**BEYOND LEBESGUE AND BAIRE: GENERIC REGULAR VARIATION**

BY

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**To Jacob (Jaap) Korevaar, on his 85th birthday**

**Abstract.** We show that the No Trumps combinatorial property (NT), introduced for the study of the foundations of regular variation in [BOst1], permits a natural extension of the definition of the class of functions of regular variation, including the measurable/Baire functions to which the classical theory restricts itself. The 'generic functions of regular variation' defined here characterize the maximal class of functions, to which the three fundamental theorems of regular variation (Uniform Convergence, Representation and Characterization Theorems) apply. The proof uses combinatorial variants of the Steinhaus and Ostrowski Theorems deduced from NT in [BOst3].

# 1 Introduction

The theory of regular variation was initiated by Karamata in 1930 (see [BGT]) for continuous functions, but began to achieve its modern form only in 1949 in the work of Korevaar et al. [KvAEdB], where it is extended to (Lebesgue) measurable functions. It may also be developed for functions which have the property of Baire (briefly, Baire functions). We refer to [BGT] for an expositioin of this classical theory, in the measurable and Baire cases, and to [Oxt] for duality between measure and category. We point out that regular variation is motivated, not only by its intrinsic mathematical interest, but by two major areas of application – Tauberian theory, for which we refer to [Kor], Ch. IV, and probability theory, for which see e.g. [BGT], Ch. 8. We point out also that the classical theory is in one dimension, but that much interest currently attaches to the multi-dimensional case, see e.g. [HLMS].

The three foundation stones of the theory or regular variation are the Uniform Convergence Theorem (UCT), the (Karamata) Representation Theorem and the Characterization Theorem, which identifies the crucial concept of the *index* of regular variation (denoted here by $\rho$). In [BOst1] we introduced a combinatorial property, called No Trumps or **NT** (see [BOst1], [BOst3] for the origin of this name, traced there to an analogy with Jensens's $\diamond$ and Ostaszewski's $\clubsuit$), which gave the UCT for slowly varying functions in a maximally general context, thus including both measurability and the Baire property as special cases. Here we extend the UCT from slow to regular variation – for which we need a strengthening of No Trumps to Strong No Trumps or **SNT** – and also obtain the Representation and Characterization Theorems in this new setting.

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We use combinatorial versions of the classical Steinhaus and Ostrowski theorems, recently obtained in [BOst3]. We call our new setting *generic,* by analogy with usage in two areas: in analysis, as it includes the measurable and Baire contexts – see e.g. [AlpPras-1], [AlpPras-2] – and in mathematical logic, where certain model-theoretic extensions are said to be generic – see e.g. [Jech1], [Jech2] where the two canonical extensions – Cohen generic and Solovay generic – have respectively category and measure connections. We mention in passing that, if we restrict from slow to 'very slow' variation one can dispense with assumptions such as measurability or the Baire property altogether, as demonstrated by [BOst2]. We restrict attention here to one dimension, for convenience and brevity; for a glimpse of what our generic approach brings to the higher-dimensional case, we refer to [BOst3], Section 5.

The theory of regular variation, or of regularly varying functions, explores the consequences of a relationship of the form

$$f(\lambda x)/f(x) \to g(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0, \tag{$RV$}$$

for functions defined on $\mathbb{R}_+$. The limit function $g$ must satisfy the Cauchy functional equation

$$g(\lambda\mu) = g(\lambda)g(\mu) \qquad \forall \lambda, \mu > 0. \tag{$CFE$}$$

Subject to a mild regularity condition, $(CFE)$ forces $g$ to be a power:

$$g(\lambda) = \lambda^\rho \qquad \forall \lambda > 0. \tag{$\rho$}$$

Then $f$ is said to be *regularly varying* with *index* $\rho$, written $f \in R_\rho$.

The case $\rho = 0$ is basic. A function $f \in R_0$ is called *slowly varying*; slowly varying functions are often written $\ell$ (for *lente*, or *langsam*). Here

$$\ell(\lambda x)/\ell(x) \to 1 \ \forall \lambda > 0 \text{ as } x \to \infty.$$

While regular variation is usually used in the multiplicative formulation above, for proofs in the subject it is usually more convenient to use an additive formulation. Writing $h(x) := \log f(e^x)$ (or $\log \ell(e^x)$ as the case may be), $k(u) := \log g(e^u)$, the relations above become

$$h(x+u) - h(x) \to k(u) \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}, \tag{1}$$

$$h(x+u) - h(x) \to 0 \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}, \tag{2}$$

$$k(u+v) = k(u) + k(v) \qquad \forall u, v \in \mathbb{R}. \tag{3}$$

Subject to some mild regularity asumptions classically based on measurability or the Baire property, one proves the characterisation theorem, that

$$k(t) = \rho t \qquad \forall \lambda > 0. \tag{4}$$

Evidently it follows that

$$h_0(t) = h(t) - \rho t$$

is slowly varying, and so in the measurable/Baire case obeys the UCT. Thus the classical functions of regular variation take the form

$$h(t) = \rho t + h_0(t). \tag{5}$$

In this paper we study the maximal possible family of functions to which the theory of regular variation could conceivably apply – the functions $h$ of the form (5) with $h_0$ satisfying UCT. We prove a characterization theorem for this family by reference to a purely combinatorial property of functions (the **SNT**-functions, for 'Strong No Trumps') shared also by the slowly varying functions. As both the measurable functions and the Baire functions have this combinatorial property (this being the content of what we call the Strong No Trumps Theorem), the theorems of the extended theory demonstrably imply their classical counterparts as special cases. It is thus appropriate to dub the functions in the maximal family **generically regularly varying**, or **GRV**.

The Karamata Representation Theorem – for which see Section 3 below – decomposes a slowly varying function $h_0$ into a sum of an integral term and a term converging to a constant, $c$ say. The integral term may be made to behave as well as desired – e.g. to be $C^\infty$, to have all its derivatives tending to 0, etc. – by use of a *de Bruijn* mollifier ([dB]; [BGT] Th. 1.3.3; see also the Smooth Variation Theorem, [BGT] Section 1.8). By contrast, the term converging to a constant may be made to do so as badly as desired. It may be taken as pathological as the Axiom of Choice allows. For many purposes in analysis the distinction between a function tending to $c$ and the constant $c$ is immaterial – in which case, one may restrict oneself to functions which are smooth and well-behaved. By contrast, from the point of view of building a theory of regular variation in maximal generality, it is just here that the main difficulty, and so interest, lies. For background see e.g. [BGT] Section 3.2.2, esp. p. 145, and the Character Theorems of [BOst1].

