New automatic properties: subadditivity, convexity, uniformity.

N. H. Bingham
Mathematics Department, Imperial College London,
South Kensington, London SW7 2AZ
n.bingham@ic.ac.uk

A. J. Ostaszewski
Mathematics Department, London School of Economics,
Houghton Street, London WC2A 2AE
a.j.ostaszewski@lse.ac.uk

October 2007, BOstCDAM6-2.tex

In memoriam Marek Kuczma (1935-1991)

CDAM Research Report LSE-CDAM-2007-23

Abstract

We examine various related instances of automatic properties of functions – that is, cases where a weaker property necessarily implies a stronger one under suitable side-conditions, e.g. connecting geometric and combinatorial features of their domains. The side-conditions offer a common approach to (mid-point) convex, subadditive and regularly varying functions (the latter by way of the uniform convergence theorem). We examine generic properties of the domain sets in the side-conditions - properties that hold typically, or off a small exceptional set. The genericity aspects develop earlier work of Kestelman [Kes] and of Borwein and Ditor [BoDi]. The paper includes proofs of three new analytic automaticity theorems announced in [BOst7].

Classification: 26A03

Keywords: Kestelman universal set, affine compactness, automatic continuity, convex function, subadditive function, additive function, regular variation, uniform convergence theorem, analytic set.

1 Introduction

The term 'automatic continuity', whose first occurrence (according to MathSciNet) is in a paper by Brooks of 1967 [Broo] (see MR0216291, 35 #7126), and most prominent in the theory of Banach algebras, is used conventionally to describe theorems identifying 'general circumstances' when group homomorphisms are continuous ([Dal1], [Dal2], [THJ]). In the context of real analysis, the analogous broader term 'automatic property' may be specifically defined. Given a set $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$, i.e. a class of functions $f : \mathbb{R} \to \mathbb{R}$ – e.g. additive functions, those with $f(x+y) = f(x) + f(y)$ – and properties (regarded as sets) of functions $\mathcal{P}$,$\mathcal{Q} \subseteq \mathbb{R}^{\mathbb{R}}$, call $\mathcal{P}$ strong and $\mathcal{Q}$ weak if all functions f in $\mathcal{F}$ with property $\mathcal{P}$ have $\mathcal{Q}$. The strong property $\mathcal{P}$ is then described as holding automatically if it is implied by the weak property $\mathcal{Q}$ holding on a *test set* $T \subset \mathbb{R}$ with some side-condition governing T. The first classic example is Darboux's theorem of 1875 ([Dar], [AD] Section 21.6) asserting that, for additive functions, boundedness on an interval T implies continuity (so that implicitly the weak property $\mathcal{Q}$ is 'local boundedness'). The second is the Bernstein-Doetsch theorem of 1915 [BeDoe], that a (midpoint) convex function, defined on $\mathbb{R}^d$ and bounded above on a non-empty open set T, is continuous.

Ostrowski's result of 1929 [Ostr], that a convex, so *a fortiori* an additive, function bounded above on some test set T of positive measure is continuous, may be regarded as weakening the Darboux, or the Bernstein-Doetsch, side condition in a first major step towards *thinning out* the assumed weak property. However, despite the break-through, this thinning out may be regarded as slight from the perspective of Littlewood's First Principle (that sets of positive measure are almost open sets, cf. [Lit] Ch. 4 and [Roy] Section 3.6 p.72).

It was not until 1942, in course of studying measure and descriptive set-theoretic properties of Hamel bases (defined by Hamel in [Ham] in 1905), that Jones [Jones] discovered that an additive function continuous on a set T, with T both *analytic* and containing a Hamel basis, is continuous on $\mathbb{R}$ (for definition and background on analytic sets see [Rog]). For convenience, let us say briefly that T is a *spanning set*, when $\mathbb{R}$ regarded as a vector space over $\mathbb{Q}$ has T as a spanning set of vectors. (In the presence of the Axiom of Choice a spanning set contains a Hamel basis.) Since an additive function is fully specified by its values on a basis, Burton Jones in his formulation concentrated on the character of the test set T and on spanning as the natural side-condition on T; his theorem thus required continuity on a spanning, analytic set T.

The Bernstein-Doetsch result (in $\mathbb{R}^d$) was taken through its first step towards a thinning out by Császár in 1958 [Csa], who showed that it is enough for the test set T to have positive measure, or to be non-meagre (see Proposition 3 below).

