeneous space. Let $\varphi = \{\varphi_n\}$ be a divergent sequence in $\mathcal{G}$. Let H be a normed group.

We say that $h : X \to H$ is *φ-regularly varying,* or if context permits, just *Fréchet regularly varying*, if for some function $k(.) = \partial_\varphi h(.)$ and, for each t,

$$h(\varphi_n(t))h(\varphi_n(z_0))^{-1} \to k(t).$$

We have thus preferred division on the right and so, strictly speaking, have defined right–regular variation (left-regular requiring division on the left); we return to this matter below. The definition of Φ-*regularly varying* follows that of Φ-*slowly varying* (for which see [BOst13]), to which this case reduces when $k(t) \equiv e_H$. In particular, for Φ generated from a divergent sequence $\varphi = \{\varphi_n\} \subseteq \mathcal{G}$ by composing φ_n with the bounded homeomorphisms of

$\mathcal{H}(X)$, we will say that h is *strongly φ-regularly varying*. We refer to the function $\partial_\varphi h$ as the *limit function*, or the Φ-*limit function.*

We will be exclusively concerned with *Baire* functions (functions with the Baire property, see below for the definition). When $h : X \to H$ and X, H are locally compact topological groups it is natural to consider h measurable in the sense of Haar measures on X and H. Then the limit function $k := \partial_\varphi h$ is also measurable. We shall soon see that k is then a homomorphism. According to Kodaira's theorem ([Kod], corollary to Satz 18. p. 98) k is measurable iff k is continuous (so iff Baire), since the Weil topology determined by a measure is the original topology of the group – see [We], and [Hal1] Ch. XII. See also [BOst12] Section 5 for an integrated treatment in the context of normed groups.

In a companion paper [BOst16], we study $\mathbb{R}$-flows, i.e. group actions specializing T to $\mathbb{R}$, and so one needs to discriminate between cases. By analogy with the theory of differentiation in functional analysis (compare [HP] Ch. III and [Ru-FA1] 1st ed., omitted in 2nd ed.) we shall there call these cases Fréchet, Gâteaux and Hadamard. The limit function k here will there be called the Fréchet limit function.

Given a bounded homeomorphism τ we will later identify the point (image) $t = \tau(z_0)$ in the definition above with τ. Thus

$$k(\tau), \text{ or } k(\tau(z_0)) = \lim h(\varphi_n(\tau(z_0)))h(\varphi_n(e_X))^{-1}. \qquad \text{(RV)}$$

This enables us to interpret k as a mapping from $\mathcal{H}(X)$ to H. Our first proposition shows the effect of changing the distinguished point.

Proposition (Concatenation Formula). *If h is φ-regularly varying for the distinguished point $z = z_0$, then for any w the corresponding Fréchet limit $k_w(x) = \lim h(\varphi_n(x))h(\varphi_n(w))^{-1}$ exists and*

$$k_z(x) = k_z(w)k_w(x).$$

Proof. We have

$$\begin{aligned} k_z(x) &= \lim h(\varphi_n(x))h(\varphi_n(w))^{-1}h(\varphi_n(w))h(\varphi_n(z))^{-1} \\ &= k_w(x)k_z(w). \qquad \square \end{aligned}$$

Our next result demonstrates that we may identify $k(x)$ and $k(\phi_x)$, despite the fact that there will be more than one homeomorphism mapping z_0 to x.

Definition. Here (as in Section 4 of [BOst13]) let $\mathcal{H}_0 = \{\phi \in \mathcal{H}(X) : \phi(z_0) = z_0\}$ be the *stabilizer subgroup* (of the distinguished null point). Note that this is conjugate to the stabilizer of any other point of the (homogeneous) space X. Thus, for σ, τ in $\mathcal{H}(X)$ with $\sigma(z_0) = \tau(z_0)$, we have $\sigma^{-1}\tau \in \mathcal{H}_0$. We will regard two homeomorphisms σ, τ in $\mathcal{H}(X)$ that are $\mathcal{H}_0$-equivalent (i.e. both in the same coset of $\mathcal{H}_0$, e.g. $\tau \in \sigma\mathcal{H}_0$) as equal. Whenever convenient we will denote by ϕ_x the unique homeomorphism (up to equivalence) taking z_0 to x. This is particularly useful when G is a topological group, where the canonical choice is

$$\phi_u(g) = \tau_u(g) = ug,$$

as we then have $\phi_u\phi_v = \phi_{uv}$. The following result justifies use of $\mathcal{H}_0$-equivalence.

Proposition. *If h is strongly φ-regularly varying and ϕ is a bounded homeomorphism with $\phi(z_0) = z_0$, then the corresponding Fréchet limit function satisfies $k(\phi(t)) = k(t)$.*

Proof. We have

$$k(\phi(t)) = \lim h(\varphi_n(\phi(t)))h(\varphi_n(z_0))^{-1} = \lim h(\varphi_n(\phi(t)))h(\varphi_n(\phi(z_0)))^{-1} = k(t).$$

□

Now consider the homeomorphism

$$\sigma(x, y) := \phi_y\phi_x^{-1}.$$

Since $\sigma(x, y)(x) = y$, this is just the canonical homeomorphism taking x to y. Moreover,

$$\phi_y = \sigma(x, y)\phi_x,$$

so that σ is a *coboundary cocycle* (the defining property being the last equation) given the present context which treats the homeomorphism t from x to y as unique so that y and xt are indistinguishable (see e.g. [Ell2]). Of course, in the group context we have $\sigma(x, y) := \phi_{yx^{-1}}$.

Proposition (Coboundary Property). *If k is strongly φ-regularly varying, then k is a homomorphism from the group of bounded homomorphisms $\mathcal{H}(X)$ into the normed group H, that is*

$$k(\sigma\tau) = k(\sigma)k(\tau).$$

In particular, k has the coboundary property,

$$k(\phi_y) = k(\sigma(x,y))k(\phi_x),$$

and hence, if X is a topological group, then

$$k(\phi_{uv}) = k(\phi_u)k(\phi_v).$$

Proof. For bounded σ, τ we have

$$\begin{aligned} k(\sigma\tau) &= \lim h(\varphi_n(\sigma(\tau(z_0))))h(\varphi_n(z_0))^{-1} \\ &= \lim\{[h(\varphi_n(\sigma(\tau(z_0))))h(\varphi_n(\sigma(z_0)))]^{-1} \cdot [h(\varphi_n(\sigma(z_0)))h(\varphi_n(z_0))]^{-1}\} \\ &= k(\tau)k(\sigma). \end{aligned}$$

The coboundary property follows from taking $\tau = \phi_x$ and $\sigma(x,y) = \phi_y\phi_x^{-1}$ so that $\sigma\tau = \phi_y$.