# 2 Definitions and assumed background

This section is devoted to basic definitions and theorems on which this paper relies.

## 2.1 Definitions of NT principles

We recall the definition from [BOst1]. It is convenient to amend the notation in the light of our present needs. We follow the set-theorists and denote the set of natural numbers by $\omega = \{0, 1, 2, ..\}$.

**Definition 1.** For $\{T_k : k \in \omega\}$ a family of subsets of $\mathbb{R}$, $\mathbf{NT}(\{T_k : k \in \omega\})$ means that, for every bounded/convergent sequence $\{u_n\}$ in $\mathbb{R}$, some $T_k$ contains a translate of a subsequence of $\{u_n\}$, i.e. there are $k \in \omega$, an infinite $\mathbb{M} \subseteq \omega$, and $t \in \mathbb{R}$ such that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq T_k.$$

In the definitions below, the subscripts $A, F$ and $L$ are meant to suggest '*almost all*', '*in full*' and '*localized*'.

**Definition 2.** For $\{T_k : k \in \omega\}$ a family of subsets of $\mathbb{R}$, $\mathbf{NT}_A(\{T_k : k \in \omega\})$ means that, for every convergent sequence $\{u_n\}$, some $T_k$ contains almost all of a translate of $\{u_n\}$, i.e. there are $k, M, t$ such that

$$\{t + u_n : n > M\} \subseteq T_k.$$

**Definition 3.** For $\{T_k : k \in \omega\}$ a family of subsets of $\mathbb{R}$, $\mathbf{NT}_F(\{T_k : k \in \omega\})$ means that, for every convergent sequence $\{u_n\}$, some $T_k$ contains all of $\{u_n\}$, i.e. there is $k$ such that

$$\{u_n : n \in \omega\} \subseteq T_k.$$

**Definition 4 (Strong No Trumps, SNT).** For $\{T_k : k \in \omega\}$ a family of subsets of $\mathbb{R}$, $(\{T_k : k \in \omega\}) = \mathbf{NT}_L(\{T_k : k \in \omega\})$ means that, for every bounded/convergent sequence $\{u_n\} \to t \in \bigcup_{k\in\omega} T_k$, some $T_k$ contains a 'neighbouring' translate of a subsequence of $\{u_n\}$, i.e. for all $\varepsilon > 0$, there are $k \in \omega$, an infinite $\mathbb{M} \subseteq \omega$ and $z \in (t - \varepsilon, t + \varepsilon)$ such that

$$\{z + u_n : n \in \mathbb{M}\} \subseteq T_k.$$

For the function $h : \mathbb{R} \to \mathbb{R}$, the (symmetric) **level sets** of $h$ are defined by

$$H^r, \text{ or } H^r(h), \;\; := \{t : |h(t)| < r\}.$$

The **difference** function $h_x(t)$ is defined by $h_x(t) = h(x+t) - h(x)$. It is a central tool; it may be helpful to think of it as a *differential operator*. Its level sets are to be denoted

$$H^r_x, \text{ or } H^r(h_x), \;\; := \{t : |h(t+x) - h(x)| < r\}.$$

The function $h : \mathbb{R} \to \mathbb{R}$ is **slowly varying** if it satisfies (2) above, i.e. its difference function tends to zero:

$$h_x(t) = h(x+t) - h(x) \to 0, \qquad (x \to \infty) \qquad \forall t \in \mathbb{R}.$$

For $\mathbf{x} = \{x_n\}$ a sequence, in $\mathbb{R}^\omega$, tending to infinity, we will write $\mathbf{x} \to \infty$.

The **x-stabilized sets**, or just the 'stabilized sets', of $h$ are defined to be

$$T^r_k, \text{ or } T^r_k(\mathbf{x}), \;\; := \bigcap_{n \ge k} H^r_{x(n)} = \bigcap_{n \ge k} \{t : |h(x_n + t) - h(x_n)| < r\},$$

with $x_n$ and $x(n)$ synonymous. They are of necessity instrumental in our analysis of the limiting behaviour of $h_x$ (cf. Proposition 1 below). Note that

$$T^r_0(\mathbf{x}) \subseteq T^r_1(\mathbf{x}) \subseteq T^r_2(\mathbf{x}) \subseteq ... \text{ and } T^r_k(\mathbf{x}) \subseteq T^s_k(\mathbf{x}) \text{ whenever } r < s.$$

For $\mathbf{x} \to \infty$ and any $\varepsilon > 0$, if $h$ is slowly varying, then

$$\mathbb{R} = \bigcup_{k \in \omega} T^\varepsilon_k(\mathbf{x}). \tag{6}$$

The function $h : \mathbb{R} \to \mathbb{R}$ is **additive** if it satisfies the *Cauchy functional equation* (3) of Section 1. In this case

$$H_x^r = H^r = \{t : |h(t)| < r\},$$

which is independent of $x$. Thus the stabilized sets $T_k^r$ coincide with the sets $H^r$ in this case. Note that, if $h$ is additive, then $t \in H^{|h(t)|}$ and so

$$\mathbb{R} = \bigcup_{k \in \omega} H^k. \tag{7}$$

The connection between results derived from No Trumps assumptions and classical measure/category considerations is given by the following theorem. For the cognoscenti, the intuition for this may be gleaned from forcing proofs due to Miller; see the cycle of papers [Mil1], [Mil2], [Mil3].

**Strong No Trumps Theorem** (Csiszár and Erdös) *If $T$ is an interval and $T = \bigcup_{k\in\omega} T_k$ with each $T_k$ measurable/Baire, then $\mathbf{NT}_L(\{T_k : k \in \omega\})$ holds. Indeed, for every convergent sequence $\{u_n\} \to u_0 \in T$, any neighbourhood of the limit $u_0$ contains a point $z$ for which there exist $K = K(z) \in \omega$ and an infinite set $\mathbb{M} = \mathbb{M}(z) \subseteq \omega$ such that*

$$z + u_m \in T_K \text{ for } m \in \mathbb{M}.$$

**Proof.** The result is contained in [CsEr] implicitly. Here is the proof, a simple adaptation of the text in [BOst1], Section 4.3.

