Generic subadditive functions

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Abstract. We prove a generalization of the 'Subadditive Limit Theorem' and of the corresponding Berz Theorem in a class of functions that includes both the measurable functions and the 'Baire functions' (those with the Baire property). The generic subadditive functions are defined by a combinatorial property previously introduced for the study of the foundations of regular variation in [**BOst1**]. By specialization we thus provide the previously unknown Baire variants of the fundamental theorems of subbaditive functions, answering an old question ([**BGT**], 2.12.4 p. 123).

1. Introduction and Definitions

The class of subadditive functions is interesting from the point of view of both theory and applications. Regarding theory – for which see e.g. [**Ros**], [**HP**], [**Kucz**]– they have connections with both additive functions and convex functions. All three classes share pathologies in general – the *Hamel pathology*; see e.g. [**BGT**], p.5, where they occur in connection with the class of regularly varying functions – but have good properties under minimal regularity assumptions. In [**BOst1**], [**BOst3**], [**BOst4**] we undertook the programme of developing the theory of regular variation under minimal assumptions; the resulting theory of regular variation was there called generic because it gave a common generalization of the measurable and Baire cases ([**Kech**] (8.5) p. 42). It turns out that the methods developed there lend themselves to the corresponding programme for subadditive functions. Accordingly, we call the resulting theory that of generic subadditive functions. Regarding applications: for analysis, the principal theorem is the – widely used – limit theorem for subadditive functions, generalized below as the 'First Limit Theorem'. This has a probabilistic version, the subadditive ergodic theorem (see [**King**], [**Lig**]), extension of which provided additional motivation for this paper.

We offer here a generalization of the 'Subadditive Limit Theorem' applicable to a class of real-valued subadditive functions defined on $\mathbb{R}^N$ that includes both the measurable functions and functions with the Baire property (briefly the Baire functions), namely the class of **WNT** functions as defined below. This is a less restrictive class of functions permitting a unified treatment of the two classical cases, and still adequate, since, as we will show, a sublinear function on $\mathbb{R}^N$ is continuous iff it is in **WNT**. The conclusion that the theorem applies to Baire subadditive

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functions appears to be new. We extend several fundamental theorems concerning subadditive functions on $\mathbb{R}^N$ to the class **WNT** and prove a Uniform Convergence Theorem as a contribution to the understanding of the Subadditive Limit Theorem. We note that other approaches are known: [**MatŚw**] consider one-to-one instead of measurable or Baire.

We begin by recalling some combinatorial definitions from [**BOst3**]. We follow the set-theorists and denote the set of natural numbers by $\omega = \{0, 1, 2, ...\}$.

DEFINITION 1. *For a family* $\{T_k : k \in \omega\}$ *of subsets* $\mathbb{R}^N$, $\boldsymbol{NT}(\{T_k : k \in \omega\})$ *means that, for every bounded/convergent sequence* $\{u_n\}$ *in* $\mathbb{R}^N$, *some* T_k *contains a translate of a subsequence of* $\{u_n\}$, *i.e. there is* $k \in \omega$, *infinite* $\mathbb{M} \subseteq \omega, t \in \mathbb{R}^N$ *such that*

$$\{t + u_n : n \in \mathbb{M}\} \subseteq T_k.$$

For the function $h : \mathbb{R}^N \to \mathbb{R}$, its (symmetric) **level sets** are defined by

$$H^r, \text{ or } H^r(h), \ := \{t \in \mathbb{R}^N : |h(t)| < r\}.$$

DEFINITION 2. ***(WNT-functions).***

(1) Let $h : \mathbb{R}^N \to \mathbb{R}$. We will say that h is a **WNT-function**, $h \in$ **WNT**, if, for each $r > 0$, $\mathbf{NT}(\{H^k : k \in \omega\})$ holds.
(2) Let $h : \mathbb{R}^N \to [-\infty, \infty]$. We will say that h is a **WNT-function**, $h \in$ **WNT**, if, for each $r > 0$, $\mathbf{NT}(\{H^k : k \in \omega\} \cup H^{\pm\infty})$ holds, where $H^{\pm\infty} = \{t \in \mathbb{R}^N : |h(t)| = \infty\}$.

THEOREM 1. ***No Trumps Theorem*** *(Csiszár and Erdös* [**CsEr**]*; see* [**BOst1**]*). If T is an interval and $T = \bigcup_{k\in\omega} T_k$ with each T_k measurable/Baire, then* $\boldsymbol{NT}(\{T_k : k \in \omega\})$ *holds.*

COROLLARY 1. *If h is measurable/Baire, then* $h \in$ ***WNT****.*

PROOF. Taking $T = \mathbb{R}$ and $T_k = H^k$, a measurable/Baire set, this follows since

$$\mathbb{R} = \bigcup_{k\in\omega} H^k.$$

□

2. Subadditive Limit Theorem

We begin by extending the basic limit theorem for subadditive functions (Rosenbaum [**Ros**], Hille and Phillips [**HP**] p. 255).