The matter of alternative side-conditions and also the convex function case lay dormant until the initiation in 1970 of the Ger-Kuczma programme for the study of automatic continuity [GerKucz], almost a response to Barry Johnson's ground-breaking result on automatic continuity in Banach algebras [Jo] of 1969. As one of the programme's important contributors it fell to Z. Kominek [KomZ] in 1981 to make explicit the generalization of Ostrowski's theorem in the ultimate thinned-out form, albeit for the additive function case: such a function is continuous if it bounded above on any analytic spanning set T (in particular, on one of measure zero).

Though the proofs of Jones's and Kominek's theorem are similar in spirit (both based on Steinhaus's theorem), their inter-relation remained unclarified. Our recent reappraisal of the foundations of regular variation (see e.g. [BOst4]) offers two new perspectives on the Jones-Kominek theorem. We showed in [BOst7] that Kominek's theorem implies Jones's Theorem, on the grounds that an analytic spanning set contains a compact spanning set. (The implication follows since a function continuous on a compact set is bounded thereon; in fact continuity on 'additively compact' test sets, as defined below, also implies boundedness, for which see [BOst8].) The other is that a related but distinctive proof exists of a stronger theorem (the analytic automaticity theorem, [BOst7]) containing as special cases both Jones' and Kominek's result. The theorem concentrates on the additive combinatorics at the heart of the side-conditions, as exhibited by those convergent sequences on which the behaviour of a function from a class $\mathcal{F}$ may be regarded as 'good'.

The current paper re-opens the Ger-Kuczma programme by proposing a broadened scope to include new classes of functions $\mathcal{F}$, new properties (on both the $\mathcal{P}$ and $\mathcal{Q}$ fronts), and a fresh approach to side-conditions. On the function class front $\mathcal{F}$, our contribution is concerned with the subadditive functions and the slowly varying functions and with a review of convex functions, given our sequential approach to side-conditions. This includes automatic analyticity theorems for the subadditive and for the convex functions, though we require additionally that some translate of the spanning test set T is symmetric (about the origin). On the properties front, we study the case $\mathcal{P} = \mathcal{Q}$ described by the uniform convergence theorem UCT of regular variation theory and obtain a corresponding analytic automaticity theorem. Altogether there are thus three, new, analytic automaticity theorems.

On the side-condition front, we pay considerable attention to the role of sets T such that for any null sequences $\{z_n\} \to 0$, some translated subsequence $\{t + z_m : m \in \mathbb{M}\}$ lies in T for an infinite $\mathbb{M}$. When $\mathbb{M}$ is co-finite Kestelman [Kes] terms such sets T *universal*. For general infinite $\mathbb{M}$ we call this property of T *subuniversal* – see [BOst8] for the alternative topological terminology *additive compactness*.

We make explicit additional geometric (affine) features for use as side-conditions in the derivation of automatic properties of convex functions. Such 'affine compactness' features are implicit in Császár's cited work as well as in Kominek [KomZ]. It follows from the theorems of Kestelman and of Borwein and Ditor that for measurable and Baire sets these features are generic. In a companion paper [BOst9] we show that these features are in fact generic in an even stronger sense.

2 Side-conditions

Given a convergent sequence of real numbers $\{u_n\} \to u$, or a null sequence, i.e. converging to zero – identified as such for clarity as $\{z_n\} \to 0$ with the convention that $u_n = u + z_n$ – we will be concerned with inclusion, in specified ways, of its images in certain sets. By 'image' we will mean the transform of some subsequence, the transform often being a simple translation (shift). Sometimes, however, some scaling may be applied to the subsequence or the shift (creating a 'similarity': see e.g. Miller [Mil]). Image-inclusion (regarded as embedding properties in the more general setting of [BOst11]) will assist weak properties of functions in entailing their strong properties 'automatically' (see Section 3). Much of this is readily generalizable to the Euclidean context. A further step is to replace the null sequences z_n of reals by null sequences of functions $z_n(.)$ subject to some local regularity restrictions (for which see [BOst9]).

For the simple kind of inclusion, we begin by recalling some combinatorial definitions from [BOst3] which have been used to unify measure/category dualities (for which see [Oxt]) in the theory of regular variation. We follow the set-theorists and denote the set of natural numbers by $\omega = \{0, 1, 2, ...\}$.