Let $\mathbf{u} = \{u_n\}$ be a bounded sequence, which we may as well assume is convergent to some $u_0$. Let $\eta > 0$. We assume that $|u_n - u_0| \le \eta$. Put

$$[-\eta + u_0, u_0 + \eta] = \bigcup_k I_k, \text{ where } I_k = [-\eta + u_0, u_0 + \eta] \cap T_k.$$

By assumption, each $I_k$ is measurable [Baire], so there is $K$ such that $I_K$ has positive measure [is non-meagre]. Let

$$Z_K = \mathbf{u}(I_K) := \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} (I_K - u_n).$$

We now quote almost verbatim from [BGT] p. 9. 'In the measurable case all the $(I_K - u_n)$ have measure $|I_K|$, and as they are subsets of the fixed bounded interval $[u_0 - 2\eta, u_0 + 2\eta]$, $Z_K$ is a subset of the same interval having measure

$$|Z_K| = \lim_{j\to\infty} \left| \bigcup_{n=j}^{\infty} (I_K - u_n) \right| \ge |I_K| > 0.$$

So $Z_K$ is non-empty.

In the Baire case $I_K$ contains some set $I \backslash M$, where $I = (t - \delta, t + \delta)$ is an open interval of length $2\delta > 0$, with $\delta < \eta$ and $M$ is meagre. So each $(I_K - u_n)$ contains $I^n \backslash M^n$, where $I^n = I - u_n$ is an open interval of length $2\delta$ and $M^n := M_n - u_n$ is meagre. Choosing $J$ so large that $|u_i - u_j| < 2\delta$ for all $i, j \geq J$, the intervals $I^J, I^{J+1}, ...$ all overlap each other, and so $\bigcup_{n=j}^{\infty} I^n$, for $j = J, J+1, ...$, is a decreasing sequence of intervals, all of length $\geq 2\delta$ and all contained in the interval $[u_0 - 2\eta, u_0 + 2\eta]$; hence $I^0 = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} I^n$ is an interval of length $\geq \delta$. Since $Z_K$ contains $I^0 \backslash \bigcup_{n=j}^{\infty} M^n$, it follows that $Z_K$ is non-meagre, so non-empty.'

Thus in either case, there is a point $z \in Z_K \subseteq [u_0 - 2\eta, u_0 + 2\eta]$. This means that $z \in (I_K - u_n)$ for infinitely many $n$. Say that

$$z \in (I_K - u_m) \text{ for } m \in \mathbb{M}.$$

Without loss of generality, $m \in \mathbb{M}$ implies $m > K$. Consider $m \in \mathbb{M}$. By definition, for some $y = y_m$, we have $z = y_m - u_m$ with $y_m \in I_K$. But this says that

$$z + u_m \in I_K \text{ for } m \in \mathbb{M},$$

as required. □

**Notes. 1.** This is the localized, and hence sharper and more useful, version of the theorem needed in [BOst1] and [BOst3]. It was gleaned from the proof in [BGT] of Theorem 2.0.1 as the strongest version capable of delivering all of the several uniformity theorems in regular variation, and goes back to [BG1]; it is also meant to motivate a forthcoming definition (of the **SNT** functions in Section 2.4).

The measure-theoretic form can be traced back to Borwein and Ditor [BD]; the combination of measure and Baire versions seems to be new. In fact much more can be said – see [BOst6]. Borwein and Ditor [BD] actually prove more than we have here (the difference is in the size of the set of possible translators); we restrict attention here to the simplest possible setting.

2. The Theorem remains true if $T$ is a set with the Baire property, as then, for some interval $S$ and meagre $M_0$, $T$ contains $S \backslash M_0$. For a proof, re-read interpreting $u_0 + [-\eta, \eta]$ as $S$, and expanding the meagre set $M$, mentioned below, to contain $M_0$.

## 2.2 The combinatorial Steinhaus and Ostrowski Theorems

We will need the next two theorems which were proved in [BOst3].

**Combinatorial Steinhaus Theorem.** *For an additive subgroup $S$ of $\mathbb{R}$, the following are equivalent:*
(i) $S = \mathbb{R}$,
(ii) $\mathbf{NT}_A(S)$,
(iii) $\mathbf{NT}(S)$.

The classical version is in [St] in the measurable case, [P] in the Baire case; see [BGT] Th. 1.1.1. For the next theorem we need a definition.

**Definition (Weak NT-functions).** Let $h : \mathbb{R} \to \mathbb{R}$. We will say that $h$ is a **weak NT-function**, $h \in \mathbf{WNT}$, if $\mathbf{NT}(\{H^k : k \in \omega\})$ holds.

**Combinatorial Ostrowski Theorem.** *For $h(x)$ an additive function: $h(x)$ is continuous and $h(x) = cx$ for some $c$ iff $h(x)$ is a* $\mathbf{WNT}$*-function, i.e.* $\mathbf{NT}(\{H^k : k \in \omega\})$ *holds.*

Recall that the classical version of this result in the measurable case is in [Ostr], the Baire case in Mehdi [Meh]; see [BGT] Theorem 1.1.8.

## 2.3 Uniform Convergence Theorem – UCT

The classical Uniform Convergence Theorem UCT ([BGT] Theorem 1.2.1), as applied to measurable/Baire functions, is the first of the three Fundamental Theorems on which the foundations of regular variation rest. The other two are the Characterisation Theorem ([BGT], Theorem 1.4.1, p. 17), and the Representation Theorem ([BGT], Theorem 1.3.1, p. 12). Our aim is to define a wider class of functions to which all three theorems apply. Here we recall, from [BOst1], the combinatorial material which constitutes the departure point for this paper, a general form of the UCT. This theorem in particular identifies its own maximal class of functions.