As to the final equation, take $v = x$, $u = yx^{-1}$ to obtain $uv = y$ and note $\sigma(x,y) = \phi_{yx^{-1}} = \phi_u$. $\square$

Our last results in this section assert continuity. One of the ingredients is an idea due to Banach (see [Ban-T] 1.3.4, p. 40 in collected works, cf. [Meh], see also the Banach-Mehdi Theorem in the companion paper [BOst14] and associated literature cited there). As there, so too here, we refer to functions with properties related to the classical property of Baire. For background on Baire sets (i.e., sets with the Baire property) we refer to Kechris ([Kech]; see section 8.F p. 47) and on Baire category and *Baire spaces*, we refer to Engelking ([Eng]; see especially p.198 Section 3.9 and Exercises 3.9.J), although we prefer 'meagre' to 'of first category'. In our more general context we need to distinguish between three possible interpretations of the Baire property in relation to functions, as follows.

Definitions.

1. Say that a function $f : X \to Y$ between two topological spaces is $\mathcal{H}$*-Baire*, for $\mathcal{H}$ a class of sets in Y, if $f^{-1}(H)$ has the Baire property (i.e. $f^{-1}(H)$ is open in X modulo the meager sets of X) for each set H in $\mathcal{H}$. Thus f is $\mathcal{F}(Y)$-Baire if $f^{-1}(F)$ has the Baire property for all closed F in Y. Since

$$f^{-1}(Y \backslash H) = X \backslash f^{-1}(H),$$

f is $\mathcal{F}(Y)$-Baire iff it is $\mathcal{G}(Y)$-Baire, when we will simply say that f is *Baire* ('f has the Baire property' is the alternative usage).

2. We distinguish between functions that are $\mathcal{F}(Y)$-Baire and those that lie in the smallest family of functions closed under pointwise limits of sequences and containing the continuous functions (for a modern treatment see [Jay-Rog] Sect. 6). We follow tradition in calling these last *Baire-measurable.*

3. We will say that a function is *Baire-continuous,* if it is continuous when restricted to some co-meagre set.

The connections between these concepts are given in the theorems below. See the cited papers for proofs.

Banach-Neeb Theorem. ([Ban-T] Th. 4 pg. 35, and Vol I p. 206; [Ne]).

(i) *A Baire-measurable $f : X \to Y$ with X a Baire space and Y metric is Baire-continuous.*

(ii) *A Borel-measurable $f : X \to Y$ with X, Y metric and Y separable is Baire-measurable.*

Remarks. In fact Banach shows that a Baire-measurable function is Baire-continuous on each perfect set ([Ban-T] Vol. II p. 206). Neeb assumes in addition that Y is arcwise connected, but, as Pestov remarks in a review of the paper, the arcwise connectedness may be dropped by referring to a result of Hartman and Mycielski [HM] that a separable metrizable group embeds as a subgroup of an arcwise connected separable metrizable group.

Baire Continuity Theorem. *A Baire function $f : X \to Y$ is Baire continuous in the following cases:*

(i) Baire condition (see e.g. [THJ] Th. 2.2.10 p. 346): *Y is a second-countable space;*

(ii) Emeryk-Frankiewicz-Kulpa ([EFK]): *X is Čech-complete and Y has a base of cardinality not exceeding the continuum;*

(iii) Hansell condition ([Han]): *f is σ-discrete and Y metric;*

(iv) Pol condition ([Pol]): *X is Borelian-K and Y metrizable and of nonmeasurable cardinality.*

Remarks. Hansell's condition, requiring the function f to be σ-discrete, is implied by f being analytic when X is absolutely analytic (i.e. Souslin-$\mathcal{F}(X)$ in any complete metric space X into which it embeds). Frankiewicz [Fr] considers implications of the axiom of constructibility.

We will say that the pair (X, Y) *enables Baire continuity* if the spaces X, Y satisfy any one of the three conditions (i), (ii), or (iv). In the applications Y is usually the additive group of reals $\mathbb{R}$, so satisfies (i).

We recall a definition from [BOst13].

Definitions.
1. Let $\psi_n : X \to X$ be auto-homeomorphisms. We say that a sequence ψ_n in $\mathcal{H}(X)$ *converges to the identity* if

$$||\psi_n|| = d^*(\psi_n, id) := \sup_{t \in X} d(\psi_n(t), t) \to 0.$$

2. Say that X has the *crimping property* at z_0 if for any null sequence $z_n \to z_0$, there is a sequence of homeomorphisms ψ_n converging to the identity (so necessarily in $\mathcal{H}$) with $\psi_n(z_0) = z_n$. We refer to the ψ_n as a *crimping sequence* at z_0. Say that X has the crimping property *globally* if it has the crimping property at all points.

Theorem (Continuous Coboundary Theorem). *Suppose that X is a Baire space with the crimping property (as in the UCT). Let H be a topological group such that the pair (X, H) enable Baire continuity. If $h : X \to H$ is Baire regularly varying with limit function k, then k is Baire, has the coboundary property*

$$k(\phi_y) = k(\sigma(x, y))k(\phi_x),$$

equivalently

$$k(\phi_x \phi_y) = k(\phi_x)k(\phi_y),$$

and is continuous.

Proof. First observe that $k(.)$ is Baire. Indeed, for each $r > 0$, the corresponding level set $T_r := \{t : |k(t)| < r\}$ may be expressed as

$$T_r = \bigcup_{k \in \omega} \bigcap_{n \geq k} \{t : |h(\varphi_n(t))h(\varphi_n(z_0))^{-1}| < r\},$$

and this is a Baire set, because the Baire sets form a σ-algebra and each set $\{t : |h(\varphi_n(t))h(\varphi_n(z_0))^{-1}| < r\}$ is Baire by the continuity of φ_n and the assumption that h is Baire. Now

$$X = \bigcup_{r \in \mathbb{Q}_+} T_r,$$

so since X is a Baire space, the set T_r for some r is non-meagre.

We have already demonstrated the coboundary property.

We first set out the proof of continuity at z_0. Take $z_n \to z_0$; we will show that $k(z_n) \to k(z_0) = 0$. By the crimping property, we may choose a sequence ψ_n converging to the identity with $z_n = \psi_n(z_0)$. Being Baire, the function k is continuous on a co-meagre set D. Now

$$T := \bigcap_{n=1}^{\infty} \{t : \psi_n(t) \in D\} = \bigcap_{n=1}^{\infty} \psi_n^{-1}(D)$$

is co-meagre and so non-empty, since each ψ_n is a homeomorphism. Let $t_0 \in T$. Select τ with $\tau(z_0) = t_0$. Put $t_n = \psi_n(t_0) = \psi_n(\tau(z_0))$. Thus $\{t_n : n \in \omega\} \subseteq D$ and $t_n \to t_0$, since ψ_n converges to the identity. Writing ψ_n for σ in the Coboundary Property, we obtain

$$k(\psi_n(\tau(z_0))) = k(\psi_n(z_0))k(\tau(z_0)),$$

or

$$k(t_n) = k(z_n)k(t_0).$$

Since k is continuous on D at t_0 we conclude that $k(z_n) \to k(t_0)k(t_0)^{-1} = e = k(z_0)$. Thus k is continuous at z_0.