THEOREM 2. ***Subadditive Limit Theorem (First Limit Theorem – at Infinity)****. If the subadditive function $f : \mathbb{R}^N \to \mathbb{R}$ is in* ***WNT,*** *then the limit function*

$$F(\mathbf{x}) := \lim_{s\to+\infty} \frac{f(s\mathbf{x})}{s}, \ \textit{for } \mathbf{x} \in \mathbb{R}^N, \tag{2.1}$$

is finite, positively homogeneous, convex and continuous. Moreover,

$$F(\mathbf{x}) = \inf_{t>0} \frac{f(t\mathbf{x})}{t}.$$

REMARK 1. *For $\mathbf{x} \neq 0$, write $v(\mathbf{x}) = \mathbf{x}/||\mathbf{x}||$; then*

$$F(\mathbf{x}) = ||\mathbf{x}|| \lim_{s\to+\infty} \frac{f(sv(\mathbf{x}))}{s} = ||\mathbf{x}||F(v(\mathbf{x})), \text{ for } \mathbf{x} \in \mathbb{R}^N.$$

*Note that, as F is in particular sublinear (subadditive and also $F(nx) = nF(x)$ for $n \in \omega$), this formula is in agreement with the theorem of Berz (see below) for measurable sublinear F ([**Kucz**] p. 415). We will see later circumstances under which this convergence is uniform. For the mean time we note:*

COROLLARY 2. *The Limit Theorem at Infinity holds if the subadditive function f is measurable/Baire.*

In preparation for the proof of the main theorem we need some auxiliary results.

LEMMA 1. *For any subadditive function $f : \mathbb{R}^N \to \mathbb{R}$, if f is locally bounded above at a point, then it is locally bounded at every point.*

PROOF. For a proof see [**Kucz**] p. 404 Th 2. □

LEMMA 2. *For any function $f : \mathbb{R}^N \to \mathbb{R}$, if f is locally bounded, then f is bounded on any bounded set.*

PROOF. If A is bounded, then its closure $\bar{A}$ is compact. Appealing to compactness, the result follows by covering $\bar{A}$ with a finite number of open sets on each of which f is bounded. □

PROPOSITION 1. *(cf.[**Kucz**] p 404 Th 3). If $f : \mathbb{R}^N \to \mathbb{R}$ in **WNT** is subadditive, then f is locally bounded.*

PROOF. By assumption, $\mathbf{NT}(\{H^k : k \in \omega\})$ holds for $H^k = \{x : |f(x)| < k\}$. Suppose that f is not locally bounded; then it is not locally bounded above at some point u, i.e. there exists $u_n \to u$ with

$$f(u_n) \to +\infty.$$

For some $k \in \omega$, $t \in \mathbb{R}$ and an infinite $\mathbb{M}$ we have

$$\{t + u_n : n \in \mathbb{M}\} \subseteq H^k.$$

For n in $\mathbb{M}$ we have

$$f(u_n) = f(t + u_n - t) \leq f(t + u_n) + f(-t) \leq k + f(-t),$$

which contradicts $f(u_n) \to +\infty$. □

Proof. (**Proof of the First Limit Theorem.**) Fix $\mathbf{x} \neq 0$. Put

$$\beta = \beta_+(\mathbf{x}) = \inf_{t>0} \frac{f(t\mathbf{x})}{t} \geq -\infty.$$

(We have adapted the notation of [**HP**] for clarity and to suit later needs.) As $\beta < \infty$, consider $b \in \mathbb{R}$ with $b > \beta$ and select t_0 with $f(t_0\mathbf{x})/t_0 < b$. Now consider any $t > 3t_0$ and let $m = [t/t_0]$. Thus

$$mt_0 < t < (m+1)t_0.$$

As f is **WNT**, we may, by Proposition 1 and Lemma 2, select an M which bounds f on the interval $[2t_0\mathbf{x}, 3t_0\mathbf{x}]$. Writing $n = m - 2$, we have

$$2t_0 \leq t - (m-2)t_0 = t - nt_0 \leq 3t_0.$$

Hence, by subadditivity, since $f(nz) \leq nf(z)$, we have

$$f(t\mathbf{x}) = f(nt_0\mathbf{x}) + f((t - nt_0)\mathbf{x}) \leq nf(t_0\mathbf{x}) + f((t-nt_0)\mathbf{x}) \leq nf(t_0\mathbf{x}) + M,$$

and so

$$\beta \leq \frac{f(t\mathbf{x})}{t} \leq \frac{nt_0}{t}\frac{f(t_0\mathbf{x})}{t_0} + \frac{M}{t} \leq \frac{nt_0}{t}b + \frac{M}{t}.$$

But, we have

$$\frac{m-2}{m+1} \leq \frac{(m-2)t_0}{t} \leq \frac{m-2}{m},$$

so, as $t \to \infty$, we see that $nt_0/t \to 1$. Thus in the limit we have

$$\beta \leq \liminf_{t\to\infty} \frac{f(t\mathbf{x})}{t} \leq \limsup_{t\to\infty} \frac{f(t\mathbf{x})}{t} \leq b.$$