Definition 1 (The Kestelman universal and subuniversal class). Let $T \subseteq \mathbb{R}$.

(i) We call T *universal,* and write $T \in \mathfrak{K}$ (Gothic 'K' for Kestelman), if for any null sequence $\{z_n\} \to 0$ in $\mathbb{R}$ there is a co-finite $\mathbb{M} \subseteq \omega$, and $t \in \mathbb{R}$ such that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq T. \tag{1}$$

(ii) If (1) holds with $\mathbb{M}$ arbitrary but infinite, we call T *subuniversal* and write $T \in \mathfrak{S}$ (Gothic 'S' for 'subsequence'). Clearly

$$\mathfrak{K} \subseteq \mathfrak{S}.$$

(iii) We will say that T is *generically subuniversal,* and write $T \in \mathfrak{S}_{\text{gen}}$, if for any null sequence of real numbers $\{z_n\} \to 0$ there are $t \in T$ and an infinite set $\mathbb{M}_t$ such that

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

The distinction here is that the translator is now required to be in T which implies that it is a limit point of T. This observation forms the basis for a useful connection with (sequential) compactness [BOst8]. Evidently, the closed members of $\mathfrak{S}$ form a subclass, $\overline{\mathfrak{S}}$ say, of $\mathfrak{S}_{\text{gen}}$.

For α non-zero, note that $T \in \mathfrak{S}$ iff $\alpha T \in \mathfrak{S}$, so $\mathfrak{S} = \alpha\mathfrak{S}$. Similarly, $T \in \mathfrak{S}_{\text{gen}}$ iff $\alpha T \in \mathfrak{S}_{\text{gen}}$, so $\mathfrak{S}_{\text{gen}} = \alpha\mathfrak{S}_{\text{gen}}$.

The property of T in definition (iii) is typical in the sense captured by the following theorem, due in the measure case in this form to Borwein and Ditor [BoDi], but already known much earlier albeit in somewhat weaker form by Kestelman ([Kes] Th. 3), and rediscovered by Trautner [Trau]. We will need a definition.

Definition 2 (Genericity). Suppose Γ is $\mathcal{L}$ or $\mathcal{B}a$, the class of measurable sets or Baire sets (i.e. sets with the Baire property). We will say that $P \in \Gamma$ holds *for generically all* t if $\{t : t \notin P\}$ is null/meagre according as Γ is $\mathcal{L}$ or $\mathcal{B}a$.

As an indication of the use we shall make of these definitions we record here:

Theorem (Kestelman-Borwein-Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null/Baire and non-meagre, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

In particular, $T \in \mathfrak{S}_{\text{gen}}$*. Furthermore, for any density point* u *of* T*, there is* $t \in T$ *arbitrarily close to* u *for which the above holds.*

A stronger form still is derived in [BOst9] (the Generic Reflection Theorem); see also [BOst3] Section 2.1 Note 3, [BOst4] Section 2.1 Note 1. For proofs see the original papers [Kes] and [BoDi]; for a unified treatment see [BOst9].

Notes.

1. Write $S \in \mathfrak{S}^k$ if S has the property that its k-fold sum satisfies $S + ... + S \in \mathfrak{S}$. Note that $kS \subseteq S + ... + S$; hence $S \in \mathfrak{S} = \frac{1}{k}\mathfrak{S}$ implies $S \in \mathfrak{S}^k$, i.e. $\mathfrak{S} \subseteq \mathfrak{S}^k$. We will see later that the classes $\mathfrak{S}^k$ broaden the scope of applicability of our theorems by permitting a thinning out of the test sets to null sets.

2. The situation of Note 1 is similar to that in probability theory, where one distinguishes between the lattice case and the density case – for example, in renewal theory or in local (central) limit theory. The obvious condition is that the probability law be absolutely continuous, but one can weaken this to the condition (called spread-out in English, étalé in French) that some convolution power have an absolutely continuous component. See e.g. [IbLi], Th. 4.4.1.

3. A wider class may be sought by requiring weak properties to hold on a finite family of sets $S_1, ...S_k$ such that a generalized sum of the form $\alpha_1 S_1 + \alpha_2 S_2 + .. + \alpha_k S_k$ lies in $\mathfrak{S}$. For results on the ability of such classes to contain an interval see [CGM] and note the remarkable example of a measurable set S such that $S + S$ is null but $S - S = [-1, 1]$.