**Uniform Convergence Theorem.** *For $h(x)$ slowly varying, the following are equivalent:*
(i) $h_x(t) = h(x+t) - h(x) \to 0$, *uniformly in $t$ on compact sets as $x \to \infty$,*
(ii)$_{\mathbf{x}}$ $\mathbf{NT}_A(\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\})$,*whenever $\varepsilon > 0$ and $\mathbf{x} \to \infty$,*
(iii)$_{\mathbf{x}}$ $\mathbf{NT}_F(\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\})$,*whenever $\varepsilon > 0$ and $\mathbf{x} \to \infty$,*
(ii) $\mathbf{NT}_A(\{T_k^\varepsilon(\mathbf{id}) : k \in \omega\})$,*whenever $\varepsilon > 0$ with* $\mathbf{x} = \mathbf{id}$ where $\mathbf{id}(n) = n$,
(iii) $\mathbf{NT}_F(\{T_k^\varepsilon(\mathbf{id}) : k \in \omega\})$,*whenever $\varepsilon > 0$ with* $\mathbf{x} = \mathbf{id}$ where $\mathbf{id}(n) = n$.

The most convenient criterion to test for uniform convergence (and on which the generalization of UCT rests) is the following result (from [BOst1]). We will need to invoke it several times.

**Bounded Equivalence Principle.** *For $h(x)$ slowly varying, the following are equivalent:*
(i) $h_x(t) = h(x+t) - h(x) \to 0$, *uniformly in $t$ on compact sets as $\mathbf{x} \to \infty$,*
(ii) $\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0$, *whenever* $\mathbf{u}$ *is a bounded sequence, and* $\mathbf{x} \to \infty$.

The following simple result plays a crucial role in the current paper.

**Proposition on sequence containment.** *Suppose the UCT holds for a function $h$. Let $\mathbf{u}$ be any bounded sequence, and let $\varepsilon > 0$. Then, for every sequence $\mathbf{x}$ tending to infinity, the stabilized $\varepsilon$-level set $T_k^\varepsilon(\mathbf{x})$ for some $k$ contains the sequence $\mathbf{u}$.*

## 2.4 Generically regularly varying functions (GRV)

**Definition (NT-functions).** Let $h : \mathbb{R} \to \mathbb{R}$. We will say that:

(i) $h$ is an **NT-function**, $h \in \mathbf{NT}$, if, for each $\mathbf{x} \to \infty$ and each $r > 0$, $\mathbf{NT}(\{T_k^r(\mathbf{x}) : k \in \omega\})$ holds.

(ii) $h$ is an **SNT-function**, $h \in \mathbf{SNT}$, if, for each $\mathbf{x} \to \infty$ and each $r > 0$, $\mathbf{NT}_L(\{T_k^r(\mathbf{x}) : k \in \omega\})$ holds.

With these definitions the main result from [BOst1] is that, for $h$ slowly varying, the Uniform Convergence Theorem holds for $h$ iff $h$ is a **NT**-function iff $h$ is an **SNT**-function. The implication from **NT** to **SNT** is a corollary of the Proposition on sequence containment. It is in the **SNT** property wherein is the key to identifying our maximal extension for the theory of regular variation.

**Two important examples.**

(i) If $h(t) \to c$, as $t \to \infty$, then $h$ is a slowly varying **NT**-function. But (as in Section 1) note that there are no restrictions on the character of $h$ here; qualitatively, $h$ could be as pathological as the Axiom of Choice allows.

(ii) Let $e : \mathbb{R} \to \mathbb{R}$ be continuous. If $e(t) \to 0$, as $t \to \infty$, define

$$h(t) = \int_0^t e(s)ds.$$

Then

$$h_x(t) = \int_x^{x+t} e(s)ds,$$

and so $h$ is a slowly varying **NT**-function. In fact given $\varepsilon > 0$, for $x$ large enough, we have

$$|h_x(t)|/t \le \varepsilon. \tag{8}$$

**Proposition 1.** *Let $h \in \mathbf{NT}$. Assume that*

$$h^*(t) = \lim_{x\to\infty} [h(t + x) - h(x)]$$

*exists (possibly as $\pm\infty$) for all $t \in \mathbb{R}$. Then $h^* \in \mathbf{WNT}$.*

**Proof.** Note that $|h^*(t)| < r$ iff

$$|\lim_n (h(t + n) - h(n))| < r,$$

iff for some $k$ we have

$$|h(t + n) - h(n)| < r \text{ for } n \ge k.$$

Thus

$$H^r(h^*) = \{t : |h^*(t)| < r\} = \bigcup_k \bigcap_{n \geq k} \{y : |h(t+n) - h(n)| < r\},$$

or

$$H^r(h^*) = \bigcup_k T_k^r. \tag{9}$$

Given $\{u_n\}$, if $\mathbf{NT}(\{T_k^r : k \in \omega\})$ holds, then, for some $k, z$ and an infinite $\mathbb{M}$, we have $\{z + u_n : n \in \mathbb{M}\} \subseteq T_k^r \subseteq H^r(h^*)$. $\square$

**Notes.**

1. In [BOst1] Section 2, the Second Character Theorem, referring to the descriptive character of functions, asserts that if $h \in \mathbf{\Delta}_2^1$ then $h^* \in \mathbf{\Delta}_2^1$ (assuming $h^*$ exists). Interest there was focussed on automatically having a slowly varying function $h$ be in $\mathbf{NT}$, by virtue of set-theoretic, axiomatic, assumptions. In this connection see Theorem 5 at the end of the paper.

2. In view of the Strong No Trumps Theorem, we may regard **WNT**-functions as also having some "uniformity" features common to measurable functions and functions having the Baire property. This viewpoint is evidently also supported by the equivalence result in our sharp form of Ostrowski's Theorem (in Section 2.2 above).

**Definition.** Let $h : \mathbb{R} \to \mathbb{R}$ be in **SNT**. Say that $h$ is **generically regularly varying** (or, $h \in \mathbf{GRV}^+$) if the limit

$$h^*(t) = \lim_{x \to \infty} [h(t+x) - h(x)]$$

exists (possibly as $\pm\infty$) for all $t \in \mathbb{R}$.

Evidently $h^*(0) = 0$. The important example is $h(x) = \rho x$; here $h^*(t) = \rho t$.

**Notes.** 1. The above is the *additive* formulation, whence the superscript $+$. As in Section 1, for applications it is the *multiplicative* formulation that is used, and there we write $f$, or $\ell \in \mathbf{GRV}^{\odot}$ – or, just as a product $a \cdot b$ is elided to $ab$, $f$, or $\ell \in \mathbf{GRV}$, as in Section 2.5 below. This then directly extends the classical usage (for which see [BGT]), where one writes $RV$ in the measurable case, or $BRV$ in the Baire case.