But $b > \beta$ was arbitrary, so we have

$$\beta = \lim_{t\to\infty} \frac{f(t\mathbf{x})}{t} = F(\mathbf{x}) < \infty.$$

Now let

$$\alpha = \alpha_-(\mathbf{x}) = \sup_{t<0} \frac{f(t\mathbf{x})}{t} \leq \infty.$$

Evidently $\alpha > -\infty$. Now substituting $t = -s$, we have

$$\alpha = -\inf_{s>0} \frac{f(-s\mathbf{x})}{s} = -F(-\mathbf{x}) > -\infty.$$

But, since $f(t\mathbf{x}) + f(-t\mathbf{x}) \geq f(t\mathbf{x} - t\mathbf{x}) = f(0) \geq 0$, we have

$$F(\mathbf{x}) + F(-\mathbf{x}) = \lim_{t\to\infty} \left(\frac{f(t\mathbf{x})}{t} + \frac{f(-t\mathbf{x})}{t} \right) \geq 0.$$

Thus $-\infty < -F(-\mathbf{x}) \leq F(\mathbf{x})$. So $F(\mathbf{x})$ is finite. Clearly $F(\mathbf{x})$ is positively homogenous. Moreover, F is subadditive; indeed $f(t\mathbf{x}) + f(t\mathbf{y}) \geq f(t(\mathbf{x}+\mathbf{y}))$, so

$$F(\mathbf{x}+\mathbf{y}) = \lim_{t\to\infty} \left(\frac{f(t(\mathbf{x}+\mathbf{y}))}{t} \right) \leq \lim_{t\to\infty} \left(\frac{f(t\mathbf{x})}{t} + \frac{f(t\mathbf{y})}{t} \right) = F(\mathbf{x}) + F(\mathbf{y}). \tag{2.2}$$

Hence $F(\mathbf{x})$ is convex in $\mathbf{x}$: since it is subadditive and positively homogenous we have

$$F(\alpha\mathbf{u} + \beta\mathbf{v}) \leq F(\alpha\mathbf{u}) + F(\beta\mathbf{v}) = \alpha F(\mathbf{u}) + \beta F(\mathbf{v}), \tag{2.3}$$

as $\alpha, \beta \geq 0$. Thus ([**Rock**], Cor. 10.1.1, or [**Kucz**], Th. 7.1.1 p. 149) $F(.)$ is thus continuous. □

REMARK 2. *We recall the proof of continuity: one shows that F is locally bounded at any point u, as follows. Take $v_1, ..., v_{N+1}$ so that $u \in int[conv(v_1, ..., v_{N+1})]$. Now F is bounded on the finite set $\{v_1, ..., v_{N+1}\}$, and hence on $conv(v_1, ..., v_{N+1})$, by virtue of (2.3). Hence, by the Bernstein-Doetsch theorem (*[**BD**] *or* [**Kucz**] *p. 145), F is continuous.*

EXAMPLE 1. *The renewal function of probability theory is always subadditive (and measurable); see* [**Dal**] *Section 4,* [**Fell**] *Ch. XI. If μ is the mean of the lifetime distribution F – of lightbulbs, say – then the renewal function*

$$U(x) := \sum_{n=0}^{\infty} F^{*n}(x),$$

*where * denotes convolution; $U(x)$, the expected number of bulbs needed by time x, satisfies*

$$U(x+h) - U(x) \sim h/\mu \qquad (x \to \infty),$$

whence the weaker result

$$U(x)/x \sim 1/\mu \qquad (x \to \infty),$$

offering a nice illustration of the Limit Theorem at Infinity.

We will need the following result in the next section.

PROPOSITION 2. *(cf.* [**Kucz**] *p 406 Th 7). Let $f : \mathbb{R}^N \to [-\infty, \infty]$ in* ***WNT*** *be subadditive. Put $\lambda = \liminf_{x\to 0} f(x)$. Then $\lambda \geq 0$, or $\lambda = \pm\infty$. If $|\lambda| = +\infty$, then f is infinitary (takes one at least of the values $\pm\infty$).*

PROOF. For some null-sequence u_n we have $f(u_n) \to \lambda$. For some $k \in \omega$, an infinite $\mathbb{M}$ and t we have

$$\{t + u_n : n \in \mathbb{M}\} \subseteq H^k.$$

For n in $\mathbb{M}$, we have again

$$f(u_n) = f(t + u_n - t) \leq f(t + u_n) + f(-t) \leq k + f(-t).$$

Passing to the limit, we have

$$\lambda \leq k + f(-t).$$

Thus if $\lambda = +\infty$, the function f is infinitary at $-t$. Also, since $f(2u) \leq 2f(u)$, we have

$$\lambda \leq \liminf f(2u_n) \leq \liminf 2f(u_n) = 2 \liminf f(u_n) = 2\lambda,$$

so for λ finite we conclude $\lambda \geq 0$. If f assumes only finite values, then by Proposition 1 f is locally bounded and so $\lambda \neq \pm\infty$. Hence if $|\lambda| = \infty$, the function f is infinitary. □

If f is infinitary it may assume either or both the values $\pm\infty$. If the subadditive f does not assume the value ∞, suppose that $f(t) = -\infty$. Then, for all x, we have

$$f(x) = f(x - t + t) \leq f(x - t) + f(t) = -\infty.$$

(Here for each $k \in \omega$, the level sets $H^k(f)$ are empty.) On the other hand the subadditive function f, with $f(t) = 1$ if $t \in \mathbb{R}\backslash\mathbb{Q}$ and $+\infty$ otherwise, is in **WNT** and plainly does not assume the value $-\infty$. See [**HP**] p. 240 for other examples. The final result, clarifying the behaviour of subadditive functions taking negative values, mimicks its classical counterpart. For subadditive f, recall that $f(0) \geq 0$.