4. In similar vein, the Cantor set C is null; but $C + C = [0, 2]$ (e.g. [Fal] p. 108, [Kucz] p. 50). For, C is the subset of reals in $[0, 1]$ containing only 0 and 2 in their ternary expansions, and $\{0, 2\}$ generates $\mathbb{Z}_3$. Thus $C + C \in \mathfrak{K}$, i.e. is universal, and so $C \in \mathfrak{S}^2$.

5. Observe that, in the conditions of the theorem, it is enough that our set, S say, should contain some measurable non-null set, T say, (i.e. that S should have positive inner measure) – S itself need not be measurable, and similarly in the Baire case; one can specialize to the result stated above by shrinking S to this measurable/Baire subset T. We retain the formulation above for convenience in expressing the conclusion: 'for generically all $t \in T$'.

6. See [MilH] for another generalization of the (Kestelman-) Borwein-Ditor Theorem.

7. The theorem implies the Strong No Trumps Theorem of [BOst4]. See also [BOst10].

While subuniversality is the key combinatorial concept for this paper, we need to rephrase it geometrically to suit the needs of various arguments which are geometric in nature. This is done below.

Averaging Lemma. *A set T is subuniversal iff it is 'averaging', that is, for any null sequence $\{z_n\} \to 0$, any given point $u \in T$, and with $u_n := u + z_n$ (thus an arbitrary convergent sequence, but with limit in T), there are $w \in \mathbb{R}$ (an averaging translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have:*

$$u_n = \frac{1}{2}w + \frac{1}{2}v_n.$$

Equivalently, there are $w \in \mathbb{R}$ (a reflecting translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have:

$$v_n = \frac{1}{2}w + \frac{1}{2}u_n.$$

Proof. In the averaging case, it is enough to show that $\frac{1}{2}T$ is subuniversal iff T is averaging. If $\frac{1}{2}T$ is subuniversal then, given $u_n \to u$, there are $w \in \mathbb{R}$ and some infinite $\mathbb{M}$ so that $\{-\frac{1}{2}w + u_n : n \in \mathbb{M}\} \subseteq \frac{1}{2}T$; hence, putting $v_n := 2u_n - w$, we have $\{v_n : n \in \mathbb{M}\} \subseteq T$. Conversely, if T is averaging and $\{z_n\} \to 0$, then for some x and some $\mathbb{M}$, $\{2x + 2z_n : n \in \mathbb{M}\} \subseteq T$, so $\{x + z_n : n \in \mathbb{M}\} \subseteq \frac{1}{2}T$ and hence $\frac{1}{2}T$ is subuniversal. Similar reasoning yields the reflecting case. $\square$

The averaging notion appears implicitly in [KomZ] and the reflecting notion in [Csa]. The latter name is suggested by the reflecting property:

$$(v_n - u) \approx (v_n - u_n) = -(v_n - w).$$

The Lemma may thus be summarized symbolically:

$$\mathfrak{S}_{\text{ref}} = \frac{1}{2}\mathfrak{S} = \mathfrak{S} = 2\mathfrak{S} = \mathfrak{S}_{\text{av}}.$$

A further definition, more general still, will be needed later.

Definition 3. Say that a set S is *strongly averaging* if some $T \subseteq S$ (henceforth a *strong core* of S) has the following property:

For any null $\{z_n\} \to 0$, and any bounded sequence $\{\gamma_n\}$ and any $\{u_n = u + z_n\} \to u \in T$, there are $w \in \mathbb{R}$ (a translator) and an increasing sequence $m(n)$ such that, for $n \in \omega$, we have

$$v_n := \gamma_n w + (1 - \gamma_n) u_{m(n)} \in T.$$

Equivalently, for any null $\{z_n\} \to 0$, and any sequence $\{\alpha_n\}$ bounded away from zero, there are points $t_n \in T$ and an increasing sequence $m(n)$ such that for $n \in \omega$ the following affine combinations are constant, i.e.

$$\alpha_n t_n + (1 - \alpha_n) u_{m(n)} := w \text{ for some } w \in \mathbb{R}.$$

Thus $\alpha_n \equiv -1$ and $\alpha_n \equiv 2$ (or $\gamma_n \equiv \frac{1}{2}$) yield respectively the reflecting and the averaging case.