To prove continuity at an arbitrary location x_0, first choose a bounded homeomorphism ξ with $\xi(z_0) = x_0$. Put $z_n = \xi^{-1}(x_n)$. Then $z_n \to z_0$, so we may choose a (crimping) sequence ψ_n converging to the identity with $z_n = \psi_n(z_0)$. As we wish to prove a topological result about k we may, by the deGroot and McDowell Lemma ([dGMc]), assume w.l.o.g. that ξ is uniformly continuous. Thus, by Lemma 2, the conjugate sequence $\bar{\psi}_n = \xi\psi_n\xi^{-1}$ converges to the identity. As before,

$$T := \bigcap_{n=1}^{\infty} \{t : \bar{\psi}_n(t) \in D\} = \bigcap_{n=1n}^{\infty} \bar{\psi}_n^{-1}(D)$$

is non-empty. For $t_0 \in T$, we have $t_n = \bar{\psi}_n(t_0) \to t_0$, since $\bar{\psi}_n$ converges to the identity. So $k(t_n) \to k(t_0)$, as $t_n \in T$. Writing $\xi\psi_n$ for σ in the Coboundary Property, and noting that $\xi\psi_n(z_0) = x_n$, we obtain

$$k(\xi\psi_n(\tau(z_0))) = k(\xi\psi_n(z_0))k(\tau(z_0)).$$

So

$$k(t_n) = k(x_n)k(t).$$

Thus $k(x_n) \to k(t_n)k(t)^{-1} \to k(t_0)k(t)^{-1} = k(x_0)$, since with σ replaced by ξ in the Coboundary Property we have

$$k(t_0) = k(\xi\tau(z_0)) = k(\xi(z_0))k(\tau(z_0)) = k(x_0)k(t).$$

So again $k(x_n) \to k(x_0)$, and k is continuous. □

In particular specializing X to topological groups, and taking $\phi_x(z) = \tau_x(z) = xz$, one has:

Corollary (Continuous Homomorphism Theorem). *Suppose that $h : X \to H$ is a Baire regularly varying function defined on a Baire topological group X with values in the topological group H, and that the pair (X, H) enables Baire continuity. If h has a limit function k, then k is a continuous homomorphism, i.e.*

$$k(xy) = k(x)k(y).$$

Comments.

1. When investigating the limit function $\partial_\varphi h$ in the topological group context one should restrict attention to divergent sequences φ that are admissible in the following sense. If $\mathcal{K}(G, \mathbb{R}) \subseteq \mathcal{C}(G, \mathbb{R})$ is the subspace of (continuous) homomorphisms from a topological group G to the additive group of the reals $\mathbb{R}$, then we say that $\varphi = \{\varphi_n\}$ is *admissible* if, for each k in $\mathcal{K}(G, \mathbb{R})$,

$$\partial_\varphi k := \lim_n k(\varphi_n(g))k(\varphi_n(z_0))^{-1} \in \mathcal{K}(G, \mathbb{R}).$$

For example, when $G = \mathbb{R}$ with Φ comprising affine homeomorphisms, a sequence $\varphi_n(x) = a_n x + b_n$ is admissible if $a_n \to a$ is convergent and $|b_n| \to \infty$. Indeed, if $k(x) = \kappa x$, then we have $\partial_\varphi k(x) = ak(x)$, as

$$\kappa(a_n x + b_n) - \kappa b_n = \kappa a_n x \to \kappa a x.$$

2. Isometries are special, but Brouwer's Plane Translation Theorem asserts that any orientation preserving fixed-point-free homeomorphism of $\mathbb{R}^2$ is topologically conjugate to a translation, e.g. $\varphi_{e_1}(x) := x + e_1 = (x_1 + 1, x_2)$. See for example [Gu].

3 The calculus

We begin by recalling that for X, H normed groups $h : X \to H$ is φ-regularly varying in the *weak sense*, for $\varphi = \{\varphi_n\}$ a divergent sequence of auto-homeomorphisms of X, if, for some function $k : X \to H$,

$$h(\varphi_n(x))h(\varphi_n(e))^{-1} \to k(x) \text{ for all } x \in X, \text{ as } n \to \infty,$$

with $e = e_X$, the identity element of X, i.e.

$$d_H(h(\varphi_n(x))h(\varphi_n(e))^{-1}, k(x)) \to 0.$$

In this section we work with this weaker form. When $\varphi_n(x) = u_n x$, we have $||\varphi_n|| = d^*_X(id_X, \varphi_n) = \sup d_X(x, u_n x) = d_X(e, u_n) = ||u_n||$, and so the definition reduces to

$$h(u_n x)h(u_n)^{-1} \to k(x) \text{ for all } x \in X, \text{ as } n \to \infty.$$

We note that by the triangle inequality (cf. Corollary in Section 2 of [BOst13])

$$||u_n|| - ||x|| \leq ||u_n x|| \leq ||u_n|| + ||x||,$$

so that, in some sense, a fixed x provides a relatively small increment to the point at infinity (however, here we do not have an upper bound on $||x||/||u_n x||$); on that basis we may think of $f(u_n x)f(u_n)^{-1}$ as a generalized differential quotient. These analogies are driven by the abelian case, when we may write additively

$$d(h(u_n + x) - h(u_n), k(x)) \to 0, \text{ for all } x \in X; \text{ as } n \to \infty.$$

Correspondingly, here $k(x)$ is linear, and thus the differential $h(u_n + x) - h(u_n)$ is linearly approximated. Passing to a normed vector space X, one has

$$||h(u_n + x) - h(u_n) - k(x)|| \to 0, \text{ for all } x \in X, \text{ as } n \to \infty,$$

which is differential calculus proper. This is the ultimate justification for borrowing differential terminology; in particular, we write $\partial_\varphi h$ for the limit function, when it exists. Indeed topological groups were taken by A. D. Michal and his collaborators as a canonical setting for differential calculus (see the review [Mich] and as instance [JMW]).

As a first application of the concept of normed group we prove the following.

