Regular variation without limits

N. H. Bingham
Mathematics Department, Imperial College London,
South Kensington, London SW7 2AZ (n.bingham@ic.ac.uk)

A. J. Ostaszewski
Mathematics Department, London School of Economics,
Houghton Street, London WC2A 2AE (a.j.ostaszewski@lse.ac.uk)

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Abstract

Karamata theory ([BGT] Ch. 1) explores functions f for which the limit function $g(\lambda) := f(\lambda x)/f(x)$ exists (as $x \to \infty$) and for which $g(\lambda) = \lambda^\rho$ subject to mild regularity assumptions on f. Further Karamata theory ([BGT] Ch. 2) explores functions f for which the upper limit $f^*(\lambda) := \limsup f(\lambda x)/f(x)$, as $x \to \infty$, remains bounded. Here the usual regularity assumptions invoke boundedness of f^* on a Baire non-meagre/measurable non-null set, with f Baire/measurable, and the conclusions assert uniformity over compact λ-sets (implying upper bounds of the form $f(\lambda x)/f(x) \leq K\lambda^\rho$ for all large λ, x). We give unifying combinatorial conditions which include the two classical cases, deriving them from a combinatorial semigroup theorem. We examine character degradation in the passage from f to f^* (using some standard descriptive set theory) and thus identify natural classes in which the theory may be established.

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1 Introduction

We recall the theory of regularly varying functions (Karamata theory: [BGT] Ch. 1), which concerns *limits.* It explores functions for which the limit function g below exists:

$$f(\lambda x)/f(x) \to g(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0. \tag{RV}$$

The theory needs some assumption on f – some regularity (or, as we shall see, uniformity) assumption. The two classical ones are measurability (traditionally regarded as the primary case) and the Baire property. In recent work ([BOst-SteinOstr], [BOst-FRV]) we have solved the long-standing problem of finding the minimal common generalization of these two conditions. Our new approach is combinatorial: it makes crucial use of infinite combinatorics. As a by-product, one can now see that in fact it is the Baire case, rather than the measurable case, which is the primary one.

Here we address the question of just how much of the fundamental theory survives 'without limits', i.e. when the limit does not necessarily exist. This is the further Karamata theory of [BGT] Ch. 2. Thus one refers to the upper limit

$$f^*(\lambda) := \limsup f(\lambda x)/f(x).$$

Passing from f to f^* degrades the properties of f. In Theorems 14 to 16 of Section 3 (the 'Character Theorems') we study in detail the extent and nature of this degradation. To formulate our results, we need a modest use of the language of descriptive set theory.

In (RV), the limit function g must satisfy the Cauchy functional equation

$$g(\lambda\mu) = g(\lambda)g(\mu) \qquad \forall \lambda, \mu > 0. \tag{CFE}$$

Subject to a mild regularity condition, (CFE) forces g to be a power:

$$g(\lambda) = \lambda^\rho \qquad \forall \lambda > 0. \tag{ρ}$$

Then f is said to be *regularly varying* with *index* ρ, written $f \in R_\rho$. The case $\rho = 0$ is basic. A function $f \in R_0$ is called *slowly varying*; slowly varying functions are often written ℓ (for *lente*, or *langsam*). The basic theorem of the subject is the Uniform Convergence Theorem (UCT), which states that if

$$\ell(\lambda x)/\ell(x) \to 1 \qquad (x \to \infty) \qquad \forall \lambda > 0, \tag{SV}$$

and some regularity condition holds – for example, measurability or the Baire property – then the convergence is *uniform* on compact λ-sets in $(0, \infty)$. Some regularity is needed here. In our recent paper [BOst-FRV], we find *exactly how much* regularity is needed for the UCT to hold.

While the multiplicative formulation above is more convenient for applications of the theory, for proofs in the subject it is usually more convenient to use an additive formulation. Writing $h(x) := \log f(e^x)$ (or $\log \ell(e^x)$ as the case may be), $\partial h(u) := k(u) := \log g(e^u)$, the relations above become

$$h(x+u) - h(x) \to k(u) \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}, \tag{RV_+}$$

$$h(x+u) - h(x) \to 0 \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}, \tag{SV_+}$$

$$k(u+v) = k(u) + k(v) \qquad \forall u, v \in \mathbb{R}, \tag{CFE_+}$$

$$k(u) = \rho u \qquad \forall u \in \mathbb{R}. \tag{ρ_+}$$

Here the functions are defined on $\mathbb{R}$, whereas in the multiplicative notation functions are defined on $\mathbb{R}_+$.

The tools needed for the solution in [BOst-FRV] of the fundamental foundational question of regular variation – how much regularity is needed to ensure the UCT – are combinatorial. The question addressed here is harder, since we do not assume that limits exist, and we accordingly need rather more tools here. In Section 2 we turn to what we need from infinite combinatorics.

Working with limits, the fundamental results needed are the theorems of Steinhaus ([BGT] Th. 1.1.1) and Ostrowski ([BGT] Th. 1.1.7); for a combinatorial treatment of these, see [BOst-SteinOstr]. Working without limits as here, the result corresponding to the UCT is [BGT] Th. 2.0.1, which rests on a result on semigroups ([BGT] Cor. 1.1.5, see Section 2 for references). Like all the other results of [BGT] Ch. 1, this (and so also its consequence, [BGT] Th. 2.0.1) does combinatorialize. However, the degradation resulting from the absence of limits exacts it price here: we need to refer explicitly to the axioms of set theory that we use. Since there are various possibilities here, the upshot is that these results disaggregate. In Section 3, we formulate and discuss our results. Proofs follow in Section 4. We close in Section 5 with some remarks and comments.

2 Combinatorial framework

The central definition is as follows. Note that in 1.2 the 'translator' s is required to be in S rather than be arbitrary as in 1.1

Definitions - 1.

1.1 Call a set S **universal** (resp. **subuniversal**) if for any null sequence $z_n \to 0$, there are s and a co-finite (resp. infinite) set $\mathbb{M}_s$ such that

$$\{s + z_m : m \in \mathbb{M}_s\} \subseteq S, \qquad s \in \mathbb{R}. \tag{SUB}$$

1.2. Call a set S is **generically universal** (resp. **subuniversal**) if for any null sequence $z_n \to 0$, there are s and a co-finite (resp. infinite) set $\mathbb{M}_s$ such that

$$\{s + z_m : m \in \mathbb{M}_s\} \subseteq S \text{ and } s \in S. \tag{GSUB}$$

We shall also say that a universal set S *includes by translation* the null sequences. (Omission of 'by translation' is not to be taken as implying translation.) We say that a subuniversal set *traps* null sequences, to abbreviate 'includes by translation a subsequence of'.

The Kestelman-Borwein-Ditor Theorem below says that generic subuniversality is implied by the standard regularity assumptions of regular variation with limits when the functions are Baire or measurable. Our work in [BOst-FRV] shows that subuniversality is enough for a development of the fundamental theory of regular variation: it contains both instances of the two formulations of the classical theory, in the language of measurable or Baire functions. Note that, since subuniversality is preserved upwards, it follows immediately from the Theorem that, e.g., sets with *positive inner measure* are also generically subuniversal. The result below is due in this form in the measure case to Borwein and Ditor [BoDi], but was already known much earlier albeit in somewhat weaker form by Kestelman ([Kes] Th. 3), and rediscovered by Trautner [Trau] (see [BGT] p. xix and footnote p. 10). See our strengthening in Theorem 9 below.

Theorem (Kestelman-Borwein-Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null (resp. Baire and non-meagre), then, for almost all (resp. for quasi-all) $t \in T$, there is an infinite set $\mathbb{M}_t$ such that (SUB) holds for T :*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

For a bitopological approach (with the Euclidean and the density topologies in play, as in Theorem 9 below) see [BOst-SteinOstr]. The sets to which we wish to apply the theorem are usually defined by reference to a function of interest, often a level set; as the work below demonstrates, for a natural class of functions and under appropriate set-theoretic axioms (regarded by some logicians as the natural axioms for the practising mathematician, cf. Section 5), we can deduce that the 'non-meagre Baire set' condition holds.

The projection of a Baire set may be, but need not be, Baire: under Gödel's Axiom, $V = L$, there are non-Baire sets which are projections of co-analytic sets (which are themselves Baire), as Theorem 4 demonstrates. However, the weaker concept of subuniversality is preserved under projection.

Theorem 1. *For a (sub)universal set in* $\mathbb{R}^2$, *its projection is (sub)universal in* $\mathbb{R}$.