The following result is implicit in [Csa]; for the proof of a more general result see [BOst9].

Theorem (Császár's Genericity Theorem, [Csa], or [Kucz] p 223**).** *If S is Baire, non-meagre/measurable, non-null, then S is strongly averaging. In a neighbourhood of the relevant limit point, generically all $t \in S$ are translators.*

In our final definition, the term *affinely compact* is used by close analogy with the term *additively compact* of [BOst8] (a synonym for 'generically subuniversal'). This notion is generalized in a functional setting in [BOst9].

Definition 4 (**Affine compactness**). For $T \subseteq \mathbb{R}$, we say that T is *affinely compact* to mean that, for any $\{z_n\} \to 0$ and any non-zero α, there are $u \in T$ and some infinite $\mathbb{M}_u$ so that, for all $n \in \mathbb{M}_u$,

$$(1 - \alpha)u + \alpha u_n \in T, \text{ for all } n \in \mathbb{M}_u, \text{ with } u_n = u + z_n.$$

Proposition 1. *For $T \in \Gamma$, T is generically subuniversal, $T \in \mathfrak{S}_{\text{gen}}$, iff T is affinely compact generically, i.e. for any $\{z_n\} \to 0$ and any α with $\alpha \neq 0$, for generically all $u \in T$, there is some infinite $\mathbb{M}_u$ such that for all $m \in \mathbb{M}_u$ the affine combinations $v_n := (1 - \alpha)u + \alpha u_n$ satisfy*

$$(1 - \alpha)u + \alpha u_m \in T, \text{ where } u_m = u + z_m.$$

Proof. Supose $T \in \mathfrak{S}_{\text{gen}}$. Then for any $\{z_n\} \to 0$ and any α, we have, for generically all $u \in T$, that for some infinite $\mathbb{M}_u$

$$u_n = u + \alpha z_n = u + \alpha(u_n - u) \in T, \text{ for all } n \in \mathbb{M}_u, \text{ with } u_n = u + z_n.$$

That is, the affine combinations $(1-\alpha)u + \alpha u_n$ satisfy

$$(1-\alpha)u + \alpha u_n \in T, \text{ for all } n \in \mathbb{M}_u.$$

Conversely, let $\{z_n\} \to 0$ be given and let α be non-zero. Suppose that, for generically all $u \in T$, there is some infinite $\mathbb{M}_u$ such that for all $n \in \mathbb{M}_u$

$$u + \alpha(u_n - u) \in T, \text{ where } u_n = u + z_n/\alpha = u + \tilde{z}_n,$$

say. Then

$$u + z_n \in T.$$

That is, $T \in \mathfrak{S}_{\text{gen}}$. $\square$

3 Automatic properties: theory

We are concerned with *properties* of functions $h : T \to \mathbb{R}$, for $T \subseteq \mathbb{R}^d$, here interpreted as a family $\mathcal{Q}(T)$ – so that h has $\mathcal{Q}$ iff $h \in \mathcal{Q}$. The perspective on sequences guiding the preceding section motivates the first few definitions.

Definition 1. Let us say that $f : T \to \mathbb{R}$ is *extensibly continuous* or, more simply, *precompact* to mean that f is continuous on T and if $\{t_n\} \subseteq T$ is Cauchy then so is $\{f(t_n)\}$.

To motivate the terms, note that the definition may be rephrased equivalently to require that f has a continuous extension to the closure $\bar{T}$; in this case f carries compacts of $\bar{T}$ to compacts. (See [Bou] Section 8.5 Th. 1, and [Eng] Lemma 4.3.16.)

If T is closed the condition demands simply that f be continuous on T. However, regarding T as a test set (testing on a restriction for global continuity), one might not wish absence from T of a limit point to reduce information carried by T. This is why we prefer this stronger definition in automatic continuity work – compare [Kucz]. Of course weaker hypotheses yield stronger theorems, ceteris paribus.