THEOREM 3. *(cf.* [**HP**] *p. 243 and* [**Kucz**] *p 406 Th 8). Let $f : \mathbb{R}^N \to \mathbb{R}$ be subadditive and in **WNT**. Suppose that there exists $\mathbf{x}_0 \in \mathbb{R}^N$ s.t. $f(\mathbf{x}_0) < 0$. Then $f(t\mathbf{x}_0) < 0$ for all $t > 0$ sufficiently large and $f(t\mathbf{x}_0) \geq 0$ for all $t \leq 0$.*

PROOF. The proof in Kuczma applies with Lemma 2 as above in place of his Th. 6. □

3. Sublinear functions: Berz's Theorem

For this section recall that a sublinear function is subadditive and satisfies

$$f(n\mathbf{x}) = nf(\mathbf{x}), \text{ for } n = 1, 2, ...$$

Such functions are characterized by Berz's Theorem ([**Berz**], see below). We note, as a canonical example, that the norm function is sublinear. For continuous sublinear functions, it turns out that $f(0) = 0$, so the alternative definition that $f(n\mathbf{x}) = nf(\mathbf{x})$, for $n \in \omega$, turns out to be equivalent.

The characterization theorem is usually deduced from Berz's Lemma, as quoted below, by way of an additive minorant lemma for measurable functions (see [**Kucz**] p. 218 for the convex minorant lemma). Such a proof is also possible in our context, since the corresponding additive minorant lemma holds also for **WNT** functions. However, we prefer to deduce the characterization theorem from Berz's Lemma by way of the First Limit Theorem, as then the relevant majorization is explicitly by way of a linear function.

LEMMA 3. ***Additive Minorant Lemma.*** *Suppose that $f, g : \mathbb{R}^N \to \mathbb{R}$ satisfy*

$$g(\mathbf{x}) \leq f(\mathbf{x}) \text{ for all } x \in \mathbb{R}^N,$$

*with g additive and f a **WNT** function. Then g is linear and continuous.*

PROOF. Suppose not. Then, by Ostrowski's Lemma (see e.g. [**BGT**] Lemma 1.1.6 p. 4 and [**BOst3**]), there is a convergent sequence u_n with $g(u_n)$ unbounded from above. For some t and infinite $\mathbb{M}$ we have

$$\{t + u_m : m \in \mathbb{M}\} \subseteq H^k = \{\mathbf{x} : |f(\mathbf{x})| < k\}.$$

Hence, for $m \in \mathbb{M}$, we have

$$g(t) + g(u_m) = g(t + u_m) \leq f(t + u_m) < k,$$

a contradiction. □

COROLLARY 3. ***(Ostrowski's Theorem).*** *A **WNT** additive function is continuous.*

PROOF. Take $f = g$. □

LEMMA 4. ***Subadditive Majorant Lemma.*** *For $T \subseteq \mathbb{R}^N$ suppose that* ***NT****(T) holds (e.g. if T is non-empty and open) and that $f, g : \mathbb{R}^N \to \mathbb{R}$ satisfy*

$$g(t) \leq f(t) \text{ for } t \in T,$$

with g additive, and f a subadditive ***WNT*** *function. Then $g(t)$ is linear and continuous.*

PROOF. Suppose not. Then there is a convergent sequence u_n with $g(u_n)$ unbounded from above. For some t and infinite $\mathbb{M} \subseteq \omega$ we have $\{t + u_m : m \in \mathbb{M}\} \subseteq T$. As $\{t + u_m : m \in \mathbb{M}\}$ is convergent there are a real z, some $k \in \omega$ and an infinite $\mathbb{M}' \subseteq \mathbb{M}$ such that

$$\{z + t + u_m : m \in \mathbb{M}'\} \subseteq H^k = \{x : |f(x)| < k\}.$$

Hence, for $m \in \mathbb{M}'$, we have

$$g(t + u_m) \leq f(t + u_m) \text{ and } |f(z + t + u_m)| < k.$$

Thus, for $m \in \mathbb{M}'$, since $t + u_m \in T$, we have

$$\begin{aligned} g(u_m) &= g(t + u_m) - g(t) \leq f(t + u_m) - g(t) \\ &\leq f(z + t + u_m) + f(-z) - g(t) \\ &\leq k + f(-z) - g(t), \end{aligned}$$

as $z + t + u_m \in H^k$, and so $g(u_m)$ is bounded, a contradiction. □

We now recall a Lemma due to Berz.