The classical treatment of regular variation begins with the Steinhaus and Piccard theorems on the existence of an interior point in the difference set $S - S$ for S non-null/Baire non-meagre, as witness its place in [BGT] as Theorem 1.1.1. Actually, it is the following corollary that is critical.

Theorem (Category [Measure] Subgroup Theorem). *For an additive Baire [measurable] subgroup* S *of* $\mathbb{R}$, *the following are equivalent:*
(i) $S = \mathbb{R}$,
(ii) S *is non-meagre [non-null].*

In fact, as we have recently established in [BOst-SteinOstr], the two cases above are subsumed in a single combinatorial version, in which either of subuniversal or universal replaces measurable or Baire. We note the result that the Steinhaus Theorem is implied by universality, a fact known to Kestelman, and also by generic subuniversality, not an issue considered by Kestelman.

Theorem 2 (Combinatorial Steinhaus-Piccard Theorem; cf. [St],[Pic], [Kes] Th. 7**).** *For* S *universal,* $S - S \supseteq (-\delta, \delta)$ *for some positive number* δ*. Likewise for* S *generically subuniversal.*

As an immediate corollary, the subgroup theorem holds for S generically subuniversal, since for S a subgroup $S = S - S$. Thus we have

Theorem 3 (Generically Subuniversal Subgroup Theorem). *For an additive subgroup* S *of* $\mathbb{R}$, *the following are equivalent:*
(i) $S = \mathbb{R}$,
(ii) S *is generically subuniversal.*

In particular, if S is Baire non-meagre [measurable, non-null], $S = \mathbb{R}$.

This result becomes less surprising when one notes the following.

Proposition. *A generically subuniversal subgroup S of $\mathbb{R}$ is locally compact.*

Proof. For $s_n \to s_0$ with $s_n \in S$, put $z_n := s_0 - s_n$. If $t + z_m \in S$ with $t \in S$ down a subsequence $m \in \mathbb{M}$, then $z_m \in S$ down the subsequence. But, then $s_0 - s_m = (s_0 + t) - (s_m + t) = z_m \in S$ and hence $s_0 = (s_0 - s_m) + s_m \in S$. So S is closed. □

The following classical result is due to Hille and Phillips [H-P] Th. 7.3.2 (cf. Beck et al. [BCS] Th. 2, [Be]) in the measurable case, and to Bingham and Goldie [BG1] in the Baire case; see [BGT] Cor. 1.1.5. For a combinatorial form see Theorem 11.

Theorem (Category [Measure] Semigroup Theorem). *For an additive Baire [measurable] semigroup S of $\mathbb{R}$, the following are equivalent:*
(i) $S \supseteq (s, \infty)$, *for some* s,
(ii) *S is non-meagre [non-null].*

In the absence of a regularity assumption on the semigroup this result fails badly.

Theorem 4 (A counterexample). *Assume the Axiom of Choice. There exists an additive semi-group $T \subseteq \mathbb{R}$ which is generically universal (and so generically subuniversal), and*
(a) *T has empty interior, although*
(b) *its closure $\bar{T}$ contains a half-infinite interval.*
Furthermore the set T may be selected so that additionally it is non-meagre/non-null.

Under Gödel's Axiom, $V = L$, the set T may be selected so that it is also the projection of a co-analytic set; as T is non-meagre and has empty interior, by the Semigroup Theorem, it is not Baire. In these circumstances T is the non-Baire projection of a co-analytic, and hence Baire, set. Working within the natural projective classes of functions and sets as above, with the addition of of Gödel's Axiom of Constructibility as a strengthening of the Axiom of Choice, the pleasant properties in evidence up to this point break down: the semigroup T above is nice enough in that it is universal but not nice enough in that it fails to have non-empty interior despite being in the natural class of sets identified in this paper. We note that the property (b) above is automatic, being an immediate corollary of a result derived by Kestelman, namely:

Theorem ([Kes], Th. 6). *For S universal, S', the set of limit points of S, contains an interval.*

Definition - 2. The **No Trumps** combinatorial principle, denoted $\mathbf{NT}(\{T_k : k \in \omega\})$, refers to a family of subsets of reals $\{T_k : k \in \omega\}$ and means the following.

For every bounded sequence of reals $\{u_m : m \in \omega\}$ there are $k \in \omega$, $t \in \mathbb{R}$ and an infinite set $\mathbb{M} \subseteq \omega$ such that

$$u_m + t \in T_k \text{ for all } m \text{ in } \mathbb{M}.$$

See Section 5 for the background on this terminology. As with universality (resp. subuniversality), we will also say that the family $\{T_k : k \in \omega\}$ *includes by translation* (resp. *traps*) *bounded sequences* when it so includes co-finitely many (resp. infinitely) many terms.

Our next theorem addresses the difference set $T - T$ when only $\mathbf{NT}(\{T_m : m \in \omega\})$ is assumed to hold. By Theorem 4, there exists a set T satisfying $\mathbf{NT}(T)$ such that $T - T$ has empty interior (see also below). So we cannot hope to replicate the result of Theorem 2. However, in a precise combinatorial sense, $T - T$ is not far from containing an interval. We will need the Ger-Kuczma class $\mathfrak{C}$ defined below ([Kucz] p. 206); note that any subuniversal set is in $\mathfrak{C}$. This follows from the Combinatorial Ostrowski Theorem ([BOst-SteinOstr] Th. 4), according to which any additive function $f : \mathbb{R} \to \mathbb{R}$, bounded (locally, above or below) on a subuniversal set S is locally bounded and hence linear.

Definition - 3. The *Ger-Kuczma class* $\mathfrak{C}$ is defined by

$$\mathfrak{C} = \{T \subseteq \mathbb{R} : \text{if } f : \mathbb{R} \to \mathbb{R} \text{ is additive and bounded on } T, \text{ then } f \text{ is continuous}\}.$$

Theorem 5. *If* $\mathbf{NT}(\{T_m : m \in \omega\})$ *holds, e.g. if some* T_m *is subuniversal, then* $T := \bigcup T_m \in \mathfrak{C}$*, and hence*

$$\text{int conv}_{\mathbb{Q}}(T - T) \neq \emptyset, \tag{1}$$

where $\text{conv}_{\mathbb{Q}}(S)$ *denotes the smallest* $\mathbb{Q}$*-convex set* C *to contain* S*. That is,* C *is closed under weighted sums with rational weights.*

Definitions - 4.

4.1. Say that $\mathbf{NT}^*(\{T_k\})$ holds, in words No Trumps holds *generically*, if for any null sequence $z_n \to 0$ there is $k \in \omega$ and an infinite $\mathbb{M}$ such that

$$\{t + z_m : m \in \mathbb{M}\} \subset T_k \text{ and } t \in T_k.$$

In distinction to $\mathbf{NT}$ (Definition 2 above) here we consider null, rather than bounded, sequences z_n and require the translator t to be in T_k. Of course $\mathbf{NT}^*(\{T_k : k \in \omega\})$ implies $\mathbf{NT}(\{T_k : k \in \omega\})$.

4.2. For a function h with domain a set T and for $x_n \to \infty$ put

$$T_k(x) := \bigcap_{n>k} \{t \in T : h(t + x_n) - h(x_n) < n\}.$$

4.3. Let us say that h is $\mathbf{NT}^*$ on T if for any $x_n \to \infty$ and any null sequence $z_n \to 0$, $\mathbf{NT}^*(\{T_k(x)\})$ holds.

Our next result may be compared with the No Trumps Theorem ([BOst-FRV] Th. 2). There, we deal with slowly varying functions h, for which the limit exist. Here, we do not assume existence of limits but deal instead with upper limits, involving h^*; the corresponding combinatorial principle needed here is $\mathbf{NT}^*$. We call the result below the 'Generic No Trumps Theorem' or the 'No Trumps* Theorem' (cf. the weak* topology of functional analysis for a similar usage).

Theorem 6 (**Generic No Trumps Theorem or No Trumps* Theorem**). *For* T *Baire non-meagre/measurable non-null and* h *Baire/measurable with* $h^*(t) < +\infty$ *on* T*,* h *is* $\mathbf{NT}^*$ *on* T*.*

This result and the following one together yield the required uniformity theorem for regular variation without limits.