Proposition (Chain Rule of Regular Variation). *Let X, G, H be normed groups. Let $g : X \to G$ and $h : G \to H$ be regularly varying with g diverging under the group-norm of G, i.e.*

$$||g(x)||_G \to \infty, \text{ as } ||x||_X \to \infty,$$

and suppose that G is locally compact. Then

$$\partial_X(h \circ g)(t) = \partial_G h \partial_X g(t).$$

Proof. Fix t. Put

$$g(tx) = a\partial_X g(t)g(x) \text{ with } a = a(x) \to e_G, \text{ as } ||x|| \to \infty.$$

Then in the limit as $||x|| \to \infty$, we have with $y = g(x)$ that $||y|| \to \infty$ and so for s in a compact set

$$h(sy)h(y)^{-1} = b\partial_G h(s) \text{ with } b = b(s, y) \to e_H, \text{ as } ||y|| \to \infty.$$

We take s such that

$$s = a(x)\partial_X g(t),$$

which, for large x, remains in a compact neighbourhood of $\partial_X g(t)$.

Now $\partial_G h$ is a continuous homomorphism, so that $\partial_G h(a) \to e_H$ as $a \to e_G$, and so

$$\begin{aligned} h(g(tx))h(g(x))^{-1} &= h(a\partial_X g(t)g(x))h(g(x))^{-1} = b\partial_G h(a\partial_X g(t)) \\ &= b\partial_G h(a)\partial_G h(\partial_X g(t)) \to e_H \partial_G h(\partial_X g(t)). \end{aligned}$$

Thus

$$\partial_X(h \circ g) = \partial_G h \circ \partial_X g,$$

as asserted. □

Our main concern in this section is with products of regularly varying functions. In the classical context of the real line it is obvious that the product of two regularly varying functions is regularly varying. This is also true in the context of functions $h : X \to H$ when the group H is abelian and the metric is invariant. What may be said if H is non-commutative? It has to be appreciated that our definition of regular variation opted for division on the right, so to be fair the question should address one-sided multiplication (in fact on the left, see below). To guess the answer, focus on the special case of two multiplicative functions $k(x)$ and $K(x)$ with $K(x) = x$; if the product $k(x)K(x)$ were to be regularly varying, one would expect it to be multiplicative, and the latter property is equivalent to

$$k(xy)xy = k(x)xk(y)y, \text{ i.e. } k(y)x = xk(y);$$

this asserts that each value $k(y)$ commutes with each element x in the group H. One guesses that the range of k must lie in the center $Z(H)$ of the group H. (We recall that the subgroup $Z(H) = \{a \in H : ah = ha \text{ for all } h \in H\}$ is the *centre*.)

Definition. A function $k : X \to H$ will be termed *central* if the range of k is in the centre $Z(H)$.

Thus if H has trivial centre $k(x) := e_H$. We show here that a non-commutative theory may be developed justifying the guess and yielding a Left Product Theorem which characterizes the admissible left factor as the product kh of a central function k with a slowly varying function h (subject to a mild regularity assumption). The theory requires that the group H exhibit a strong metric property, one satisfied in the usual abelian case of $\mathbb{R}$ and $\mathbb{C}$, namely *bi-invariance* (two-sided invariance). Thus our theory extends the classical case of $\mathbb{R}$ and $\mathbb{C}$. Bi-invariance is equivalent, as Klee [Klee] shows, to the existence of a metric possessing what we term *Klee's property*:

$$d_H(ab, xy) \leq d_H(a, x) + d_H(b, y). \tag{1}$$

This is equivalent (see [BOst12] Section 2) to the norm property

$$||ab(xy)^{-1}|| \leq ||ax^{-1}|| + ||by^{-1}||.$$

We recall also Klee's result [Klee] that, when the group H is topologically complete and abelian, then it admits a metric which is *bi-invariant* (i.e. both right- and left-invariant). However, we work with the assumption of bi-invariance occasionally only, and sometimes also require completeness.

Definitions. We call a metric with Klee's property (1) a *Klee metric* for H. We call H a *Klee group* if its metric d_H is a Klee metric.

The bi-invariance property acts as a replacement for commutativity, and is exactly the condition which allows a proper development of the calculus of regularly varying functions, mimicking the non-commutative development of the Haar integral (see e.g. [Na]). Traditionally regular variation finds its uses in probability theory, where $H = \mathbb{R}$ (the result of probabilities being real), so our restriction offers an expansion of the theory which, in particular, takes in its stride applications to complex analysis. For a discussion of bi-invariance in the context of matrices see e.g. [Bha] Section 3. We begin with basic factorization theorems where one factor, a right-factor, is slowly varying. This way round is easy by virtue of the definition of regular variation on the 'right' (the division term being on the right). The other way about requires the presence of some 'central' features, as we shall see later.

Proposition (Preservation under inversion). *Suppose H has a bi-invariant metric. If $h : X \to H$ is φ-slowly varying, then the mapping $h^{-1} : x \to h(x)^{-1}$ is φ-slowly varying. Hence the product of two φ-slowly varying functions is φ-slowly varying.*

Proof. Indeed, we have

$$d_H(h(\varphi_n(t))^{-1}h(\varphi_n(z_0), e_H) = d_H(h(\varphi_n(z_0), h(\varphi_n(t))) = d_H(e_H, h(\varphi_n(t))h(\varphi_n(z_0)^{-1}),$$

so $h(.)$ is slowly varying iff $h(.)^{-1}$ is slowly varying. Using this we see that for h, h' slowly varying we have

$$\begin{aligned} & d_H(h(\varphi_n(t))h'(\varphi_n(t))h'(\varphi_n(z_0)^{-1}h(\varphi_n(z_0)^{-1}, e_H) \\ = \; & d_H(h(\varphi_n(t))h'(\varphi_n(t))h'(\varphi_n(z_0)^{-1}, h(\varphi_n(z_0)) \\ = \; & d_H(h'(\varphi_n(t))h'(\varphi_n(z_0)^{-1}, h(\varphi_n(t))^{-1}h(\varphi_n(z_0)) \\ \to \; & d(e_H, e_H) = 0. \end{aligned}$$

Thus hh' is slowly varying. $\square$

First Factorization Theorem. *Suppose H has a bi-invariant metric. If $h : X \to H$ is φ-regularly varying, then, with $k = \partial_\varphi h(t)$,*

(i) *$k(t)$ is φ-regularly varying and $k(t) = \partial_\varphi k(t)$;*

(ii) *$\bar{h}(t) := k(t)^{-1}h(t)$ is φ-slowly varying. Thus $h(t)$ is the left product of its limit function with a slowly varying function $\bar{h}$:*

$$h(t) = \partial_\varphi h(t) \cdot \bar{h}(t).$$

Proof. For fixed n, since φ_n is bounded $\varphi_m(\varphi_n(.))$ is, by Lemma 3 of [BOst13], a divergent sequence, so

$$\begin{aligned}k(\varphi_n(t))k(\varphi_n(z_0))^{-1} &= \lim_m [h(\varphi_m(\varphi_n(t)))h(\varphi_m(z_0))^{-1}][h(\varphi_m(\varphi_n(z_0)))h(\varphi_m(z_0))^{-1}]^{-1} \\ &= \lim_m [h(\varphi_m(\varphi_n(t)))h(\varphi_m(\varphi_n(z_0)))^{-1} \\ &= k(t).\end{aligned}$$