LEMMA 5. ***Berz's Lemma.*** *(***[Berz]***) For $f : \mathbb{R}^N \to \mathbb{R}$ sublinear*

$$f(x) = \sup\{g(x) | g : \mathbb{R}^N \to \mathbb{R} \text{ additive and } g \leq f\}.$$

PROOF. [**Kucz**] p. 414. □

THEOREM 4. ***Berz's Characterization Theorem.*** *(***[Berz]***) For sublinear $f : \mathbb{R}^N \to \mathbb{R}$ in* ***WNT****:*

$$f(\mathbf{x}) = ||\mathbf{x}||f(v(\mathbf{x})) = ||\mathbf{x}||F(v(\mathbf{x})), \text{ for } \mathbf{x} \neq 0, \tag{3.1}$$

where F is the limit function defined by (2.1), and so f is positively homogeneous, so $f(0) = 0$, and continuous.

PROOF. Fix $\mathbf{x} \neq \mathbf{0}$. For any additive $h \leq f$, define $g_h(t) = h(t\mathbf{x})$. Then, $g_h : \mathbb{R} \to \mathbb{R}$ is additive.
Let $\varepsilon > 0$. By the First Limit Theorem we have for all $t > 0$

$$F(v(\mathbf{x})) \leq \frac{f(t\mathbf{x})}{t}, \text{ i.e. } F(v(\mathbf{x}))t \leq f(t\mathbf{x}), \tag{3.2}$$

and also for some $t_0 > 0$ we have, for $t \geq t_0$, that

$$F(v(\mathbf{x})) \leq \frac{f(t\mathbf{x})}{t} \leq F(v(\mathbf{x})) + \varepsilon. \tag{3.3}$$

Thus, for $t \geq t_0$, we have

$$g_h(t) = h(t\mathbf{x}) \leq f(t\mathbf{x}) \leq t(F(v(\mathbf{x})) + \varepsilon).$$

Thus the additive function g_h is locally bounded above far enough to the right by a linear form, and so, by the Subadditive Majorant Lemma, is continuous. Write $g_h(t) = c_h(\mathbf{x})t$. Thus we now have, for all t, that

$$c_h(\mathbf{x})t = g_h(t) = h(t\mathbf{x}) \leq f(t\mathbf{x}).$$

It follows from (3.3) that $c_h(\mathbf{x}) \leq F(v(\mathbf{x}))$. Hence, by the Berz Lemma, we have for fixed $t \geq 0$ that

$$\begin{aligned} f(t\mathbf{x}) &= \sup\{c_h(\mathbf{x})t | c_h(\mathbf{x})s \ \leq f(s\mathbf{x}) \text{ all } s\} \\ &= t\sup\{c_h(\mathbf{x}) | c_h(\mathbf{x})s \ \leq f(s\mathbf{x}) \text{ all } s\} \\ &\leq tF(v(\mathbf{x})). \end{aligned}$$

From here and (3.2) we have, as asserted, that

$$f(t\mathbf{x}) = tF(v(\mathbf{x})) \text{ for } t \geq 0. \tag{3.4}$$

For $\mathbf{x}$ with $\mathbf{x} = v(\mathbf{x})$ we obtain $F(v(\mathbf{x})) = f(v(\mathbf{x}))$, and so (3.4) implies

$$f(\mathbf{x}) = f(||x||v(x)) = ||x||F(v(\mathbf{x})) \text{ for } t \geq 0,$$

whence (3.1) and positive homogeneity. Since $v(\mathbf{x})$ is continuous for $\mathbf{x} \neq 0$, and since $F(.)$ is continuous, so is $f(\mathbf{x})$ for all $\mathbf{x}$. □

As a corollary, we have again an equivalence result.

THEOREM 5. ***Equivalence Theorem.*** *For f sublinear, f is continuous iff $f \in$* ***WNT.***

PROOF. Immediate from Berz's Theorem, since continuous functions are in **WNT**. □

REMARK 3. *In the sublinear case, even if the definition does not require so, $f(0) = 0$, by Berz's Theorem. By contrast, a general subadditive function f in* ***WNT*** *may satisfy $f(0) = 0$, be locally bounded and yet not continuous. Proposition 2 puts this in perspective. It is the case however, that such a function f is continuous iff it is continuous at the origin. (See* [**Kucz**] *Th. 1 p. 404, or* [**HP**] *p. 247.)*

4. Differentiability, Lipschitz condition, uniform convergence

Several fundamental theorems on subadditive functions concerning differentiability (see e.g. [**Kucz**] Ch. XVI or [**HP**] Ch. VII for a review of these) remain valid when measurability is replaced by membership of the class **WNT**. We note some examples in this section and sketch the proofs where these differ in a significant detail from the classical setting; we refer to results and ideas of the last section. We begin by extending the notation of the Subadditive Limit Theorem, to take in the following quantities with which we are concerned in this section, namely

$$\beta_-(\mathbf{x}) = \inf_{t<0} \frac{f(t\mathbf{x})}{t}, \qquad \alpha_+ = \sup_{t>0} \frac{f(t\mathbf{x})}{t},$$

where

$$-\infty \leq \beta_-(\mathbf{x}) < \infty \text{ and } -\infty < \alpha_+(\mathbf{x}) \leq \infty.$$

THEOREM 6. ***Theorem (Second Limit Theorem – at Zero).*** *Let* $f : \mathbb{R}^N \to \mathbb{R}$ *in* ***WNT*** *be subadditive. Then*

(1) *The following inequality holds:*

$$\beta_-(\mathbf{x}) \leq -\beta_-(-\mathbf{x}) = \sup_{t>0} \frac{f(t\mathbf{x})}{t} = \alpha_+(\mathbf{x}),$$

in which the left-hand side may be $-\infty$ *and the right* $+\infty$.