Theorem 7 (Combinatorial Uniform Boundedness Theorem). *Suppose that* $h^*(t) < \infty$ *on a set* T *on which* h *is* $\mathbf{NT}^*$. *Then for compact* $K \subset T$

$$\limsup_{x\to\infty} \sup_K |h(x+u) - h(x)| < \infty.$$

As a corollary we obtain the following theorem, which extends results of Delange [Del] and Csiszár and Erdős [CsEr] (compare [BajKar] Th. 3 and [Ost-knit] Th. 3 in the setting of topological groups), which combinatorializes those parts of [BGT] Th. 2.0.1 for which this does not involve set-theoretic complications.

Theorem 8 (Uniform Boundedness Theorem). *For h Baire/measurable, suppose that* $h^*(t) < \infty$ *on a Baire non-meagre/measurable non-null set* T. *Then for compact* $K \subset T$

$$\limsup_{x\to\infty} \sup_K |h(x+u) - h(x)| < \infty.$$

To complete our disaggregation, we next capture a key similarity (their topological 'common basis', adapting a term from logic) between the Baire and measure cases. Recall ([Rog] p. 460) the usage in logic, whereby a set B is a basis for a class $\mathcal{C}$ of sets whenever any member of $\mathcal{C}$ contains a point in B. Recall also the density topology (see e.g. [LMZ]), a classic example of a fine topology, and in particular finer than the Euclidean topology.

Theorem 9 (Common Basis Theorem). *For* V, W *Baire non-meagre in the line* $\mathbb{R}$ *equipped with either the Euclidean or the density topology there is* $a \in \mathbb{R}$ *such that* $V \cap (W + a)$ *contains a non-empty open set modulo meagre sets common to both, up to translation. In fact, in both cases, up to translation, the two sets share a Euclidean* $\mathcal{G}_\delta$ *subset which is non-meagre in the Euclidean case and non-null in the density case.*

This leads to a strengthening of the Kestelman-Borwein-Ditor Theorem, which concerns two sets rather than one.

Theorem 10. *For* V, W *Baire non-meagre/measurable non-null, there is* $a \in \mathbb{R}$ *such that* $V \cap (W + a)$ *is Baire non-meagre/ measurable non-null* *and for any null sequence* $z_n \to 0$ *and quasi all (almost all)* $t \in V \cap (W + a)$ *there exists an infinite* $\mathbb{M}_t$ *such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subset V \cap (W + a).$$

This result motivates a further strengthening of generic subuniversality.

Definitions - 5. Let S be generically subuniversal.

5.1. Call T *similar* to S if for every null sequence $z_n \to 0$ there is $t \in S \cap T$ and $\mathbb{M}_t$ such that

$$\{t + z_m : m \in \mathbb{M}_t\} \subset S \cap T.$$

Thus S is similar to T and both are generically subuniversal.

Call T *weakly similar* to S if if for every null sequence $z_n \to 0$ there is $s \in S$ and $\mathbb{M}_s$ such that

$$\{s + z_m : m \in \mathbb{M}_s\} \subset T.$$

Thus again T is subuniversal.

5.2. Call S *subuniversally self-similar,* or just *self-similar,* (up to reflected translation) if for some $a \in \mathbb{R}$ and some $T \subset S$, S is similar to $a - T$.

Call S *weakly self-similar* (up to reflected translation) if for some $a \in \mathbb{R}$ and some $T \subset S$, S is weakly similar to $a - T$.

As an immediate corollary of Theorem 9 or 10, taking $V = S, W = -S$, we may now formulate the following.

Theorem 11 (Self-similarity Theorem). *For S Baire non-meagre/measurable non-null, S is self-similar.*

Self-similarity is the additional feature needed to establish the Semigroup Theorem.

Theorem 12 (Semigroup Theorem). *If S, T are generically subuniversal with T (weakly) similar to S, then $S - T$ contains an interval about the origin. Hence if S is generically subuniversal and (weakly) self-similar, then $S + S$ contains an interval. Hence, if additionally S is a semigroup, then S contains an infinite half-line.*

By the Common Basis Theorem, replacing T by $-T$, we obtain as an immediate corollary of Theorem 12 a new proof of two classical results, extending the Steinhaus and Piccard Theorems (cf. Theorem 2 above).

Theorem (**Vector Sum Theorem**, Steinhaus [St] measure case, cf. [Be]; Pettis [Pet] Baire case, cf. [Kom]). *If S, T are Baire non-meagre/measurable non-null, then $S+T$ contains an interval.*

We are now able to aggregate our results along the lines of [BGT] Th. 2.0.1, written in additive form as above.

Theorem 13 (Combinatorial Uniformity Theorem). *Suppose that $h^*(t) < +\infty$ on a self-similar set T on which h is $\mathbf{NT}^*$. Then there exists A_0 with $h^*(u) < \infty$ for $u \geq A_0$, and for every A, B with $2A_0 < A < B$,*

$$\limsup_{x\to\infty} \sup_{u\in[A,B]} h(u+x) - h(x) < \infty.$$

Also for $A > 2A_0$ there exists $x_1 = x_1(A)$ and a constant K such that

$$h(u+x) - h(x) < Ku \qquad (u \geq A, x \geq x_1)$$

and h is bounded away from $-\infty$ and ∞ on finite intervals sufficiently far to the right.

We now have as a corollary of Theorems 6, 7 and 13 the classical result.

Uniformity Theorem. *For h Baire/measurable, suppose that $h^*(t) < +\infty$ on a Baire non-meagre/measurable non-null set of t. Then there exists A_0 with $h^*(u) < \infty$ for $u \geq A_0$, and for every A, B with $2A_0 < A < B$,*

$$\limsup_{x\to\infty} \sup_{u\in[A,B]} h(u+x) - h(x) < \infty.$$

Also for $A > 2A_0$ there exists $x_1 = x_1(A)$ and a constant K such that

$$h(u+x) - h(x) < Ku \qquad (u \geq A, x \geq x_1)$$

and h is bounded away from $-\infty$ and ∞ on finite intervals sufficiently far to the right.

3 Descriptive character of limits

For the most part we work with the mind-set of the practising analyst, that is in 'naive' set theory. As usual, we work in the standard mathematical framework of Zermelo-Fraenkel set theory (ZF), i.e. we do not make use of the Axiom of Choice unless we say so explicitly. Our interest in the complexities induced by the limsup operation points us in the direction of definability and descriptive set theory because of the question of whether certain specific sets, encountered in the course of analysis, have the Baire property. The answer depends on what further axioms one admits. For us there are two alternatives yielding the kind of decidability we seek: Gödel's Axiom of Constructibility, as an appropriate strengthening of the Axiom of Choice which creates definable sets without the Baire property (without measurability), or, at the opposite pole, the Axiom of Projective Determinacy (see [MySw], or [Kech] 5.38.C) which guarantees the Baire property in the kind of definable sets we encounter. Thus to decide whether sets of the kind we encounter below have the Baire property, or are measurable, the answer is: it depends on the axioms of set theory that one adopts.

To formulate our results we need the language of descriptive set theory, for which see e.g. [JayRog], [Kech], [Mos]. Within such an approach we will regard a function to be a set: namely its *graph*. We need the beginning of the *projective hierarchy* in Euclidean space (see [Kech] S. 37.A), in particular the following classes:

the *analytic* sets $\mathbf{\Sigma}_1^1$;

their complements, the *co-analytic* sets $\mathbf{\Pi}_1^1$;

the common part of the previous two classes, the ambiguous class $\mathbf{\Delta}_1^1 := \mathbf{\Sigma}_1^1 \cap \mathbf{\Pi}_1^1$, that is, by Souslin's Theorem ([JayRog], p. 5, and [MaKe] p.407 or [Kech] 14. C) the *Borel* sets;

the *projections* (continuous images) of $\mathbf{\Pi}_1^1$ sets, forming the class $\mathbf{\Sigma}_2^1$;

their complements, forming the class $\mathbf{\Pi}_2^1$;

the ambiguous class $\mathbf{\Delta}_2^1 := \mathbf{\Sigma}_2^1 \cap \mathbf{\Pi}_2^1$;

and then: $\mathbf{\Sigma}_{n+1}^1$, the projections of $\mathbf{\Pi}_n^1$; their complements $\mathbf{\Pi}_{n+1}^1$; and the ambiguous class $\mathbf{\Delta}_{n+1}^1 := \mathbf{\Sigma}_{n+1}^1 \cap \mathbf{\Pi}_{n+1}^1$.

Throughout we shall be concerned with the cases $n = 1, 2$ or 3.