So k is regularly varying and, as $n \to \infty$,

$$\begin{aligned}& d_H(k(\varphi_n(t))^{-1}h(\varphi_n(t))[k(\varphi_n(z_0))^{-1}h(\varphi_n(z_0))]^{-1}, e_H) \\ = \; & d_H(k(\varphi_n(t))^{-1}h(\varphi_n(t))h(\varphi_n(z_0))k(\varphi_n(z_0)), e_H) \\ = \; & d_H(h(\varphi_n(t))h(\varphi_n(z_0))]^{-1}, k(\varphi_n(t))k(\varphi_n(z_0))^{-1}) \\ \to \; & d_H(k(t), k(t)) = 0.\end{aligned}$$

That is, $k(t)^{-1}h(t)$ is slowly varying. □

As a converse result, we have the following.

Second Factorization Theorem *If H is a Klee group, g is regularly varying and h is slowly varying, then $g(t)h(t)$ is regularly varying with limit $\partial_\varphi g$.*

Proof. Put $h_n(t) = h(\varphi_n(t))$ and $h_n = h_n(e)$ and let $k = \partial_\varphi g$. Then

$$\begin{aligned}d_H(g_n(t)h_n(t)h_n^{-1}g_n^{-1}, k) &= \lim d_H(g_n(t)h_n(t)h_n^{-1}g_n^{-1}, g_n(t)g_n^{-1}) \\ &= \lim d_H(h_n(t)h_n^{-1}, e) = 0. \qquad \square\end{aligned}$$

To progress further we need the idea of asymptotic conjugacy in a group (cf. [KiKu] in the context of a C^*-algebra where approximate inner automorphism are obtained from a sequence of unitary elements). Our analysis is inspired by the non-commutative theory of the Haar integral (cf. [Na], Ch. 2.5). To motivate our definition we first consider a number of special cases.

Proposition. *In a locally compact Klee group H, there exist divergent sequences $\eta = \{h_n : n \in \omega\}$ for which $\{h_n a h_n^{-1}\}$ is convergent for some $a \neq e_H$.*

Proof. We begin by observing that, for any (divergent) sequence $\eta = \{h_n : n \in \omega\}$ in a Klee group H and for any $a \neq e_H$,

$$||h_n a h_n^{-1}|| = d_H(h_n a h_n^{-1}, e) = d_H(h_n a h_n^{-1}, h_n h_n^{-1}) = d_H(a, e) = ||a||.$$

Thus, for any $a \neq e_H$, the sequence $\{h_n a h_n^{-1}\}$ has a convergent subsequence; passage to a convergent subsequence yields the conclusion. □

Of course, in an abelian group, asymptotic conjugacy is just the identity, so convergence of the sequence $\{h_n a h_n^{-1}\}$ holds at each a, likewise when $\{h_n\}$ lies in the centre $Z(H)$; more significantly, convergence holds at all points when $\{h_n\}$ is *centrally asymptotic* (i.e. asymptotic to the centre) in the two senses captured in (i) and (ii) of the Proposition below. The summability assumption in (ii) is motivated by a condition occurring in Kendall's Theorem ([BGT], Th. 1.9.2 and its variants 1.9.3 & 4), namely

$$\limsup x_n = \infty \text{ and } \text{limsup } x_{n+1}/x_n = 1.$$

We recall Kendall's Theorem: a continuous function $f : \mathbb{R} \to \mathbb{R}$ for which $\{a_n f(x_n t)\}$ converges to a continuous function of t, for some sequences $\{a_n\}$ and $\{x_n\}$ as above, is regularly varying. Thus here f is φ-regularly varying for the sequence $\varphi_n(t) = tx_n$. We recall that $||h|| := d_H(h, e)$, so that the condition $||h_{n+1}h_n^{-1}|| \to 0$ (implied by the summability condition) is equivalent to the second Kendall condition $d_H(h_{n+1}h_n^{-1}, e) \to 0$, when H is interpreted as $\mathbb{R}_+^*$, the multiplicative group of strictly positive reals.

Proposition (Centrally asymptotic sequences).
(i) *If $k_n \in Z(H)$ and $d_H(k_n, h_n) \to 0$, then, for all a, $\lim h_n a h_n^{-1} = a$.*
(ii) *If H is complete and h_n satisfies the summability condition*

$$\sum_n ||h_{n+1}h_n^{-1}|| < \infty, \tag{2}$$

then, for each a, $\{h_n a h_n^{-1}\}$ is convergent, as are $\{(h_n k_n)a(h_n k_n)^{-1}\}$ and $\{(k_n h_n)a(k_n h_n)^{-1}\}$ for $k_n \in Z(H)$.

Proof. (i) Since $k_n^{-1}h_n \to e$ and $k_n \in Z(H)$, we have

$$\begin{aligned} d_H(h_n a h_n^{-1}, a) &= d_H(h_n a h_n^{-1}, k_n a k_n^{-1}) \\ &= d_H(k_n^{-1} h_n a h_n^{-1} k_n, a) \to d_H(a, a) = 0. \end{aligned}$$

(ii) Here H is complete. Using the Klee property we obtain

$$\begin{aligned} d_H(h_n a h_n^{-1}, h_{n-1} a h_{n-1}^{-1}) &\leq 2 d_H(h_n, h_{n-1}) = 2 d_H(h_n h_{n-1}^{-1}, e_H) \\ &= 2||h_n h_{n-1}^{-1}||. \end{aligned}$$

For general $n > m$, we have

$$\begin{aligned} d_H(h_n a h_n^{-1}, h_m a h_m^{-1}) &\leq d_H(h_n a h_n^{-1}, h_{n-1} a h_{n-1}^{-1}) + ... + d_H(h_{m+1} a h_{m+1}^{-1}, h_m a h_m^{-1}) \\ &\leq 2\sum_{j=m}^{n-1} ||h_{j+1} h_j^{-1}||. \end{aligned}$$

Thus by the summability condition $\{h_n a h_n^{-1}\}$ is a Cauchy sequence and hence convergent (as H is complete). When k_n is in the centre, $h_n k_n a k_n^{-1} h_n^{-1} = h_n a h_n^{-1}$ and so again the sequence $\{(h_n k_n) a (h_n k_n)^{-1}\}$ is convergent. Likewise $\{(k_n h_n) a (k_n h_n)^{-1}\}$ is convergent, as $k_n(h_n a h_n^{-1}) k_n^{-1} = h_n a h_n^{-1}$. $\square$

We now show that in the non-commutative case the points a of convergence of a sequence $\{h_n a h_n^{-1}\}$ are well-structured. The choice of sign in the notation below is motivated by the Modular Flow Theorem to be established subsequently.