2. The qualifier 'generic' borrows from the usage in analysis whereby 'behaviour is generic' when it occurs on a set large in the measure or category sense. Our context includes both the measurable and the Baire functions.

3. We will see in the Characterization Theorem that, under the 'mild' additional condition $h \in \mathbf{SNT}$ (see the comment below), $h^*(t)$ is the linear function $\rho t$ for some constant $\rho$. To aid the intuition, one may think of the *function* $h^*$ as the 'derivative' of $h$, at infinity.

4. **Comment on the SNT condition.** When $h \in$ **SNT,** by (9), we assert that if $t$ satisfies $|h^*(t)| < r$, and $u_n \to 0$, then, for each $\varepsilon > 0$, there are $k \in \omega$,

an infinite $\mathbb{M} \subseteq \omega$ and $z \in \mathbb{R}$ with $|t - z| < \varepsilon$, such that

$$\{z + u_n : n \in \mathbb{M}\} \subseteq \bigcap_{n \geq k} \{y : |h(t + n) - h(n)| < r\}.$$

For the important case $h(x) = \rho x$, where $h^*(t) = \rho t$ (to which all other cases of interest reduce), the displayed condition simplifies considerably. The hypothesis, $|h^*(t)| < r$ means $|\rho t| < r$. The **SNT** condition then requires that there are $k \in \omega$, an infinite $\mathbb{M} \subseteq \omega$ and $z \in \mathbb{R}$ with $|t - z| < \varepsilon$, such that

$$\{z + u_n : n \in \mathbb{M}\} \subseteq H^r = (-r/|\rho|, r/|\rho|).$$

In the cases of interest, this is indeed a very mild restriction on $h$.

5. Notice that the **SNT** condition relates only to the *local* behaviour of $h^*$ at the origin (recall $h^*(0) = 0$). This should come as no surprise: we have said that the existence of $h^*$ may be regarded as 'a condition of differentiability at infinity', and as such naturally restricts attention to approximations for small increments $t$.

6. In the classical context when $h$ is measurable, according to *Littlewood's 2nd Principle*, $h$ is 'nearly continuous' (see [Lit], Section 4). In this case, the Strong No Trump Theorem confirms the $h \in$ **SNT** condition. Indeed the use of SNT makes possible this deduction of classical theory from the current setting for regular variation.

## 2.5 Generic regular variation with index $\rho$

Our initial definition of a 'hierarchy' of classes for the functions of generic regular variation is motivated by technical concerns. We are led to identify first the **regularly-varying NT-functions.** The payoff is a transparent argument leading to a Characterisation Theorem, which describes the more natural classes of (**SNT**) functions of '**generic regular variation'.**

The two definitions follow, starting with the more natural one. The superscript $+$ in the definitions is to suggest the additive formulation of regular variation theory, and similarly $\odot$ is to suggest the multiplicative formulation. Once introduced, the latter will suffer the natural elision associated with the dot of multiplication.

**Definition (GRV$_\rho$).** A function $h \in$ **SNT** such that

$$h^*(t) = \lim_n |h(t + n) - h(n)| = \rho t$$

is said to be of **generic regular variation with index** $\rho$ (in the additive sense), $h \in \mathbf{GRV}_\rho^+$.

The corresponding function $f$ with $h(x) = \log f(e^x)$ is then said to be of **generic regular variation with index** $\rho$ (in the multiplicative sense), $h \in \mathbf{GRV}_\rho^\odot$.

A function $h \in \mathbf{GRV}_0^+$ is just **generic-slowly varying** (in the additive sense), meaning that $h$ satisfies the **UCT**.

The corresponding $f$ is **generic-slowly varying** (in the multiplicative sense).

**Definition (NT-regular variation – additive formulation).**

(i) For $h_0$ slowly varying, we will say that $h_0$ is **NT-slowly varying** (in the additive sense) if $h_0 \in \mathbf{NT}.$

Recall that by UCT, for $h_0$ slowly varying, $h_0 \in \mathbf{SNT}$ iff $h_0 \in \mathbf{NT}$, so this agrees with generic slow variation.

(ii) We will say that $h$ is $\mathbf{NTR}_\rho^+$, or **NT-regularly-varying function with index** $\rho$ (in the additive sense), if

$$h(x) = \rho x + h_0(x) \text{ with } h_0 \text{ slowly varying in } \mathbf{NT}.$$

Thus the case $\rho = 0$ reduces to the NT-slowly varying functions, i.e.

$$\mathbf{NTR}_0^+ = \mathbf{NT}.$$

(Note that the symbol **NT** applies only to the additive formulation.)

**Definition (NT-regular variation – multiplicative formulation).**

(i)′ We will say that $f(x)$ is $\mathbf{NTR}_0^\odot$, or **NT-slowly-varying** (in the multiplicative sense), if $h(x) = \log f(e^x)$ is in **NT.**

(ii)′ We will say that $f(x)$ is $\mathbf{NTR}_\rho^\odot$, or **NT-regularly-varying function of index** $\rho$ (in the multiplicative sense), if

$$h(x) = \log f(e^x) \in \mathbf{NTR}_\rho^+.$$

This case reduces to (i)′ when $\rho = 0$.

To sum up: our objective is to show that generic regular variation with index $\rho$ is the same as **NT**-regular-variation with index $\rho$, i.e. $\mathbf{GRV}_\rho = \mathbf{NTR}_\rho$. This will be the content of Theorem 3 in Section 3.

**Proposition 2.** *If $h_0, k_0$ are* **NT***-slowly varying, then so is $h_0 + k_0$, and*

$$\text{if } h \in \mathbf{NTR}_\rho^+ \text{ and } k \in \mathbf{NTR}_\sigma^+, \text{ then } h + k \in \mathbf{NTR}_{\rho+\sigma}^+.$$

**Proof.** Plainly $h_0 + k_0$ is slowly varying. By the UCT for slowly varying functions, as $h_0, k_0$ satisfy bounded equivalence (see Section 2.2), so does $h_0+k_0$, and so $h_0 + k_0$ is **NT**-slowly varying. The conclusion follows by the definition of $\mathbf{NTR}_\rho^+$. □

We may now generalize the Uniform Convergence Theorem to a form which applies to functions of regular variation with index $\rho$. We will see later (after we have proved the Equivalence Theorem) that there is an alternative formulation replacing $\mathbf{NTR}_\rho^+$ by $\mathbf{GRV}_\rho^+$. Our result is most conveniently formulated as a 'local uniformity' (which, via compactness, implies uniformity on compact sets).