The notation reflects the fact that the canonical expression of the logical structure of their definitions, that is with the quantifiers (ranging over the reals, hence the superscript 1, as reals are type 1 objects - integers are of type 0) all at the front, is determined by a string of alternating quantifiers starting with an existential or universal quantifier (resp. $\mathbf{\Sigma}$ or $\mathbf{\Pi}$). Here the subscript accounts for the number of alternations.

Interest in the character of a function h is motivated by an interest within the theory of regular variation in the character of the level sets

$$H^k := \{s : |h(s)| < k\} = \{s : (\exists t)[(s,t) \in h \ \& \ |t| < k]\},$$

for $k \in \mathbb{N}$ (where as above h is referred to here by way of its graph). The set H^k is thus the projection of $h \cap (\mathbb{R} \times [0,k])$ and hence is $\mathbf{\Sigma}^1_n$ if h is $\mathbf{\Sigma}^1_n$, e.g. it is $\mathbf{\Sigma}^1_1$, i.e. analytic, if h is analytic (in particular, Borel). Also

$$H^k = \{s : (\forall t)[(s,t) \in h \implies |t| \leq k]\} = \{s : (\forall t)[(s,t) \notin h \text{ or } |t| \leq k]\},$$

and so this is also $\mathbf{\Pi}^1_n$ if h is $\mathbf{\Sigma}^1_n$. Thus if h is $\mathbf{\Sigma}^1_n$ then H^k is $\mathbf{\Delta}^1_n$. So if $\mathbf{\Delta}^1_n$ sets are Baire, for some k the set H^k is Baire non-null, and hence subuniversal, as

$$\mathbb{R} = \bigcup_{k \in \omega} H^k.$$

With this in mind, it suffices to consider upper limits; as before, we prefer to work with the additive formulation. Consider the definition:

$$h^*(x) := \limsup_{t \to \infty} [h(t+x) - h(t)]. \qquad (*)$$

Thus in general h^* takes values in the extended real line. The problem is that the function h^* is in general less well behaved than the function h – for example, if we assume h measurable, h^* need not be measurable, and similarly if h has the Baire property, h^* need not. The problem we address here is the extent of this degradation – saying *exactly how much less regular* than h the limsup h^* may be. The nub is the set S on which h^* is finite. This set S is an additive semi-group on which the function h^* is subadditive (see [BOst5]) – or additive, if limits exist (see [BOst4]). Furthermore, if h has Borel graph then h^* has $\mathbf{\Delta}^1_2$ graph (see below). But in the presence of certain axioms of set-theory (for which see below) the $\mathbf{\Delta}^1_2$ sets have the Baire property and are measurable; hence if S is large in either of these two senses then in fact S contains a half-line. The extent of the degradation in passing from h to h^* is addressed in the following result, which we call the First Character Theorem, and then contrast it with two alternative character theorems. Undefined terms are explained below in the course of the proof (as in BGT, we reserve the name Characterization Theorem, CT, for a result identifying the g of (RV) and (CFE) as a power function, as in (ρ).)

Theorem 14 (First Character Theorem). (i) *If h is Borel (has Borel graph), then the graph of the function $h^*(x)$ is a difference of two analytic sets, hence is measurable and $\mathbf{\Delta}^1_2$. If the graph of h is $\mathcal{F}_\sigma$, then the graph of $h^*(x)$ is Borel.*

(ii) *If h is analytic (has analytic graph), then the graph of the function $h^*(x)$ is $\mathbf{\Pi}^1_2$.*

(iii) *If h is co-analytic (has co-analytic graph), then the graph of the function $h^*(x)$ is $\mathbf{\Pi}^1_3$.*

In our next theorem we assume much more than in the First Character Theorem.

Theorem 15 (Second Character Theorem). *Suppose $h \in \mathbf{\Delta}^1_2$ and the following limit exists.*

$$\partial h(x) := \lim_{t\to\infty}[h(t+x) - h(t)].$$

Then the graph of ∂h is $\mathbf{\Delta}^1_2$.

The point of the next theorem is that it may be applied under the assumption of Gödel's Axiom $V = L$ (see [Dev]), as the axiom implies that $\mathbf{\Delta^1_2}$ ultrafilters on ω exist (see for instance [Z], where Ramsey ultrafilters are considered). Note that here sets of natural numbers are identified with real numbers (via their indicator functions) and so ultrafilters are regarded as sets of reals. For information on various types of ultrafilter on ω see [CoNe]. In particular this means that we have a midway position between the results of the First and Second Character Theorems.

Theorem 16 (Third Character Theorem). *Suppose the following are of class $\mathbf{\Delta}^1_2$: the function h and an ultrafilter $\mathcal{U}$ on ω. Then the following is of class $\mathbf{\Delta}^1_2$:*

$$\partial_{\mathcal{U}} h(t) := \mathcal{U}\text{-}\lim_n [h(t+n) - h(n)].$$

Comment 1. In the circumstances of Theorem 16 $\partial_{\mathcal{U}} h(t)$ is an additive function, whereas in those of Theorem 14 one has only sub-additivity. See BGT p. 62 equation (2.0.3).

Comment 2. One may also consider replacing $h(t+n)-h(n)$ by $h(t+x(n))-h(x(n))$, as in the Equivalence Theorem of [BOst-FRV], so as to take limits along a specified sequence $\mathbf{x}:\omega\to\omega^\omega$, in which case to have an 'effective' version of Theorem 16 one would need to specify the effective descriptive character of $\mathbf{x}$. (Here again ω^ω is identified with the reals via indicator functions.)

4 Proofs

4.1 Proof of Theorem 1

Assume S is subuniversal (universal) in $\mathbb{R}^2$, put T :=proj(S), and let $z_n\to 0$. Then $\mathbf{z}_n:=(z_n,0)\to 0$, so for some $\mathbf{t}:=(t_1,t_2)$. there is an infinite (co-finite) set $\mathbb{M}_t$ such that

$$\{t+\mathbf{z}_m : m\in\mathbb{M}_s\}\subseteq S.$$

But then

$$\{t_1+z_m : m\in\mathbb{M}_s\}\subseteq T. \qquad \square$$

4.2 Proof of Theorem 2

(i) For S universal, it is enough to show that $S-S\supseteq(0,\delta)$ for some positive number δ. Suppose the contrary. Then for each $n\in\omega$ there is z_n in $(0,1/(n+1)^2)$ such that $z_n\notin S-S$. Put

$$u_n=z_0+z_1+...+z_{n-1},$$

with $u_0=0$. Thus $\{u_n : n\in\omega\}$ is convergent. By assumption, there is M,t such that

$$\{t+u_n : n>M\}\subseteq S.$$

Hence for $n>M$ we have

$$v_n=(t+u_{n+1})-(t+u_n)\in S-S.$$

But this is a contradiction. $\square$

(ii) For S generically subuniversal argue as before to construct z_n. Now for some $s\in S$ and infinite $\mathbb{M}$ we have

$$\{s+z_m : m\in\mathbb{M}\}\subseteq S.$$

But now for any m in $\mathbb{M}$ we have

$$s + z_m \in S \text{ or } z_m \in S - S,$$

again a contradiction. $\square$

4.3 Proof of Theorem 4

The proof of Theorem 4 relies on the Axiom of Choice and the observation that the set of sequences of reals is equinumerous with $\mathbb{R}$ (i.e. $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$), as is the family of all $\mathcal{F}_\sigma$-subsets. Let γ be the least ordinal of cardinality $\mathfrak{c}$.

We will construct the required set T by induction so that $T = \{t_\alpha : \alpha \in Lim \cap \gamma\}$ where Lim denotes the class of limit ordinals.

Let $\{\mathbf{z}_\alpha : \alpha \in Lim \cap \gamma\}$ enumerate all null-sequences of $\mathbb{R}$. We also let $\{H_\alpha : \alpha \in Lim \cap \gamma\}$ enumerate either all the null $\mathcal{G}_\delta$-subsets of $\mathbb{R}$, or all the meagre $\mathcal{F}_\sigma$-subsets of $\mathbb{R}$. For $\alpha < \gamma$, suppose that $T_\alpha = \{t_\beta : \beta < \alpha\}$, with $t_0 = 0$, has already been defined for limit $\alpha < \gamma$ in such a way that:

(i) $T_\beta = \{t_\delta : \delta < \beta\}$, for limit $\beta \leq \alpha$, is an additive semigroup disjoint from $\mathbb{Q}_0 = \mathbb{Q} \backslash \{0\}$,

(ii) $T_{\beta+\omega}$ contains a translate of $\mathbf{z}_\beta$ for limit $\beta < \alpha$,

(iii) $T_{\alpha+\omega}$ contains a point not in H_α.