Asymptotic Conjugacy Theorem. *Let $\eta = \{h_n\}$ be any sequence of elements in a Klee group H. The sets of the points of convergence defined by*

$$\begin{aligned} D_+(\eta) &:= \{a \in H : h_n a h_n^{-1} \text{ is convergent}\}, \\ D_-(\eta) &:= \{a \in H : h_n^{-1} a h_n \text{ is convergent}\} \end{aligned}$$

are subgroups of H which are closed if H is complete. On $D_\pm(\eta)$ respectively define the asymptotically inner automorphisms:

$$A_+(\eta, a) := \lim h_n a h_n^{-1}, \qquad \text{and} \qquad A_-(\eta, a) := \lim h_n^{-1} a h_n.$$

Then $A_+(\eta, \cdot)$ is a continuous isomorphism from $D_+(\eta)$ onto $D_-(\eta)$ and

$$A_-(\eta, A_+(\eta, a)) = a.$$

In particular, $a \in D_\pm(\eta)$ iff $A_\pm(\eta, a) \in D_\mp(\eta)$.

Proof. We work with the plus versions. For a, b in $D_+(\eta)$ we have

$$\begin{aligned} \lim h_n a h_n h_n^{-1} b h_n^{-1} &= \lim h_n a h_n^{-1} \lim h_n b h_n^{-1} = A_+(\eta, a) A_+(\eta, b), \\ h_n a^{-1} h_n^{-1} &= (h_n a h_n^{-1})^{-1} \to A_+(\eta, a)^{-1}, \end{aligned}$$

showing that $D_+(\eta)$ is a subgroup of H on which $A_+(\eta,)$ is a homomorphism. Next we show that $A_+(\eta, a) = e$ has only one solution, namely $a = e$. Indeed we have

$$d_H(A_+(\eta, a), e) = d_H(\lim h_n a h_n^{-1}, e) = \lim d_H(h_n a h_n^{-1}, h_n h_n^{-1}) = \lim d_H(a, e) = ||a||.$$

Thus if $A_+(\eta, a) = e$, then $||a|| = 0$, i.e. $a = e$. Finally, we deduce that the homomorphism is onto $D_-(\eta)$, since

$$d_H(h_n a h_n^{-1}, A_+(\eta, a)) = d_H(a, h_n^{-1} A_+(\eta, a) h_n).$$

Suppose that a_m is a convergent sequence in $D_+(\eta)$ with limit a. Continuity of $A_+(\eta, \cdot)$ at a follows as

$$\begin{aligned} 0 &\leq d_H(h_n a_n h_n^{-1}, A_+(\eta, a)) \leq d_H(a_n, h_n^{-1} A(\eta, a) h_n) \\ &\leq d_H(a_n, a) + d_H(a, h_n^{-1} A(\eta, a) h_n) \\ &= d_H(a_n, a) + d_H(h_n a h_n^{-1}, A(\eta, a)) \to 0. \end{aligned}$$

Finally, suppose that $a_m \in D_+(\eta)$ and that $a_m \to a$. Put $A_m = A(\eta, a_m)$ and choose N_m so that for $n \geq N_m$

$$d_H(h_n a_m h_n^{-1}, A(a_m)) \leq 2^{-m}.$$

As

$$d_H(h_n a_s h_n^{-1}, h_n a_t h_n^{-1}) \leq d_H(a_s, a_t),$$

by bi-invariance, we deduce that $\{A_m\}$ is Cauchy. For given integers s, t, consider any $n > \max(N_s, N_t)$; here

$$\begin{aligned} d_H(A_s, A_t) &\leq d_H(A_s, h_n a_s h_n^{-1}) + d_H(h_n a_s h_n^{-1}, h_n a_t h_n^{-1}) \\ &\quad + d_H(h_n a_t h_n^{-1}, A_t) \\ &\leq d_H(A_s, h_n a_s h_n^{-1}) + d_H(a_s, a_t) + d_H(h_n a_t h_n^{-1}, A_t) \\ &\leq 2^{-s} + d_H(a_s, a_t) + 2^{-t}. \end{aligned}$$

Thus $\{A_m\}$ is Cauchy. Suppose now that H is complete; then $\{A_m\}$ has a limit, say, A. Now note that, for any m and any $n > N_m$, we have

$$\begin{aligned} d_H(h_n a h_n^{-1}, A) &\leq d_H(h_n a h_n^{-1}, h_n a_m h_n^{-1}) + d_H(h_n a_m h_n^{-1}, A_m) + d_H(A_m, A) \\ &\leq d_H(a, a_m) + 2^{-m} + d_H(A_m, A). \end{aligned}$$

So $\lim h_n a h_n^{-1} = A$. That is, $a \in D_+(\eta)$. $\square$

Definition. We say that the Klee group H is *asymptotically-invariant* for $\eta = \{h_n\}$ if $D_+(\eta) = H$, i.e., $h_n a h_n^{-1}$ converges for all $a \in H$ to an automorphism of X. We say that η is *inner* for A_+ if for some $h_\eta \in H$

$$A_+(\eta, a) = h_\eta a h_\eta^{-1},$$

in which case η will be said *asymptotically equivalent* to h_η for A_+. The latter condition implies that $h_\eta^{-1} h_n$ is inner and equivalent to the identity map id_H, since

$$d_H(\lim h_n a h_n^{-1}, h_\eta a h_\eta^{-1}) = \lim d_H(h_n a h_n^{-1}, h_\eta a h_\eta^{-1}) = \lim d_H(h_\eta^{-1} h_n a (h_\eta^{-1} h_n)^{-1}, a).$$

Definition. Let $g, h : X \to H$. In what follows we write $h_n = h(\varphi_n(e_X))$ and $g_n = g(\varphi_n(e_X))$. We say h is *modular* if H is asymptotically invariant for $\eta = \{h_n\}$, i.e., if for each a in H the sequence of conjugates of a in H

$$h_n a h_n^{-1}$$

is convergent. Note that

$$d_H(h_n^{-1} a h_n, b) = d_H(a, h_n b h_n^{-1}),$$

so h^{-1} is modular if h is. Consider the case $X = H$. Here id_X is modular iff X is asymptotically invariant for $\varphi = \{\varphi_n(e_X)\}$. We will see later that when H is non-abelian this cannot happen. This places a restriction on which functions $h : X \to X = H$ that can be modular; their range must be in the centre $Z(H)$.