Definition 2. Let us denote a convergent sequence with limit $\mathbf{x}_0$, by $\{\mathbf{x}_n\} \to \mathbf{x}_0$. We say the property $\mathcal{Q}$ of functions is *sequential on* T if

$$f \in \mathcal{Q} \text{ iff } (\forall\{\mathbf{x}_n : n > 0\} \subseteq T)[(\{\mathbf{x}_n\} \to \mathbf{x}_0) \implies f|\{\mathbf{x}_n : n > 0\} \in \mathcal{Q}(\{\mathbf{x}_n : n > 0\})]. \tag{2}$$

$\mathcal{Q}$ is *completely sequential on* T if

$$f \in \mathcal{Q} \text{ iff } (\forall\{\mathbf{x}_n\} \subseteq T)[(\{\mathbf{x}_n\} \to \mathbf{x}_0) \implies f|\{\mathbf{x}_n\} \in \mathcal{Q}(\{\mathbf{x}_n\})].$$

Since the former definition (2) does not require the limit point to be enumerated, sequential includes completely sequential.

Note that if $\mathcal{Q}$ is sequential then $f|\{\mathbf{x}_n\} \in \mathcal{Q}(\{\mathbf{x}_n\})$ iff $f|\{\mathbf{x}_n : n \in \mathbb{M}\} \in \mathcal{Q}(\{\mathbf{x}_n : n \in \mathbb{M}\})$, for every infinite $\mathbb{M}$.

Motivating examples: The property that f is locally bounded on T, in symbols $f \in \mathcal{B}_{\text{loc}}(T)$, is sequential and completely sequential (does not depend on the limit point being enumerated). The property that f is continuous on T is completely sequential, whereas the property that f is extensibly continuous on T, $f \in \overline{\mathcal{C}}(T)$, is sequential since such a function f is continuous on the closure $\bar{T}$.

Definition 3. Let $\mathcal{F}$ be any family of functions $h : \mathbb{R}^N \to \mathbb{R}$. Let $\mathcal{F}|T$, or $\mathcal{F}(T)$, denote the functions of $\mathcal{F}$ with domains restricted to T, so that $\mathcal{F} = \mathcal{F}(\mathbb{R}^N)$. We are concerned with the following families (below 'convex' means mid-point convex, see Section 6):

$$\begin{aligned}
\mathcal{K} &= \text{convex}, \quad \mathcal{A}dd = \text{additive}, \quad \mathcal{S}ub = \text{subadditive},\\
\mathcal{C} &= \text{continuous}, \quad \overline{\mathcal{C}} = \text{precompact}, \quad \mathcal{S}v = \text{slowly varying},\\
\mathcal{B} &= \text{bounded on precompacts}, \quad \mathcal{B}^{\pm} = \text{bounded above/below on precompacts},\\
\mathcal{B}_{\text{loc}} &= \text{locally bounded}, \quad \mathcal{B}^{\pm}_{\text{loc}} = \text{locally bounded above/below},\\
\mathcal{L} &= \text{measurable}, \quad \mathcal{B}a = \text{Baire},\\
\mathcal{U}(T) &= \{h : (\forall \text{ bounded } \{u_n\} \subseteq T)(\forall\{x_n\} \to \infty)[h(u_n + x_n) - h(x_n) \to 0]\}.
\end{aligned}$$

Definition 4. Let $\mathcal{P} \subseteq \mathcal{Q}$. Say that a property $\mathcal{P}$ is *automatic* for $\mathcal{F}$ in $\mathcal{Q}$, or that $\mathcal{F}$ is *automatically* $\mathcal{P}$ in $\mathcal{Q}$, if every $h \in \mathcal{F} \cap \mathcal{Q}$ is in $\mathcal{P}$. Thus, for f in $\mathcal{F}$, $\mathcal{P}$ iff $\mathcal{Q}$.

Example. Recall the following.

Ostrowski's Theorem. ([Ostr], [BOst3], [BGT] Th. 1.1.7) *An additive function bounded above on a set of positive measure is continuous.*

Accordingly, continuity is automatic for measurable additive functions, i.e. additive functions are automatically continuous if measurable, or in symbols: $\mathcal{A}dd$ is *automatically* $\mathcal{C}$ in $\mathcal{L}$.

Definition 5. Generalizing the approach of Ger and Kuczma (see [GerKucz], or [Kucz] p. 206), *automatic classes* corresponding to the function classes of Definition 3 may be defined in unified fashion thus:

$$\mathfrak{A}_{\mathcal{F}}(\mathcal{Q},\mathcal{P}),\ \text{or}\ \mathfrak{A}_{\mathcal{F}}(\mathcal{Q}\Longrightarrow\mathcal{P}),:=\{T:(\forall f\in\mathcal{F})[f|T\in\mathcal{Q}|T\Longrightarrow f\in\mathcal{P}]\}.$$

That is, domains taken from this class guarantee that the functions in $\mathcal{F}$ having $\mathcal{Q}$ on a domain automatically have $\mathcal{P}$ on $\mathbb{R}$.