Let S_α be the additive semigroup generated by $\mathbf{z}_\alpha$ and 0. It is countable.

We will put $T_{\alpha+\omega} = T_\alpha \cup \{t_{\alpha+n} : n \in \omega\}$ with $\{t_{\alpha+2n} : n \in \omega\} = \tau + \mathbf{z}_\alpha$ for some suitable τ.

Now for any τ, we claim that the following set, which has cardinality less than continuum, is a semi-group:

$$T_\alpha(\tau) = T_\alpha \cup \{m\tau + s + t : s \in S_\alpha, t \in T_\alpha, 0 < m \in \omega\}.$$

Note that $\tau \in T_\alpha(\tau)$ (as $0 \in S_\alpha \cap T_\alpha$ and we may take $m = 1$). By hypothesis T_α is a semigroup, so it is enough to consider the sum $x + y$ for $x, y \in T_\alpha(\tau) \backslash T_\alpha$. But

$$x + y = (m\tau + s + t) + (m'\tau + s' + t') = (m + m')\tau + s'' + t'',$$

and $m + m' > 0$ when $m, m' > 0$.

To ensure that the set we construct has empty interior we will select τ so that $T_\alpha(\tau)$ is disjoint from $\mathbb{Q}_0 \backslash \{0\}$, i.e. so that for all $s \in S_\alpha, t \in T_\alpha$ and all positive integers m

$$m\tau + t + s \notin \mathbb{Q}_0, \text{ equivalently } \tau \notin \mathbb{Q}_0 - \frac{1}{m}(t + s).$$

Thus τ may be selected arbitrarily in the complement of

$$\bigcup\{\mathbb{Q}_0 - \frac{1}{m}(s+t) : s \in S_\alpha, t \in T_\alpha, m = 1, 2, 3, ..\}$$

which is the union of less than continuum many countable sets.

For $\alpha < \gamma$ we may guarantee that T is not covered by H_α by requiring that the point τ, which will lie in T, is in the complement of H_α, again a set of cardinality continuum as H_α is either meagre or null. For such a choice of τ, we let $T_{\alpha+\omega} = T_\alpha(\tau)$. By construction (i), (ii) and (iii) are satisfied with α replaced by $\alpha + \omega$.

To complete the induction consider the case of $\alpha \leq \gamma$ when $\alpha = \sup\{\beta \in Lim : \beta < \alpha\}$ with each T_β a semigroup for $\beta < \alpha$. Then $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ is a semigroup.

The set $T = T_\gamma$ satisfies the properties of the theorem, since by (i) T is a semigroup, by (ii) T is generically universal, and by (iii) T is non-null/non-meagre, as no null/meagre set H_α covers T. □

4.4 Proof of Theorem 5

By assumption, the sets $\{T_k : k \in \omega\}$ trap bounded sequences. Then for each bounded sequence $\{u_n\}$ there is some z and some k and infinite $\mathbb{M} \subseteq \omega$ such that the translated sequence $\{u_m + z : m \in \mathbb{M}\}$ is contained in T_k.

Let $f : \mathbb{R} \to \mathbb{R}$ be additive and bounded on $T = \bigcup T_m$. Suppose that f is not continuous. By an application of Ostrowski's Lemma ([Ostr]; [BGT] Th. 1.1.7) there is a convergent sequence u_n with $f(u_n)$ unbounded. There is therefore some z, some k, and infinite $\mathbb{M} \subseteq \omega$ such that the sequence $\{u_m + z : m \in \mathbb{M}\}$ is contained in T_k. Now f is bounded on T, so

$$f(u_m + z) = f(u_m) + f(z)$$

is bounded; hence, after all, $f(u_m)$ is bounded. Thus f is after all continuous, hence $T \in \mathfrak{C}$. By the Smítal-Mościcki characterization theorem ([Sm], [Moscicki], or see [Kucz], Th. 2, p. 240]), (1) is a condition on T equivalent to $T \in \mathfrak{C}$. □

4.5 Proof of Theorem 6

Since $T_k(x)$ are Baire/measurable with union T, some $T_k(x)$ is subuniversal by the Kestelman-Borwein-Ditor Theorem. □

4.6 Proof of Theorem 7

Suppose not: then for some $\{u_n\} \subset K \subset T$ and x_n unbounded we have, for all n,

$$h(u_n + x_n) - h(x_n) > 3n. \tag{2}$$

W.l.o.g. $u_n \to u \in K$. As $x_n + u \to \infty$ we may put $y_n := x_n + u$, then

$$T_k(y) := \bigcap_{n>k} \{t \in T : h(t + x_n + u) - h(x_n + u) < n\}.$$

and $\mathbf{NT}^*(T_k(y))$ holds. Now $z_n := u_n - u$ is null. So for some $k \in \omega$, $t \in T_k(y)$ and infinite $\mathbb{M}$ infinite

$$\{t + (u_m - u) : m \in \mathbb{M}\} \in T_k(y).$$

So

$$h(t + u_m + x_m) - h(x_m + u) < m \text{ and } t \in T.$$

Now $x_n + u_n \to \infty$ and $t \in T$ so, as before since $h^*(t) < \infty$, for all n large enough

$$h(t + x_n + u_n) - h(x_n + u_n) < n.$$

Now also $u \in K \subset T$. So for all n large enough

$$h(u + x_n) - h(x_n) < n.$$

But

$$\begin{aligned} h(x_n + u_n) - h(x_n) &= h(x_n + u_n) - h(t + x_n + u_n) \\ &\quad + h(t + x_n + u_n) - h(x_n + u) \\ &\quad + h(x_n + u) - h(x_n). \end{aligned}$$

Then for m large enough and in $\mathbb{M}_t$ we have

$$h(x_m + u_m) - h(x_m) < 3m,$$

a contradiction for such m to (2). $\square$

4.7 Proof of Theorem 9

In the Euclidean case if V, W are Baire non-meagre, we may suppose that $V = I \backslash M_0 \cup N_0$ and $W = J \backslash M_1 \cup N_1$, where I, J are open intervals. Take $V_0 = I \backslash M_0$ and $W_0 = J \backslash M_1$. If v and w are points of V_0 and W_0, put $a := v - w$. Thus $v \in I \cap (J + a)$. So $I \cap (J + a)$ differs from $V \cap (W + a)$ by a meagre set. Since $M_0 \cup N_0$ may be expanded to a meagre $\mathcal{F}_\sigma$ set M, we deduce that $I \backslash M$ and $J \backslash M$ are non-meagre $\mathcal{G}_\delta$-sets.

In the density case, if V, W are measurable non-null let V_0 and W_0 be the sets of density points of V and W. If v and w are points of V_0 and W_0, put $a := v - w$. Then $v \in T := V_0 \cap (W_0 + a)$ and so T is non-null and v is a density point of T. Hence if T_0 comprises the density points of T then $T \backslash T_0$ is null, and so T_0 differs from $V \cap (W + a)$ by a null set. Evidently T_0 contains a non-null closed, hence $\mathcal{G}_\delta$-subset (as T_0 is measurable non-null, by regularity of Lebesgue measure). $\square$

4.8 Proof of Theorem 10

In either case applying Theorem 9, for some a the set $T := V \cap (W + a)$ is Baire non-meagre/measurable non-null. We may now apply the Kestelman-Borwein-Ditor Theorem to the set T. Thus for almost all $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{t + z_m : m \in \mathbb{M}_t\} \subset T \subset V \cap (W + a).$$

4.9 Proof of Theorem 11

Fix a null sequence $z_n \to 0$. If S is Baire non-meagre/measurable non-null then so is $-S$; thus we have for some a that $T := S \cap (a - S)$ is likewise Baire non-meagre/measurable non-null and so for quasi all (almost all) $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{t + z_m : m \in \mathbb{M}_t\} \subset T \subset S \cap (a - S),$$

as required. $\square$

4.10 Proof of Theorem 12

For S, T (weakly) similar, we claim that $S-T$ contains $(0,\delta)$ for some $\delta>0$. Suppose not: then for each positive n there is z_n with

$$z_n \in (0, 1/n)\backslash(S-T).$$

Now $-z_n$ is null so there is s in S and infinite $\mathbb{M}_s$ such that

$$\{s - z_m : m \in \mathbb{M}_t\} \subset T.$$

For any m in $\mathbb{M}_t$ pick $t_m \in T$ so that $s - z_m = t_m$; then we have

$$s - z_m = t_m \quad \text{so} \quad z_m = s - t_m,$$

a contradiction. Thus for some $\delta > 0$ we have $(0,\delta) \subset S - T$.