Let $\mathcal{M} = \{h \in \mathcal{C}(H, H) : h \text{ is modular}\}$. We give $\mathcal{M}$ the supremum metric. Referring to the H-valued indicator function $\mathbf{1}_H(a) = e_H$, we have $\mathbf{1}_H \in \mathcal{M}$. We put

$$\begin{aligned} \Delta_+(h, a) \ &: \ = A_+(\{h_n\}, a) = \lim h_n a h_n^{-1}, \\ \Delta_-(h, a) \ &: \ = A_-(\{h_n\}, a) = \lim h_n^{-1} a h_n, \end{aligned}$$

and term these the *forward and backward (differential) moduli* of h (to distinguish them from the Haar integral moduli). Evidently $\Delta_+(\mathbf{1}_H, a) = a$, and

$$\Delta_\pm(h^{-1}, a) = \lim h_n^{\mp 1} a h_n^{\pm 1} = \Delta_\mp(h, a).$$

Lemma. *Under a bi-invariant Klee metric, for all* $a, b, g, h \in H$,

$$d_H(a,b) - 2d_H(g,h) \leq d_H(gag^{-1}, hbh^{-1}) \leq 2d_H(g,h) + d_H(a,b).$$

Proof. Referring to Klee's property, we have via the cyclic property

$$\begin{aligned} d_H(gag^{-1}, hbh^{-1}) &= ||gag^{-1}hb^{-1}h^{-1}|| = ||h^{-1}gag^{-1}h^{-1}b^{-1}|| \\ &\leq ||h^{-1}g|| + ||ag^{-1}h^{-1}b^{-1}|| \\ &\leq ||h^{-1}g|| + ||ab^{-1}|| + ||h^{-1}g||. \end{aligned}$$

Hence substituting $g^{-1}ag$ for a etc., then g^{-1} for g etc., we obtain

$$d_H(a,b) \leq 2d_H^*(g^{-1}, h^{-1}) + d_H(gag^{-1}, hbh^{-1}).$$

But d_H is bi-invariant, so

$$d_H(g^{-1}, h^{-1}) = \tilde{d}_H(g,h) = d_H(g,h). \qquad \square$$

Proposition. *Under a bi-invariant Klee metric on* H *the moduli,* $\Delta_\pm(.,.)$ *are uniformly jointly continuous on* $\mathcal{M} \times H$*, when* $\mathcal{M}$ *is given the supremum metric.*

Proof. By the Lemma

$$d_H(a,b) - 2d_H^*(g,h) \leq d_H(g_n a g_n^{-1}, h_n b h_n^{-1}) \leq 2d_H^*(g,h) + d_H(a,b). \qquad \square$$

Modular Flow Theorem. *Let* H *have bi-invariant Klee metric. Then, for* $h : X \to H$ *modular (in* $\mathcal{M}$*) the modular functions* $\Delta_\pm(h, \cdot)$ *are both isomorphisms of* H*.* $\mathcal{M}$ *is a group with identity* $\mathbf{1}_H$ *and* Δ_+ *is an* $\mathcal{M}$*-flow on* H*, that is, for all* a *and all* g, h *in* $\mathcal{M}$

$$\Delta_+(gh, a) = \Delta_+(g, \Delta_+(h,a)), \text{ and } \Delta_+(\mathbf{1}_H, a) = a;$$

moreover,

$$\Delta_+(h, \Delta_-(h,a)) = a.$$

Proof. We may solve for a the equation $\Delta_+(h,a) = b$. The solution is $a = \Delta_+(h^{-1}, b)$. Thus

$$\Delta_\pm(h, ab) = \lim h_n^{\pm 1} ab h_n^{\mp 1} = \lim h_n^{\pm 1} a h_n^{\mp 1} \lim h_n^{\pm 1} b h_n^{\mp 1} = \Delta_\pm(h,a)\Delta_\pm(h,b).$$

Moreover

$$a = \lim h_n h_n^{-1} a h_n h_n^{-1} = \Delta_+(h, \Delta_-(h, a)).$$

Since $k_n a k_n^{-1} \to \Delta_+(k, a)$, we have by continuity of $\Delta_+(h, \cdot)$ that

$$\Delta_+(gh, a) = \lim g_n h_n a h_n^{-1} g_n^{-1} = \Delta_+(g, \Delta_+(h, a)).$$

This implies first that gh is modular, secondly that, since h^{-1} is modular, $\mathcal{M}$ is a group, and thirdly that Δ_+ is an algebraic flow (i.e. without asserting continuity). Finally, by the previous Proposition it is a continuous flow (whereas Δ_- is the reversed flow). □

Left Product Theorem. *Suppose that g, h are φ-regularly varying with limit functions k and K, with g modular. Then gh is φ-regularly varying with limit*

$$\lim g(\varphi_n(xz_0))h(\varphi_n(xz_0))[g(\varphi_n(z_0))h(\varphi_n(z_0))]^{-1} = k(x)\Delta_+(g, K(x)).$$

Proof. Writing $g_n(x) = g(\varphi_n(x))$, $h_n(x) = h(\varphi_n(xe_X))$ and $k = k(x)$, $K = K(x)$, then, for any z,

$$\begin{aligned} \lim d_H(g_n(x)h_n(x)h_n^{-1}g_n^{-1}, kz) &= \lim d_H(g_n(x)h_n(x)h_n^{-1}g_n^{-1}, g_n(x)g_n^{-1}z) \\ &= \lim d_H(g_n h_n(x)h_n^{-1}g_n^{-1}, z) \\ &= d_H(\Delta_+(g, K(x)), z). \end{aligned}$$

Taking $z = \Delta_+(g, K(x))$, we obtain our result. □

Corollary 1 (Third Factorization Theorem). *If H is a complete Klee group, g is φ-regularly varying and h is φ-slowly varying and modular, in particular if $h_n = h(\varphi_n e_H)$ satisfies the summability condition* (2)*, then $h(t)g(t)$ is φ-regularly varying with limit $\Delta_+(h, k(t))$.*

Proof. Since h is modular and regularly varying we may apply the theorem. But we get more information by arguing directly as in the Second Factorization Theorem, aided this time by the modulus of h. As before, put $h_n(t) = h(\varphi_n(t))$ and $h_n = h_n(e)$ and let $k = \partial_X g$. Now h^{-1} is slowly varying, so with $\Delta_+ = \Delta_+(h, \cdot)$ and since $a = \Delta_-(h, \Delta_+(h, a))$ we have

$$\begin{aligned} d(h_n(t)g_n(t)g_n^{-1}h_n^{-1}, \Delta_+(k(t))) &= \lim d(g_n(t)g_n^{-1}, h_n^{-1}(t)\Delta_+(k(t))h_n) \\ &= \lim d(g_n(t)g_n^{-1}, [h_n^{-1}(t)h_n]h_n^{-1}\Delta_+(k(t))h_n) \\ &= \lim d(k(t), e\Delta_-(\Delta_+ k(t))) = 0. \qquad \square \end{aligned}$$