**Theorem 1 (UCT-$\rho$ : Uniform Convergence Theorem for regular variation).**
*We have $h \in \mathbf{NTR}_\rho^+$ iff the following uniformity condition holds*

$$\lim_{\delta\to 0+} \lim_{m\to\infty} \sup \sup_{|u|\le\delta} |h(t+u+m) - h(m) - \rho t| = 0.$$

**Proof.** For the direct implication, we may take $\rho \neq 0$, as the case $\rho = 0$ has already been proved in [BOst1]. Suppose $h \in \mathbf{NTR}_\rho^+$, i.e. that for some $h_0 \in \mathbf{NT}$

$$h(t) = \rho t + h_0(t).$$

Suppose not. Then there are $\varepsilon > 0$, $u_n \to 0$, $m_n \to \infty$ so that

$$|h(t + u_n + m_n) - h(m_n) - \rho t| > 4\varepsilon.$$

By UCT, since $h_0 \in \mathbf{SNT}$ there are $z$ with $|\rho(z-t)| < \varepsilon, k \in \omega$, and an infinite $\mathbb{M}$ such that

$$\{z + u_n : n \in \mathbb{M}\} \subseteq T_k^\varepsilon(\{m_n\}) = \bigcap_{n\ge k} \{y : |h_0(y + m_n) - h_0(m_n)| < \varepsilon\}.$$

So for large enough $n$ in $\mathbb{M}$,

$$|h_0(z + u_n + m_n) - h_0(m_n)| < \varepsilon.$$

Also since $h_0$ is slowly varying, reference to the pointwise limits at $t$ and at $z-t$ shows that for all $n$ large enough

$$|h_0(t + (z-t) + u_n + m_n) - h_0((z-t) + u_n + m_n)| < \varepsilon,$$

and

$$|h_0((z-t) + u_n + m_n) - h_0(m_n)| < \varepsilon,$$

by the bounded equivalence principle. Now we may write

$$\begin{aligned}
& h(z + u_n + m_n) - h(m_n) - \rho t \\
= \;& [\rho(z + u_n + m_n) + h_0(z + u_n + m_n)] - [\rho m_n + h_0(m_n)] - \rho t \\
= \;& \rho(z-t) + \rho u_n + [h_0(t + (z-t) + u_n + m_n) - h_0((z-t) + u_n + m_n)] \\
& + [h_0((z-t) + u_n + m_n) - h_0(m_n)].
\end{aligned}$$

But, for all $n$ large enough $|\rho u_n| < \varepsilon$, so for large enough $n$ in $\mathbb{M}$

$$\begin{aligned}
& |h(z + u_n + m_n) - h(m_n) - \rho t| \\
\le \;& |\rho(z-t)| + |\rho u_n| \\
& + |h_0(t + (z-t) + u_n + m_n) - h_0((z-t) + u_n + m_n)| \\
& + |h_0((z-t) + u_n + m_n) - h_0(m_n)| \\
< \;& 4\varepsilon,
\end{aligned}$$

a contradiction.

For the converse, assume the uniformity condition holds. Then $h^*(t) = \rho t$. Define $h_0(t) = h(t) - \rho t$. The condition may now be rewritten thus:

$$\begin{aligned} 0 &= \lim_{\delta\to 0+} \lim\sup_{m\to\infty} \sup_{|u|\le\delta} |h_0(t+u+m) - h_0(m) - \rho u| \\ &= \lim_{\delta\to 0+} \lim\sup_{m\to\infty} \sup_{|u|\le\delta} |h_0(t+u+m) - h_0(m)|. \end{aligned}$$

Thus $h_0$ is slowly varying and by the bounded equivalence principle we have $h_0$ in **NT**. This establishes the converse. □

**Corollary.** *Let $h \in$ **SNT**. Suppose that $h^*(t) = \lim_{x\to\infty}[h(t+x) - h(x)]$ exists with $h^*(t) = \rho t$. Then $h_0(t) = h(t) - \rho t$ is in **SNT**.*

**Proof.** Taking $h_0(t) = h(t) - \rho t$ we obtain

$$\lim_{\delta\to 0+} \lim\sup_{m\to\infty} \sup_{|u|\le\delta} |h_0(t+u+m) - h_0(m)| = 0,$$

so by the bounded equivalence principle $h_0$ satisfies any one of the clauses in the UCT, and especially the $\mathbf{NT}_F$ version: some $T_k$ contains the sequence $\{u_n\}$ ('in full'). We conclude from this, or directly from the proposition on sequence containment, that $h_0 \in$ **SNT.** □

# 3 Characterisation and Represention

In this section we generalize, to a combinatorial form, the other two of the three fundamental theorems of the classical theory of regular variation: the Characterisation Theorem ([BGT], Th. 1.4.1, p. 17), and then the Representation Theorem ([BGT], Th. 1.3.1, p. 12 especially formula 1.3.2 ), which for us is a corollary of the Characterisation result. As a first step, we prove Theorem 2, in which only the inclusion $\mathbf{GRV}_\rho^+ \subseteq \mathbf{NT}_\rho^+$ is asserted. The reverse inclusion forms the substance of Theorem 3.

**Theorem 2 (Characterisation Theorem for GRV).** *Let $h \in$ **GRV,** i.e. $h$ is an **SNT** -function and*

$$h^*(t) = \lim_{x\to\infty} (h(t+x) - h(x))$$

*is assumed to exists (possibly as $\pm\infty$) for all $t$. Then:*

(i) *$h^*(t)$ is finite for all $t$,*

(ii) *for some constant $\rho$, $\quad h^*(t) \equiv \rho t$ and $h \in \mathbf{GRV}_\rho^+$,*

(iii) *Thus:*

$$\mathbf{GRV}^+ = \bigcup_\rho \mathbf{GRV}_\rho^+ \text{ and } \mathbf{GRV}_\rho^+ \subseteq \mathbf{NT}_\rho^+.$$

**Comment.** To place this in context: this result says, for any $\varepsilon > 0$ and all $z$ large enough, that

$$\rho t - \varepsilon \leq h(t+z) - h(z) \leq \rho t + \varepsilon.$$

Put $h(z) = \log f(e^z)$, $\lambda = e^t, x = e^z$ and $\eta = e^\varepsilon - 1$. Then

$$(1-\eta)\lambda^\rho \leq f(\lambda x)/f(x) \leq e^{\rho t} e^\varepsilon = (1+\eta)\lambda^\rho,$$

for all large enough $x$. This justifies the definitions that preceeded the theorem.