For S self-similar, say S is similar to $T := a - S$, for some a, then $a + (0,\delta) \subset a + (S-T) = a + S - (a - S) = S + S$, i.e. $S + S$ contains an interval. □

4.11 Proof of Theorem 13

By the Semigroup Theorem the set T contains an interval $[A_0,\infty)$. By the Combinatorial Boundedness Theorem applied to compact sets $K := [A,B] \subset [A_0,\infty) \subset T$, we have

$$\limsup_{x\to\infty} \sup_{u\in[A,B]} h(u+x) - h(x) < \infty.$$

Now we argue as in [BGT] page 62-3, though in additive notation. Fix $A > 2A_0$ and choose x_1 and C such that

$$h(u+x) - h(x) < C \qquad (A \le u \le 2A, \quad x_1 \le x < \infty).$$

For arbitrary $v \ge A$, find $m \ge 1$ such that $mA \le v < (m+1)A$. Thus, for any x,

$$A \le (v+x) - [(m-1)A + x] \le 2A.$$

So, for $x \ge x_1$,

$$\begin{aligned} h(v+x) - h(x) &= h(v+x) - h((m-1)A + x) \\ &\quad + \sum_{k=1}^{m-1} [h(kA + x) - h((k-1)A + x)] \\ &\le mC \le \frac{C}{A} v := Kv. \end{aligned}$$

This proves the first claim of the Theorem with $K = C/A$. Next with $A + x_1 \le a < t < b$, first use the substitutions $v = A + b - t \ge A$ and $x = t > x_1$, and thereafter the substitutions $v = t - x_1 \ge A$ and $x = x_1$ to obtain the inequalities

$$h(A + b) - K(A + b - t) \le h(t) \le K(t - x_1) + h(x_1) \qquad (a \le t \le b).$$

So

$$h(A + b) - K(A + b - a) \le h(t) \le K(b - x_1) + h(x_1) \qquad (a \le t \le b).$$

This proves the local boundedness claim far enough to the right. □

4.12 Proof of Theorem 14

(i) Let us suppose that h is Borel (that is, h has a Borel graph). As a first step consider the graph of the function of two variables: $h(t + x) - h(t)$, namely the set

$$G = \{(x, t, y) : y = h(t + x) - h(t)\}.$$

One expects this to be a Borel set and indeed it is. For a proof, we must refer back to the set h itself, and to do this we must re-write the defining clause appropriately. This re-writing brings out explicitly an implicit use of quantifiers, a common enough occurrence in analysis, often missed by the untrained eye (see at the end of the paper for another important example). We have:

$$y = h(t + x) - h(t) \Leftrightarrow (\exists u, v, w \in \mathbb{R}) r(x, t, y, u, v, w),$$

where

$$r(x, t, y, u, v, w) = [\, y = u - v \;\&\; w = t + x \;\&\; (w, u) \in h \;\&\; (t, v) \in h]. \quad (3)$$

From a geometric viewpoint, the set of points

$$\{(x, t, y, u, v, w) : r(x, t, y, u, v, w)\}$$

is Borel in $\mathbb{R}^6$, hence the set $G = \{(x, t, y) : (\exists u, v, w \in \mathbb{R}) r(x, t, y, u, v, w)\}$, being a projection of a Borel set, is an analytic set in $\mathbb{R}^3$, and in general not Borel. However, in the particular present context the ‘sections’

$$\{(u, v, w) : r(x, y, z, t)\},$$

corresponding to fixed $(x, t, y) \in G$, are single points (since u, v, w are defined uniquely by the values of x and t). In consequence, the projection here is Borel. The reason for this is that any Borel set is a continuous injective image of the irrationals ([JayRog] Section 3.6, p.69), and so a continuous injective image, as here under projection, of a Borel set is Borel. (So here the hidden quantifiers are 'innocuous', in that they do not degrade the character of G.) The current result may also be seen as the simplest instance of a more general result, the Rogers-Kunugui-Arsenin Theorem, which asserts that if the sections of a Borel set are $\mathcal{F}_\sigma$ (that is, countable unions of closed sets), then its projection is Borel ([JayRog] p. 147/148).

By abuse of notation, let us put $h(t, x) = h(t+x) - h(t)$ and think of t as parametrizing a family of functions. By assumption, the family of functions $h(t, x)$ is Borel, that is, the graph $\{(x, y, t) : y = h(t, x)\}$ is a Borel set (we will weaken this restriction appropriately below).

As a second step, we now consider the formal definition of $h^*(x)$, again written out in a predicate calculus using a semi-formal apparatus. The definition comes naturally as a conjunction of two clauses:

$$y = h^*(x) \Leftrightarrow P(x, y) \ \&\ Q(x, y),$$

where

$$\begin{aligned} P &= (\forall n)(\forall q \in \mathbb{Q}^+)(\exists t \in \mathbb{R})(\exists z \in \mathbb{R})[t > n \ \&\ z = h(t, x) \ \&\ |z - y| < q], \\ Q &= (\forall q \in \mathbb{Q}^+)(\exists m)(\forall t \in \mathbb{R})(\forall z \in \mathbb{R})[t > n \ \&\ (t, x, z) \in h \implies z < y + q]. \end{aligned}$$

The first clause (predicate) asserts that y is a limit point of the set $\{h(t, x) : t \in \mathbb{R}\}$ and this requires an existential quantifier; the second clause asserts that, with finitely many exceptions, no member of the set exceeds y by more than q and this requires a universal quantifier.

From a geometric viewpoint, for fixed $q > 0$ the set of points

$$G_1 = \{(x, y, z, t) : p(x, y, z)\}, \text{ where } p(x, y, z, t) = [(t, x, z) \in h \ \&\ |z - y| < q],$$

is Borel in $\mathbb{R}^4$, hence again the set $\{(x, y) : (\exists z, t \in \mathbb{R})p(x, y, z)\}$, being a projection of a Borel set, is an analytic set in $\mathbb{R}^2$. Again, for fixed (x, y) we look at the section of G_1. Evidently $\{z : |z - y| < q\}$ is an open set, so $\mathcal{F}_\sigma$. However, only if we assume that h is $\mathcal{F}_\sigma$ can we deduce that $\{(x, y) : (\exists t \in \mathbb{R})(\exists z \in \mathbb{R})[t > n \ \&\ z = h(t, x) \ \&\ |z - y| < q\}$ is Borel. Otherwise it is merely analytic.

From the viewpoint of mathematical logic, since the quantifiers in $(\exists z \in \mathbb{R})(\exists t \in \mathbb{R})p(x, y, z, t)$ are at the front of the defining formula, that formula is said to be $\mathbf{\Sigma}^1_1$ (read: bold-face sigma-1-1) as above, where Σ refers to the opening quantifier block being existential, the superscript identifies that the quantification is of order 1 (i.e. ranging over reals rather than integers), and the subscript refers to the fact that there is only one (existential) block of quantifiers at the front. (That is, P may be written out without using any further order 1 quantifiers.) See [Rog] for a modern side-by-side exposition of the two viewpoints of mathematical logic and geometry.

Finally, the set $\{(x, y) : P(x, y)\}$ is seen to be obtainable from analytic set (or Borel in the special case) by use of countable union and intersection operations. It is thus an analytic set (or Borel as the case may be).

By contrast, the set $\{(x, y) : (\forall z, t \in \mathbb{R})q(x, y, z, t)\}$, where $q(x, y, z, t) = [[z = h(t, x)] \implies z < y + q]$, is said to be co-analytic, since its complement is the analytic set $\{(x, y) : (\exists z, t \in \mathbb{R})[z = h(t, x) \ \&\ z \geq y + q]\}$. Again for given q and for arbitrary fixed (x, y) the sections of $\{(x, y, z, t) : [z = h(t, x) \ \&\ z \geq y + q]\}$ will be be $\mathcal{F}_\sigma$ if the graph of h is $\mathcal{F}_\sigma$, but is otherwise analytic. Thus $\{(x, y) : Q(x, y)\}$ is seen to be obtainable from co-analytic sets (or at best Borel sets) by use of countable union and intersection operations. It is thus co-analytic (or Borel as the case may be).