Corollary 2. *Suppose g is φ-regularly varying and modular with limit function k. Then for every φ-regularly varying function h and for all x, y, each element $\Delta_-(h, k(y))$ commutes with each element $\partial_\varphi h(x)$. In particular, k is central, i.e. the range$\{k(x) : x \in X\}$ is a subset of the centre $Z(H)$.*

Proof. With the assumptions as stated, we have, for all x, y,

$$\Gamma(x) := k(x)\Delta_+(g, K(x)).$$

Write $\Delta(.)$ for $\Delta_+(g, .)$. Now Γ is multiplicative, so since k and K are multiplicative we have, for all x, y,

$$\Gamma(xy) = k(xy)\Delta(K(xy)) = k(x)k(y)\Delta(K(x))\Delta(K(y))$$

and

$$\Gamma(x)\Gamma(y) = k(x)\Delta(K(x))k(y)\Delta(K(y)).$$

These equations together imply that, for all x, y,

$$k(x)\Delta(K(x))k(y)\Delta(K(y)) = k(x)k(y)\Delta(K(x))\Delta(K(y)).$$

Hence for all x, y

$$\Delta(K(x))k(y) = k(y)\Delta(K(x)). \tag{3}$$

Applying the result that Δ_+ and Δ_- are inverse isomorphisms, we obtain

$$K(x)\Delta_-(h, k(y)) = \Delta_-(h, k(y)K(x). \tag{4}$$

According to (3), for all x, y, each $K(x)$ commutes with each $\Delta_-(h, k(y))$. Taking $h(x) = x$ which is regularly varying with limit $K(x) = x$, we deduce that, since $\{\Delta_+(h, K(x)) : x \in X\} = H$, we have $\{k(y) : y \in X\} \subseteq Z(H)$. Likewise, according to (4), we see that $\{\Delta_-(h, k(x)) : x \in X\} \subseteq Z(H)$. $\square$

Remark. The corollary justifies the initial guess that the product theorem is valid when the left factor is central. If h_e is inner and equivalent to γ the corollary says that each $K(x)$ commutes with each $\gamma k(y)\gamma^{-1}$. From here it is easy to see that if the choices $k(x) = x$ were admitted, it would follow that H is abelian. Thus the theorem demonstrates how restrictive modularity is.

Corollary 3. *If H is asymptotically invariant for φ, then H is abelian.*

Proof. Indeed, then $g = h = k = K = id_H$ is modular; but then x^2 is φ-regularly varying which implies that for all x, y in H we have $(xy)^2 = xyxy = x^2y^2$ i.e. $yx = xy$, as asserted. □

We now restate the Proposition on centrally asymptotic sequences as a partial converse to the Product Theorem, thereby characterizing modularity for the regularly varying functions with our Kendall-like condition.

Theorem ('Nearly central' is modular). *Let H be a complete Klee group. Then, for $h : X \to H$ φ-slowly varying such that $h_n = h(\varphi_n e_H)$ satisfies the summability condition (2) and for k central, both hk and kh are modular.*

Theorem (Modular means 'nearly central'). *Let H be a complete Klee group. Then, for $h : X \to H$ φ-slowly varying such that $h_n = h(\varphi_n e_H)$ satisfies the summability condition (2), kh is modular iff k is central.*

4 Application: Seneta's sequential criterion

As an application of these ideas we deduce a generalization of Seneta's version of Kendall's Theorem concerning a sequential criterion for regular variation.

Definition. $\{x_n\}$ is a *divergent C-net* in X if $||x_n||$ diverges monotonically to infinity and, for each x, there is n with $||xx_n^{-1}|| < C$, i.e.

$$d_X(x_n, x) < C.$$

It is clear that Euclidean spaces have a divergent 1-net built from the corners of an expanding sequence of cubes.

Seneta's Theorem ([Sen], [BGT] Th. 1.9.3). *Let X be a locally compact group with right-invariant norm and let $\{x_n\}$ be a divergent C-net in X. Let H be a Klee group and let $f : X \to H$. Suppose that, for some modular sequence a_n in H,*

$$a_n f(\lambda x_n) \to k(\lambda),$$

convergence being uniform on compacts, and that $k : X \to H$ is multiplicative. Then f is regularly varying with limit function

$$\partial_X f(\lambda) = \Delta_-(a, k(\lambda)) = \lim_n a_n^{-1} k(\lambda) a_n.$$

Proof. Let λ be arbitrary. For any t, choose $n = n(t)$ such that

$$d(t, x_n) < C.$$

Now since

$$a_m f(\lambda x_m) \to k(\lambda)$$

on compact λ sets and tx_n^{-1} lies in the C-ball around e, we may make the substitution replacing λ with $\lambda t x_n^{-1}$ (as $||\lambda t x_n^{-1}|| \leq ||\lambda|| + ||tx_n^{-1}|| = ||\lambda|| + d(e, tx_n^{-1}) < ||\lambda|| + C$). Thus with $n = n(t)$

$$a_n f(\lambda t) = a_n f(\lambda t x_n^{-1} x_n) \to k(\lambda t x_n^{-1}),$$

as $||t|| \to \infty$ since $||x_{n(t)}|| \to \infty$. We thus have uniformly in t that

$$h_n(t) := a_n f(\lambda t) k(\lambda t x_{n(t)}^{-1})^{-1} \to e.$$

Likewise replacing λ now with tx_n^{-1} (again since $||tx_n^{-1}|| = d(e, tx_n^{-1}) < C$) we have

$$g_n(t) = a_n f(t) k(t x_{n(t)}^{-1})^{-1} \to e.$$

Thus $h_n(t)$ and $g_n(t)$ are asymptotically central sequences. Finally,

$$\begin{aligned} f(\lambda t) f(t)^{-1} &= a_n^{-1} a_n \cdot f(\lambda t)[a_n f(t)]^{-1} a_n = a_n^{-1} h_n(t) k(\lambda t x_n^{-1})[g_n(t) k(t x_n^{-1})]^{-1} a_n \\ &= a_n^{-1} h_n(t) k(\lambda t x_n^{-1}) k(t x_n^{-1})^{-1} g_n(t)^{-1} a_n, \end{aligned}$$

or, since k is multiplicative,

$$\begin{aligned} f(\lambda t) f(t)^{-1} &= a_n^{-1}[h_n(t) g_n(t)^{-1} \cdot g_n(t) k(\lambda) g_n(t)^{-1}] a_n \\ &\to \Delta_-(a, k(\lambda)), \end{aligned}$$

since $\Delta_-(a, \cdot)$ is a continuous homomorphism. Note that this is multiplicative in λ, as both $\Delta_-(a, \cdot)$ and k are multiplicative. □

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