(2) *If* $\beta_-(\mathbf{x})$ *is finite, then*

$$\beta_-(\mathbf{x}) = G(\mathbf{x}) := \lim_{t\to 0-} \frac{f(t\mathbf{x})}{t}. \tag{4.1}$$

Similarly, if $-\beta_-(-\mathbf{x})$ *is finite, then*

$$-\beta_-(-\mathbf{x}) = -G(-\mathbf{x}) = \lim_{t\to 0+} \frac{f(t\mathbf{x})}{t}.$$

(3) *The equation (4.1) is also valid under either of the hypotheses*

$$\lim_{x\to 0} f(\mathbf{x}) = 0, \text{ or } \liminf_{x\to 0} f(\mathbf{x}) > 0.$$

(4) *If* $G(\mathbf{x})$ *is well-defined for all* $\mathbf{x}$ *(i.e.* $\beta_-(\mathbf{x})$ *is finite for all* $\mathbf{x}$*), then* $G(\mathbf{x})$ *is positively homogenous and subadditive, hence convex and continuous.*

PROOF. The proof of (i)-(iii) depends on Proposition 2 that $\lambda = \liminf_{x\to 0} f(x) \geq 0$ and may be taken verbatim from [**Kucz**] p. 410. Now note, as in the Subadditive Limit Theorem, that $\beta_-(\mathbf{x})$ is positively homogeneous, so (iv) follows from (ii) and a calculation of subadditivity similar to that in (2.2). □

DEFINITION 3. *The four Dini derivatives (upper-right, upper-left etc.) in direction* $\mathbf{h}$ *are as follows:*

$$D^+_{\mathbf{h}} f(\mathbf{x}) = \limsup_{t\to 0+} \frac{f(\mathbf{x}+t\mathbf{h}) - f(\mathbf{x})}{t}, \qquad D^-_{\mathbf{h}} f(\mathbf{x}) = \limsup_{t\to 0-} \frac{f(\mathbf{x}+t\mathbf{h}) - f(\mathbf{x})}{t},$$
$$d^+_{\mathbf{h}} f(\mathbf{x}) = \liminf_{t\to 0+} \frac{f(\mathbf{x}+t\mathbf{h}) - f(\mathbf{x})}{t}, \qquad d^-_{\mathbf{h}} f(\mathbf{x}) = \liminf_{t\to 0-} \frac{f(\mathbf{x}+t\mathbf{h}) - f(\mathbf{x})}{t}.$$

The notation here, adapted from Kuczma [**Kucz**], is more convenient for identifying directions than that in [**HP**].

THEOREM 7. *(**Theorem on Dini derivative bounds,*** [**HP**] *p. 251). Let* $f : \mathbb{R}^N \to \mathbb{R}$ *in* ***WNT*** *be subadditive. Then*

(1) *The Dini derivatives are bounded as follows:*

$$D^+_{\mathbf{h}} f(\mathbf{x}) \leq -\beta_-(-\mathbf{h}), \qquad D^-_{\mathbf{h}} f(\mathbf{x}) \leq -\beta_-(-\mathbf{h}),$$
$$\beta_-(\mathbf{h}) \leq d^+_{\mathbf{h}} f(\mathbf{x}), \qquad \beta_-(\mathbf{h}) \leq d^-_{\mathbf{h}} f(\mathbf{x}).$$

(2) *If* $\beta_-(\mathbf{x}) = -\beta_-(-\mathbf{x})$, *then*

$$f(t\mathbf{x}) = \beta_-(\mathbf{x})t.$$

PROOF. (i) Only the finite-valued versions require checking. All four cases require an identical approach, so we do just the first. Evidently,

$$f(\mathbf{x}+t\mathbf{h}) \leq f(\mathbf{x}) + f(t\mathbf{h}),$$

so for $t > 0$ we have

$$\frac{f(\mathbf{x}+t\mathbf{h}) - f(\mathbf{x})}{t} \leq \frac{f(t\mathbf{h})}{t}.$$

But if $\beta(-\mathbf{x}) \neq -\infty$, then $-\beta(-\mathbf{x})$ is finite, so by the last theorem we have

$$D^+_{\mathbf{h}} f(\mathbf{x}) = \limsup_{t\to 0+} \frac{f(\mathbf{x}+t\mathbf{h}) - f(\mathbf{x})}{t} \leq \lim_{t\to 0+} \frac{f(t\mathbf{h})}{t} = -\beta_-(-\mathbf{h}). \tag{4.2}$$