We shall see later that the asserted inclusion in (iii) may be improved to an equality.

**Proof of Theorem 2.** Let

$$S = \{t : |h^*(t)| < \infty\} = \bigcup_{k\in\omega}\{t : |h^*(t)| < k\} = \bigcup_k H^k.$$

For $s, t \in S$ we have that

$$\begin{aligned} h^*(s+t) &= \lim_n([h(s+t+n) - h(t+n)] + [h(t+n) - h(n)]) \\ &= \lim_n([h(s+t+n) - h(t+n)] + \lim_n[h(t+n) - h(n)]) \\ &= h^*(s) + h^*(t). \end{aligned}$$

Thus $h^*$ is additive on $S$, and so $S$ is a subgroup of $\mathbb{R}$. By Proposition 1, since $h \in \mathbf{NT}$, $h^* \in \mathbf{WNT}$, i.e. $\mathbf{NT}(H^k)$ holds for each $k > 0$, and so $\mathbf{NT}(S)$ holds. Hence $S = \mathbb{R}$ by the Combinatorial Steinhaus Theorem. Thus by the Combinatorial Ostrowski Theorem we see that for some $\rho$ we have

$$h^*(t) = \rho t.$$

Now put $h_0(t) = h(t) - \rho t$; then evidently

$$\begin{aligned} h_0^*(t) &= \lim_{n\to\infty}[h_0(t+x) - h_0(x)] \\ &= \lim_{n\to\infty}[h(t+x) - h(x) - \rho t] = 0. \end{aligned}$$

So $h_0$ is slowly-varying. By the Corollary to the UCT of Section 2.5, we deduce that $h_0$ is $\mathbf{NT}$. So $h(t) = \rho t + h_0(t) \in \mathbf{NT}_\rho^+$. □

As a corollary we now have the following result.

**Theorem 3 (Equivalence Theorem).** *The functions of generic regular variation with index $\rho$ coincide with their* **NT**-*counterparts, i.e.*

$$\mathbf{GRV}_\rho^+ = \mathbf{NTR}_\rho^+, \qquad \mathbf{GRV}_\rho = \mathbf{NTR}_\rho.$$

**Proof of Theorem 3.** We know from the last theorem that functions of generic regular variation with index $\rho$ are in $\mathbf{NT}_\rho^+$.

Now if $h$ is in $\mathbf{NT}_\rho^+$, put

$$h(t) = \rho t + h_0(t),$$

with $h_0$ **NT**-slowly varying. We are to show that $h$ is in **SNT** and that

$$h^*(t) = \lim_{x\to\infty}[h(t+x) - h(x)] = \rho t.$$

Evidently

$$\begin{aligned} h^*(t) &= \lim_x [h(t+x) - h(x)] = \lim_x [\rho t + h_0(t+x) - h_0(x)] \\ &= \rho t + \lim_x [h_0(t+x) - h_0(x)] = \rho t. \end{aligned}$$

To show that $h$ is in **SNT**, we are to show, that given $t$ with $|h^*(t)| < r$, and $\mathbf{u} \to 0$, $\mathbf{x} \to \infty$, and $\varepsilon > 0$, there are $z$ with $|z - t| < \varepsilon$, $k \in \omega$, and an infinite $\mathbb{M}$ such that

$$\{z + u_n : n \in \mathbb{M}\} \subseteq T_k^r = \bigcap_{n\geq k} \{y : |h(y + x_n) - h(x_n)| < r\}.$$

To understand the proof consider first the case $h_0 = 0$. In this case we are to show that

$$\{z + u_n : n \in \mathbb{M}\} \subseteq T_k^r = \bigcap_{n\geq k} \{y : |\rho y| < r\} = (-r/|\rho|, r/|\rho|).$$

For some $M$ we have

$$\{u_n : n > M\} \subseteq (-r/|\rho|, r/|\rho|).$$

The requirement may thus be met iff $t \in (-r/|\rho|, r/|\rho|)$.

For general slowly varying $h_0$ in **NT**, we have

$$|h(t + x_n) - h(x_n)| = |\rho t + h_0(t + x_n) - h_0(x_n)|.$$

Here again we show the same result for a fixed $t$ under the hypothesis that $|\rho t| < r$. In this case $r - |\rho t| > 0$, so we restrict attention to $\varepsilon$ with $0 < \varepsilon < r - |\rho t|$.

Now, since $h_0$ is in **SNT** there are $k \in \omega$, an infinite $\mathbb{M} \subseteq \omega$ and $z \in \mathbb{R}$ with $|t - z| < \varepsilon$, such that $|\rho(z - t)| < \varepsilon/3$ and

$$\{z + u_n : n \in \mathbb{M}\} \subseteq T_k^r = \bigcap_{n\geq k} \{y : |h_0(y + x_n) - h_0(x_n)| < \varepsilon/3\}. \tag{10}$$

We may assume that for $n \in \mathbb{M}$ we have $n > k$, and that $k$ is so large that

$$|\rho u_n| < \varepsilon/3.$$

For such $n$, by (10), we have

$$|h_0(z + u_n + x_n) - h_0(x_n)| < \varepsilon/3.$$

Hence, for $n \in \mathbb{M}$, we have

$$\begin{aligned}
& |h(z+u_n+x_n)-h(x_n)| \\
= \ & |\rho z+\rho u_n+h_0(z+u_n+x_n)-h_0(x_n)| \\
= \ & |\rho(z-t)+\rho t+\rho u_n+h_0(z+u_n+x_n)-h_0(x_n)| \\
\leq \ & |\rho(z-t)|+|\rho u_n|+|\rho t|+|h_0(z+u_n+x_n)-h_0(x_n)| \\
\leq \ & \varepsilon+|\rho t|<r.
\end{aligned}$$