On a syntactic, logical analysis the formula $(\forall z \in \mathbb{R})q(x, y, z, t)$ is said to be $\mathbf{\Pi}^1_1$, since the opening quantifier is universal of order 1.

The set $\{(x, y) : Q(x, y)\}$ is seen to be obtainable from co-analytic sets by use of countable union and intersection operations. It is thus co-analytic since such operations preserve this character. Finally, note that the sets which are differences of analytic sets are both in the classes $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$, and so are in the common part of the two classes denoted $\mathbf{\Delta}^1_2$. We have of course neglected the possibility that the lim sup is infinite, but for this case we need only note that

$$\begin{aligned} h^*(x) &= \infty \Leftrightarrow (\forall n)(\exists t \in \mathbb{R})(\exists z \in \mathbb{R})[t > n \ \&\ z = h(t, x) \ \&\ z > n], \\ h^*(x) &< \infty \Leftrightarrow (\exists y \in \mathbb{R})(y = h^*(x)), \end{aligned}$$

so that this case is simultaneously $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

We have thus proved part (i) of the Character Theorem. □(i)

(ii) Now assume that h has an analytic graph. It follows from (3) that G, being the projection of an analytic set, is now analytic. That is, we may write

$$y = h(t,x) \Leftrightarrow (\exists w \in \mathbb{R}) F(t,x,y,w),$$

where the set $\{(t,x,y,w) : F(t,x,y,w)\}$ is Borel. Then

$$\{(x,y) : (\exists z \in \mathbb{R})(\exists w \in \mathbb{R})[F(t,x,z,w) \;\&\; |z-y| < q]\}$$

is only analytic, since we have no information about special sections; however, the set

$$\{(x,y) : (\forall z \in \mathbb{R})(\exists w \in \mathbb{R})[t > n \;\&\; F(t,x,z,w) \implies z < y+q]\},$$

requires for its definition a quantifier alternation which begins with a universal quantifier, so is said to be $\mathbf{\Pi}^1_2$ (read: bold-face pi-1-2). Since $\mathbf{\Sigma}^1_1$ sets are necessarily a subclass of $\mathbf{\Pi}^1_2$ sets, the graph of $\limsup_t f(t,x)$ in this case is $\mathbf{\Pi}^1_2$. $\Box$(ii)

(iii) Now, suppose that the function $h(x)$ has a co-analytic graph. Then by (3) the set G is of class $\mathbf{\Sigma}^1_2$, i.e. the function $h(t,x)$ has a $\mathbf{\Sigma}^1_2$ graph. That is, we now have to write

$$y = h(t,x) \Leftrightarrow (\exists u \in \mathbb{R})(\forall w \in \mathbb{R}) F(t,x,y,u,w),$$

where as before the set $\{(t,x,y,w) : F(t,x,y,u,w)\}$ is Borel. Then

$$\{(x,y) : (\exists z,u \in \mathbb{R})(\forall w \in \mathbb{R})[F(t,x,z,u,w) \;\&\; |z-y| < q]\}$$

is now $\mathbf{\Sigma}^1_2$. On the other hand the set

$$\{(x,y) : (\forall z \in \mathbb{R})(\exists u \in \mathbb{R})(\forall w \in \mathbb{R})[F(t,x,z,u,w) \implies z < y+q]\}$$

is $\mathbf{\Pi}^1_3$. Since $\mathbf{\Sigma}^1_1$ sets are necessarily a subclass of $\mathbf{\Pi}^1_3$ sets, the graph of $\limsup_t h(t,x)$ in this case is $\mathbf{\Pi}^1_3$. $\Box$(iii)

Remark. The theory of analytic sets dates from work of Souslin in 1916, Luzin in 1917, Luzin and Sierpiński in 1918. For monograph treatments, see [Lu], [Rog]. The historical origins, in an error of Lebesgue in 1905, are given there - in Lebesgue's preface to [Lu] and in [JayRog] Section 1.3: projections of Borel sets need not be Borel, whence the degradation studied above.

4.13 Proof of Theorem 15

Here we have

$$y = \partial h(x) \iff (\forall q \in \mathbb{Q}^+)(\exists n \in \omega)(\forall t > n)(\forall zuvw)P,$$

where

$$P = [[z = u - v \ \& \ w = t + x \ \& \ (t, v) \in h \ \& \ (w, u) \in h\ ] \implies \ |z - y| < q]],$$

and

$$y \neq \partial h(x) \iff (\forall q \in \mathbb{Q}^+)(\exists n \in \omega)(\forall t > n)(\forall zuvw)Q,$$

where

$$Q = [[z = u - v \ \& \ w = t + x \ \& \ (v, t) \in h \ \& \ (u, w) \in h\ ] \implies \ |z - y| \geq q]].$$

□

4.14 Proof of Theorem 16

By (3) the function $y = h(t, x)$ is of class $\mathbf{\Sigma}^1_2$. We show that $y = \partial_{\mathcal{U}} h(t)$ is of class $\mathbf{\Sigma}^1_2$. The result will follow since the negation satisfies

$$y \neq \partial_{\mathcal{U}} h(t) \iff \exists z[z \neq y \ \& \ [z = h^*(t) \text{ or } h^*(t) = \pm\infty]],$$

and so is of class $\mathbf{\Sigma}^1_2$. Finally,

$$y = \partial_{\mathcal{U}} h(t) \iff (\forall \varepsilon \in \mathbb{Q}^+)(\exists U)(\forall n \in \omega)(\exists t)P,$$

where

$$P = [U \in \mathcal{U} \ \& \ n \in U \ \& \ (n, t) \in \mathbf{x} \ \& \ |t - y| < \varepsilon],$$

and

$$\partial_{\mathcal{U}} h(t) = \infty \iff (\forall M \in \mathbb{Q}^+)(\exists U \in \mathcal{U})(\forall n \in U)(\exists t)[(n, t) \in \mathbf{x} \ \& \ t > M].$$

□

5 Complements

Bitopology. We remarked in the Introduction that it is in fact the Baire case, rather than the measurable case, which is primary. This is the theme of our paper [BOst11], where we use a bitopological approach. The Baire case is handled using the Euclidean topology. The measurable case is handled using the density topology. Recall ([Kech], 17.47) that a set is (Lebesgue) measurable iff it has the Baire property under the density topology (and a function is approximately continuous in the sense of Denjoy iff it is continuous under the density topology: [LMZ], p.1).

Combinatorialization of regular variation. The contribution of [BOst-SteinOstr] and [BOst-FRV] is to subsume the measurable and Baire cases of basic Karamata theory – the content of Chapter 1 of [BGT], on the class R of regularly varying functions – under a minimal common generalization, based on infinite combinatorics. There, limits exist, and one has an index of regular variation, ρ.

The contribution of this paper is to extend this programme to the more difficult situation of further Karamata theory – Chapter 2 of [BGT] – where limits do not exist. The relevant class is now the class OR of O-regularly varying functions (Aljančić and Arandelović [AA]; [BG1], [BG2], [BGT]), and one now has a pair of indices, the Matuszewska indices. As above, the key results from [BGT] are Th. 2.0.1 and Cor. 1.1.5. With these now combinatorialized as above, the combinatorialization of the rest of Chapter 2 of [BGT] is immediate – one just replaces appeals to these results by appeals to their new combinatorial forms. This applies also to the class ER of extended regular variation ([BG1], [BG2]; [BGT], Ch. 2: $R \subset ER \subset OR$), where one has instead a pair of Karamata indices.

There remains the task of combinatorializing de Haan theory (see [BGT] Ch. 3 for background and references), where one deals with relations such as

$$[f(\lambda x) - f(x)]/g(x) \to h(\lambda) \qquad (x \to \infty)(\forall \lambda > 0),$$

giving the class Π_g, and its extensions $E\Pi_g$ and $O\Pi_g$. Here, without loss of generality, the denominator g is regularly varying, and one may take g slowly varying, as the other cases are easily dealt with ([BGT], p.145). For Π_g, the combinatorialization may be carried out from that of [BOst-SteinOstr], [BOst-FRV] by implementing the 'double-sweep' procedure of [BGT], 3.13.1. For $O\Pi_g$ (and $E\Pi_g$), the combinatorialization may be carried out similarly from the results of this paper, and matters disaggregate as here. The remaining parts of the book (theory – Abelian, Tauberian and Mercerian theorems, and applications – to analytic number theory, complex analysis and probability theory) – combinatorialize immediately from this.