(ii) If $\beta(\mathbf{x}) = -\beta(-\mathbf{x})$, then both these quantities are finite. By positive homogeneity, $\beta(t\mathbf{x}) = t\beta(\mathbf{x})$ for *all* t. Hence f restricted to the linear span of $\mathbf{x}$ is differentiable and

$$\frac{d}{dt} f(t\mathbf{x})|_{t=0} = \beta_-(\mathbf{x}).$$

So, for some constant c, we have

$$f(s\mathbf{x}) = s\beta_-(\mathbf{x}) + c.$$

But, if $c \neq 0$, suppose w.l.o.g. that $c > 0$, then we have the contradiction

$$\beta_-(\mathbf{x}) = \lim_{t\to 0-} \frac{f(t\mathbf{x})}{t} = \lim_{t\to 0-} \left(\beta_-(\mathbf{x}) + \frac{c}{t}\right) = -\infty.$$

□

THEOREM 8. ***(The Lipschitz condition).*** *Let $f : \mathbb{R}^N \to \mathbb{R}$ in **WNT** be subadditive. Suppose also that $\beta_-(\mathbf{x})$ is finite for all $\mathbf{x}$. Then f is a Lipschitz function with constant L provided*

$$L > \sup\{|G(\mathbf{h})| : ||\mathbf{h}|| = 1\}, \tag{4.3}$$

with G as in Theorem 6.

PROOF. In view of Second Limit Theorem, part (iv), under the current circumstances G is a continuous function. Fix a direction $\mathbf{h}$ and a number L satisfying the condition (4.3). Let $\varepsilon = L - \sup\{|G(\mathbf{h})| : ||\mathbf{h}|| = 1\}$. Then $\varepsilon > 0$. Let $V = span\{\mathbf{h}\}$. We now follows [**Kucz**] p. 413. As in (4.2), given $\mathbf{x} \in V$, we have, for some $\delta = \delta(\mathbf{x}) > 0$, that

$$\beta_-(\mathbf{h}) - \varepsilon \leq \frac{f(\mathbf{x} + t\mathbf{h}) - f(\mathbf{x})}{t} \leq -\beta_-(-\mathbf{h}) + \varepsilon,$$

whenever $0 < |t| < \delta(\mathbf{x})$. That is,

$$\left|\frac{f(\mathbf{x} + t\mathbf{h}) - f(\mathbf{x})}{t}\right| \leq \max\{|\beta_-(\mathbf{h})|, |\beta_-(-\mathbf{h})|\} + \varepsilon \leq L.$$

Let $\mathbf{y} \in V$. Appealing to the compactness of the line segment from $\mathbf{x}$ to $\mathbf{y}$, we see that there is a finite sequence $\mathbf{x}_i, \mathbf{y}_i$ such that

$$\mathbf{x} = \mathbf{x}_0 < y_0 < \mathbf{x}_1 < ... < y_{n-1} < \mathbf{x}_n = \mathbf{y},$$

with $\mathbf{y}_i \in (\mathbf{x}_i - \delta(\mathbf{x}_i), \mathbf{x}_i + \delta(\mathbf{x}_i))$ and the segments $(\mathbf{x}_i - \delta(\mathbf{x}_i), \mathbf{x}_i + \delta(\mathbf{x}_i))$ covering the line segment from $\mathbf{x}$ to $\mathbf{y}$. Hence

$$\begin{aligned} |f(\mathbf{y}) - f(\mathbf{x})| &\leq \sum_{i=0}^{n-1} |f(\mathbf{x}_{i+1}) - f(\mathbf{y}_i)| + \sum_{i=0}^{n-1} |f(\mathbf{x}_{i+1}) - f(\mathbf{y}_i)| \\ &\leq L\sum_{i=0}^{n-1} |\mathbf{x}_{i+1} - \mathbf{y}_i| + L\sum_{i=0}^{n-1} |\mathbf{x}_{i+1} - \mathbf{y}_i| \\ &\leq L|\mathbf{y} - \mathbf{x}|. \end{aligned}$$

Thus f has Lipschitz constant L on any line segment from $\mathbf{x}$ to $\mathbf{y}$ lying in V. This establishes the theorem, since $\mathbf{h}$ was arbitrary. □

The previous result has the following corollary.

Corollary 4. *Let $f : \mathbb{R}^N \to \mathbb{R}$ in **WNT** be subadditive. Suppose also that $\beta_-(\mathbf{x})$ is finite for all $\mathbf{x}$. Then f is absolutely continuous.*

Our next result is inspired by the fundamental result of regular variation, the uniform convergence theorem ([**BGT**] Section 1.2 and [**BOst1**]).

Theorem 9. *Let $f : \mathbb{R}^N \to \mathbb{R}$ in **WNT** be subadditive. Suppose also that $\beta_-(\mathbf{x})$ is finite for all $\mathbf{x}$. Then the convergence*

$$F(\mathbf{x}) := \lim_{s\to+\infty} \frac{f(s\mathbf{x})}{s}, \text{ for } \mathbf{x} \in \mathbb{R}^N,$$

is uniform on compacts.