Thus

$$\{z+u_n : n \in \mathbb{M}\} \subseteq T_k^r=\bigcap_{n\geq k}\{y : |h(y+x_n)-h(x_n)|<r\},$$

that is, $h$ is in **SNT**. This completes the proof. □

**Theorem 4 (Karamata Representation Theorem for GRV).** *A function $h:\mathbb{R}\to\mathbb{R}$ is of generic regular variation iff, for some constants $\rho$ $c$,*

$$h(t)=\rho t+h_c(t)+\int_0^t e(x)dx, \tag{11}$$

*where $h_c(t)\to c$, so is* **NT,** *and $e(x)\to 0$ in $C^\infty(\mathbb{R})$ as $x\to\infty$.*

**Proof of Theorem 4.** By Theorem 3, one readily checks that any function with this representation is generic of regular variation with index $\rho$ (see the two important examples in Section 2.4). For the other direction: by the Characterization theorem $h(t)=\rho t+h_0(t)$, for some constant $\rho$ and slowly varying $h_0$ in **NT**. After de Bruijn [dB] (see [BGT] theorem 1.3.3 p. 14) we will apply (as mollifier) any $p(x)$ in $C^\infty[0,1]$ which is a probability density on $[0,1]$. Put

$$e(x)=(h_0([x]+1)-h_0([x])p(x-[x]),$$

where the first factor is constant in any interval $n\leq x<n+1$ with $n$ an integer. This is a mollification of $h$, as $e$ is actually $C^\infty$ ([BGT] ibid.). Now write

$$h_1(t)=h_0(0)+\int_0^t e(x)dx.$$

Noting that

$$\begin{aligned}
h_1(t) &= h_0(0)+[h_0(1)-h_0(0)]+\int_1^t e(x)dx \\
&= ... = h_0([t])+\int_{[t]}^t e(x)dx,
\end{aligned}$$

we have, by the Bounded Equivalence Principle, as $h_0$ is slowly varying and in **NT**, that

$$\bar{h}_0(t):=h_0(t)-h_1(t)=h_0(t)-h_0([t])-\int_{[t]}^t e(x)dx\to 0,$$

as $t \to \infty$. The result follows on taking $c = h_0(0)$ and $h_c(t) = c + \bar{h}_0(t)$. $\square$

**Notes.** 1. As the proof shows, the Representation Theorem is primarily about slowly varying functions.

2. The generic functions of regular variation are thus the *largest* class of functions to which the three fundamental theorems of regular variation apply.

3. Returning to our comment in Section 2.4 about the qualitative character of $h_c(t)$, we note that Theorem 4 has, as an immediate corollary of particular relevance to the (descriptive) set-theoretic identification of the natural context (family of functions) for regular variation theory, Theorem 5 below. See the discussion in Section 2 of [BOst1] (where the notation below is fully explained).

**Theorem 5 (GRV Character Theorem for $\mathbf{\Delta}_2^1$).** *If $h \in \mathbf{\Delta}_2^1$ and $h \in \mathbf{GRV}_\rho^+$, then*

$$h(t) = \rho t + h_c(t) + \int_0^t e(x)dx,$$

*where*

$$h_c(t) \to c \text{ is in } \mathbf{\Delta}_2^1 \text{ and } e(x) \to 0 \text{ in } C^\infty[b, \infty) \text{ as } x \to \infty.$$

Indeed, the result is a corollary of the following general statement.

**General Character Theorem:** *For $h$ slowly varying satisfying UCT and $\mathbf{\Gamma}$ a pointclass of functions closed under addition of continuous functions, $h \in \mathbf{\Gamma}$ holds iff the representation equation*

$$h(t) = h_c(t) + \int_0^t e(x)dx, \text{ with } e(x) \to 0 \text{ in } C^\infty[b, \infty) \text{ as } x \to \infty,$$

*holds with $h_c \in \mathbf{\Gamma}$.*

The proof of General Character Theorem is immediate from the Representation Theorem and the closure hypothesis:

$$h = h_c + g \in \Gamma \iff h_c = h - g \in \Gamma,$$

where $g(t)$ denotes the continuous function $\int_0^t e(x)dx$. The closure condition is met in the cases where $\mathbf{\Gamma}$ is either the class of measurable functions or the class of Baire functions. In turn this yields the character information in the corresponding Representation Theorems for measurable/Baire regular variation.

To deduce the $\mathbf{\Delta}_2^1$ case (Theorem 5) we need to check the closure hypothesis when $\Gamma = \mathbf{\Delta}_2^1$. Identify functions with their graphs. Thus $y = h(t) + g(t)$ iff $(y, t) \in h + g$. The assumption is that $h$ has a $\mathbf{\Delta}_2^1$ graph and that $g$, being continuous, has a closed graph. The two formulas defining the graph of $g + h$ and its complement, namely

$$\begin{aligned} y &= h(t) + g(t) \iff (\exists u, v)[(t, u) \in h \;\&\; (t, v) \in g \;\&\; y = u + v], \\ y &\neq h(t) + g(t) \iff (\exists u, v)[(t, u) \in h \;\&\; (t, v) \in g \;\&\; y \neq u + v], \end{aligned}$$

show both sets to be $\mathbf{\Sigma}^1_2$, because $(t,u) \in h$ is a $\mathbf{\Sigma}^1_2$ statement (see [BOst1] for an explanation). Thus. $h+g$ is in $\mathbf{\Delta}^1_2$, giving the closure hypothesis. $\square$

Our final result affirms what is self-evident in the classical context – that the product of two regularly varying functions is regularly varying (working in the multiplicative formulation). For the generic variation context, this follows by an application of the UCT, so is less obvious.

**Proposition 3.** *If* $h,k \in$**GRV**$^+$, *then* $h+k \in$**GRV**$^+$.

**Proof.** This follows from Proposition 2. Indeed, if $h \in \mathbf{GRV}^+_\rho$ and $k \in \mathbf{GRV}^+_\sigma$, then writing $h_0(t) = h(t) - \rho t$ and $k_0(t) = k(t) - \sigma t$, we obtain

$$h(t) + k(t) = (\rho + \sigma)t + [h_0(t) + k_0(t)].$$

But $h_0 + k_0$ is slowly varying and satisfies **UCT**, so is in **NT**. Hence $h+k$ is $\mathbf{GRV}^+_{\rho+\sigma}$. $\square$

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