Representation theorems. For all the function classes, mentioned above, that appear in regular variation, representation theorems exist: see [BG2], [BGT] (where these are listed on p.471). Always, two functions are involved, one inside an integral (and this may be taken arbitrarily smooth), and one outside (and this has the same amount of regularity as the function being represented). See [BOst4] Th. 5 for a combinatorialized form. The point here is that there are lots of such functions, and we know exactly what they are.

In both the traditional (measurable/Baire) and combinatorial formulations, in regular variation one has a dichotomy. Either things are very bad (this is bound up with the Hamel pathology; see e.g. [BGT] 1.1.4), or very good (when, in particular, the above representation theorems apply). See [BOst12] for a development of this pathology from our combinatorial viewpoint.

The reader may be struck by the antithesis in the two aspects of character in the Uniformity Theorem which ends Section 2: on the one side h^*, a function with character potentially very much degraded from the originating Baire function h, on the other the assumption that the level set corresponding to finiteness contains a non-meagre *Baire* set. The representation theorems confirm that functions combining these two properties exist – even outside the class of regularly varying functions, in our setting where limits do not exist. Equally, Projective Determinacy asserts that the level set is Baire, and the largeness condition one is imposing really is natural.

Literature on universality. Universality (in the context of including null sequences by translation, a term introduced and hitherto used only by Kestelman [Kes]) occupied combinatorialists for its limitations. Thus Borwein and Ditor [BoDi] constructed a measurable T of positive measure and a null sequence z_n such that no shift of the sequence is almost contained in T (thereby answering a question of Erdős). Under Martin's Axiom (MA) Komjáth [Komj-1] constructs a measure zero, first category set T such that T is universal, and in fact contains a translated copy of every set of cardinality less than continuum; in [Komj-2], generalizing [BoDi], he constructs a measurable set T of positive measure and a null sequence z_n such that T fails to contain almost all of any translate of any scalar multiple λz_n. (See [Mil1] for the associated literature and for 'forcing' connections with genericity.)

Subuniversality. Our term is coined from Kestelman's; generic subuniversality is linked both to compactness and additivity through 'shift-compactness' (cf. [Par]), a notion familiar in the probability theory context of semigroups of measures under convolution. See [BOst8] for a topological analysis on the line and [BOst12] for a topological group context.

No Trumps. The term No Trumps in Definition 2, employed also in [BOst5], denotes a combinatorial principle, which is used in close analogy with earlier combinatorial principles, in particular Jensen's Diamond $\diamondsuit$ [Je] and Ostaszewski's Club $\clubsuit$ [Ost] and its weakening in another direction: 'Stick' in [FSS]. The argument in the proof of the No Trumps Theorem is implicit in [CsEr] and explicit in [BG1], p.482 and [BGT], p.9. The intuition behind our formulation may be gleaned from forcing arguments in [Mil1], [Mil2], [Mil3].

Self-similarity. Though this definition is applied to 'large sets' (generic subuniversal ones) and rests only on shifts and negligible sets, the idea is motivated by the observation that a Cantor set is self-similar when affine transformations are used in place of shifts. The Cantor set is a prototypical example of a 'small set' which is self-similar in the sense of the theory of fractals; our abbreviation of 'subuniversally self-similar' to 'self-similar' in this paper should cause no confusion with this usage.

Conditions on graphs rather than preimages. In the current context, there is a subtlety in play in regard to classifying functions according to the character of graphs rather than according to the character of their preimages. Recall that if $\{I_n^k : n \in \omega\}$ is for each k a family of disjoint intervals of diameter $1/k$ covering $\mathbb{R}$, then

$$f = \bigcap_{k=1}^{\infty} \bigcup_{n\in\omega} f^{-1}(I_n^k) \times I_n^k \text{ and } f^{-1}(I_n^k) = \mathrm{proj}(f \cap \mathbb{R} \times I_n^k).$$

Thus f has Borel (analytic) graph iff the preimages $f^{-1}(I_n^k)$ are Borel (analytic). However, if the preimages $f^{-1}(I_n^k)$ are all co-analytic, then the complementary sets $\mathbb{R}\backslash f^{-1}(I_n^k) = f^{-1}(\mathbb{R}\backslash I_n^k) = \bigcup_{m\neq n} f^{-1}(I_m^k)$ are also co-analytic.

Thus $f^{-1}(I_n^k)$ is both co-analytic and analytic and hence Borel by Souslin's Theorem. This implies that each $f^{-1}(I_n^k)$ is Borel.

Effective character. In Section 3 Comment 2 we mention effective versions of the Third Character Theorem. The projective hierarchy as introduced in Section 3 classifies sets in Euclidean space according to logical complexity (in terms of quantifiers), starting with the analytic sets, which are projections of Borel sets. This approach may be refined to take into account how effective (computable, or recursive, cf. [Rog] p. 435) are the various union and complementation operations that generate the Borel sets from opens sets (regarded as defined by an enumeration of basic intervals, whose effectiveness may be analyzed). See [Mos] Ch. 3 and [MaKe] Section 6.

Character complexity induced by hidden quantifiers. We offer the promised example of a 'far from innocuous' hidden occurrence of quantifiers. The vector sum of two sets S, T is formally defined by

$$S + T = \{r : (\exists s, t)[s \in S \ \& \ t \in T \ \& \ r = s - t]\}.$$

It is the occurrence of the quantifier here that is responsible for altering the complexity of the sum well beyond the complexity of the summands. Thus if the summands are co-analytic sets the vector sum need not be measurable. A specific example may be constructed by appeal to Gödel's Axiom $V = L$ and taking for the summands co-analytic Hamel bases; see [Kucz] p. 256. For further details of the vector sum see [NSW].

Comment on the virtues of the class $\mathbf{\Delta}_2^1$**.** It seems to us that the class $\mathbf{\Delta}_2^1$ offers an attractive class within which to carry out the analysis of regularly varying functions. It admits a pluralist interpretation. Either the members of the class $\mathbf{\Delta}_2^1$ may be taken to be measurable in the highly regular world governed by the Axiom of Projective Determinacy, or else the limit function $\partial h(t)$, or $k(t)$, is guaranteed to exist in a world otherwise filled with Hamel-type pathology governed by Gödel's Axiom.

In summary, regular variation theory has occasion in a natural way to make use of the 'projective sets' of level 2. We suggest that therefore a natural setting for the theory of regular variation is slowly varying functions of class $\mathcal{H}$, where $\mathcal{H}$ may be taken according to need to be one of the classes $\mathbf{\Sigma}_2^1$, or $\mathbf{\Pi}_2^1$, or their intersection $\mathbf{\Delta}_2^1$. The latter class is the counterpart of the

Borel sets thought of as $\mathbf{\Delta}_1^1$, namely the intersection of the classes $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ (according to Souslin's characterization of Borel sets as being simultaneously analytic and co-analytic).

In certain axiom schemes for set theory, the sets in these three classes are all measurable and have the Baire property. The notable case is Zermelo-Fraenkel set theory enriched with the Axiom of Projective Determinacy (PD), which asserts the existence of winning strategies in Banach-Mazur games with projective target sets (see [Tel] and [MaKe] for surveys); this axiom is a replacement for the Axiom of Choice (AC), some of whose reasonable consequences it upholds, at the same time negating consequences that are sometimes held to be glaringly counter-intuitive (such as the paradoxical decompositions, for which see [Wag]). Woodin's seminal work made it possible for Martin and Steel ([MaSt] and [MaSt-pr]) to find just the right strong axiom of infinity implying PD (precisely, the existence of infinitely many Woodin cardinals, cf. [Wo1]). In [Wo2] he argues that PD plays the same role for second-order number theory as the Peano Axioms do for first-order number theory.

Though somewhat inadequate from the point of view of the lim-sup operation (recall the emergence of the $\mathbf{\Pi}_3^1$ sets in the First Character Theorem), the class $\mathbf{\Delta}_2^1$ is quite rich. In Zermelo-Fraenkel set theory enriched with Gödel's Axiom of Constructibility $V = L$ (a strong form of AC), the class $\mathbf{\Delta}_2^1$ contains a variety of singular sets. In particular, the class $\mathbf{\Delta}_2^1$ is rich enough to contain the well-known Hamel pathologies (see [BGT] p. 5 and 11), since the axiom furnishes a $\mathbf{\Pi}_1^1$ set of reals which is a Hamel basis. On this latter point see [Mil1], and for a classical treatment of Hamel bases see [Kucz] or the recent [CP].

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