Proof. Recall that, for all $t > 0$, we have

$$F(\mathbf{x}) \leq \frac{f(t\mathbf{x})}{t},$$

and, according to the First Limit Theorem, for each $\varepsilon > 0$, there is $s = s(\mathbf{x})$ such that, for $s > s(\mathbf{x})$,

$$F(\mathbf{x}) \leq \frac{f(s\mathbf{x})}{s} \leq F(\mathbf{x}) + \varepsilon.$$

Suppose the Proposition is false. Then, for some $\varepsilon > 0$, there are $s_n > n$ and $\mathbf{x}_n$, with $||\mathbf{x}_n|| = 1$, such that

$$\frac{f(s_n\mathbf{x}_n)}{s_n} > F(\mathbf{x}_n) + 3\varepsilon.$$

Since $||\mathbf{x}_n|| = 1$, we may as well assume that $\mathbf{x}_n \to \mathbf{x}_0$. By the continuity of F at $\mathbf{x}_0$, we may and will restrict n to be so large that $F(\mathbf{x}_n) > F(\mathbf{x}_0) - \varepsilon$. Thus, for such n, we have

$$\frac{f(s_n\mathbf{x}_n)}{s_n} > F(\mathbf{x}_0) + 2\varepsilon. \tag{4.4}$$

According to the limit theorem, we may choose t_0 so that for $t > t_0$

$$F(\mathbf{x}_0) \leq \frac{f(t\mathbf{x}_0)}{t} \leq F(\mathbf{x}_0) + \varepsilon.$$

By the last Theorem, for any fixed L satisfying (4.3), we may appeal to the Lipschitz condition to choose N so large that, for $n > N$,

$$\left|\frac{f(s_n\mathbf{x}_n) - f(s_n\mathbf{x}_0)}{s_n}\right| \leq L|\mathbf{x}_n - \mathbf{x}_0| \leq \varepsilon.$$

From here we deduce that, for all large enough n,

$$\frac{f(s_n\mathbf{x}_n)}{s_n} \leq \frac{f(s_n\mathbf{x}_0)}{s_n} + \varepsilon \leq F(\mathbf{x}_0) + 2\varepsilon,$$

and this contradicts (4.4). □

Example 2. *As an example, note that, for f not only subadditive but also sublinear, Berz's Theorem yields an identity*

$$F(\mathbf{x}) = \frac{f(t\mathbf{x})}{t}, \text{ for } t > 0,$$

and so here convergence to the limit F is trivially uniform on compact sets.

References

[BD] F. Bernstein and G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Ann. 76 (1915), 514-526.

[Berz] E. Berz, *Sublinear functions on* $\mathbb{R}$, Aeq. Math. 12 (1975), 200-206.

[BG1] N. H. Bingham, C.M. Goldie, *Extensions of regular variation, I: Uniformity and quantifiers*, Proc. London Math. Soc., (3) 44 (1982), 473-496.

[BGT] N. H. Bingham, C.M. Goldie, J.L. Teugels, *Regular variation,* Encycl. Math. Appl. 27, Cambridge University Press, Cambridge, 1987.

[BOst1] N. H. Bingham and A. J. Ostaszewski, *Foundations of Regular Variation*, CDAM Research Report Series, LSE-CDAM-2006-22.

[BOst3] N. H. Bingham and A. J. Ostaszewski, *Beyond the Theorems of Steinhaus and Ostrowski: Combinatorial Versions*, LSE-CDAM Report, LSE-CDAM-2007-15.

[BOst4] N. H. Bingham and A. J. Ostaszewski, *Beyond Lebesgue and Baire: generic regular variation*, LSE-CDAM Report, LSE-CDAM-2007-16.

[CsEr] I. Csiszár and P. Erdös, *On the function* $g(t) = \limsup_{x\to\infty}(f(x+t) - f(x))$, Magyar Tud. Akad. Kut. Int. Közl. A 9 (1964), 603-606.

[Dal] D. J. Daley, *Upper bounds for the renewal function via Fourier methods*, Annals of Probability 6 (1987), 876-884.

[Fell] W. Feller, *An introduction to probability theory and its applications*, Volume II, 2nd ed., Wiley, 1971.

[GerKucz] R. Ger and M. Kuczma, *On the boundedness and continuity of convex functions and additive functions,* Aeq. Math. 4 (1970), 157-162.

[HLP] G. H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities.* Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

[HP] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*,. rev. ed. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957.

[Kech] A. S. Kechris, *Classical descriptive set theory*, Graduate texts in Mathematics 156, Springer, 1995.

[King] J. F. C. Kingman, *Subadditive processes*, École d'Été de Probabilités de Saint-Flour, V–1975, pp. 167–223. Lecture Notes in Math., Vol. 539, Springer, Berlin, 1976.

[Kucz] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's functional equation and Jensen's inequality,* PWN, Warsaw, 1985.

[Lig] T. M. Liggett, *Interacting particle systems*, Springer 1985/2005.

[MatŚw] J. Matkowski and T. Świątkowski, *On subadditive functions*, Proc. Amer. Math. Soc., 119 (1993), 187-197.

[Rock] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.

[Ros] R. A. Rosenbaum, *Sub-additive functions*, Duke Math. J. 17, (1950), 227–247.

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