Examples. The class $\mathfrak{B}_{\overline{\mathcal{C}}}=\mathfrak{A}_{\mathcal{A}dd}(\overline{\mathcal{C}},\mathcal{C})$ thus denotes the family of sets T such that any additive function precompact on T is necessarily continuous on $\mathbb{R}$. The class $\mathfrak{C}=\mathfrak{A}_{\mathcal{A}dd}(\mathcal{B},\mathcal{C})$ denotes the family of sets T such that any additive function bounded on T is necessarily continuous on $\mathbb{R}$.

Ger-Kuczma classes

In the terminology of Definition 5, the various Ger-Kuczma classes of [GerKucz] can be summarized symbolically as

$$\begin{aligned}
\mathfrak{A} &= \mathfrak{A}_{\mathcal{K}}(\mathcal{B}^{+},\mathcal{C}), \qquad \mathfrak{B}=\mathfrak{A}_{\mathcal{A}dd}(\mathcal{B}^{+},\mathcal{C}), \qquad \mathfrak{C}=\mathfrak{A}_{\mathcal{A}dd}(\mathcal{B},\mathcal{C})\\
\mathfrak{A}_{\mathcal{C}} &= \mathfrak{A}_{\mathcal{K}}(\mathcal{C},\mathcal{C}), \qquad \mathfrak{B}_{\mathcal{C}}=\mathfrak{A}_{\mathcal{A}dd}(\mathcal{C},\mathcal{C}),\\
\mathfrak{A}_{\overline{\mathcal{C}}} &= \mathfrak{A}_{\mathcal{K}}(\bar{\mathcal{C}},\mathcal{C}), \qquad \mathfrak{B}_{\overline{\mathcal{C}}}=\mathfrak{A}_{\mathcal{A}dd}(\overline{\mathcal{C}},\mathcal{C}),\\
\mathfrak{D} &= \mathfrak{A}_{\mathcal{A}dd}(\mathcal{B}^{-},\mathcal{B}^{-}_{\mathrm{loc}}), \qquad \mathfrak{B}_{\mathrm{sub}}=\mathfrak{A}_{\mathcal{S}ub}(\mathcal{B}^{+},\mathcal{B}^{+}_{\mathrm{loc}}),\\
\mathfrak{U} &= \mathfrak{A}_{\mathcal{S}v}(\mathcal{U},\mathcal{U}),
\end{aligned}$$

or in words,

$$\begin{aligned}
\mathfrak{A} &= \mathfrak{A}_{\mathcal{F}}(\text{Bounded above}\Longrightarrow\text{Continuous}), \qquad \mathcal{F}=\mathcal{K}=\text{Convex},\\
\mathfrak{B} &= \mathfrak{A}_{\mathcal{F}}(\text{Bounded above}\Longrightarrow\text{Continuous}), \qquad \mathcal{F}=\mathcal{A}dd=\text{Additive},\\
\mathfrak{C} &= \mathfrak{A}_{\mathcal{F}}(\text{Bounded}\Longrightarrow\text{Continuous}), \qquad \mathcal{F}=\mathcal{A}dd=\text{Additive},\\
\mathfrak{A}_{\mathcal{C}} &= \mathfrak{A}_{\mathcal{F}}(\text{Continuous}\Longrightarrow\text{Continuous}), \qquad \mathcal{F}=\mathcal{K}=\text{Convex},\\
\mathfrak{B}_{\mathcal{C}} &= \mathfrak{A}_{\mathcal{F}}(\text{Continuous}\Longrightarrow\text{Continuous}), \qquad \mathcal{F}=\mathcal{A}dd=\text{Additive},\\
\mathfrak{A}_{\overline{\mathcal{C}}} &= \mathfrak{A}_{\mathcal{F}}(\text{Precompact}\Longrightarrow\text{Continuous}), \qquad \mathcal{F}=\mathcal{K}=\text{Convex},\\
\mathfrak{B}_{\overline{\mathcal{C}}} &= \mathfrak{A}_{\mathcal{F}}(\text{Precompact}\Longrightarrow\text{Continuous}), \qquad \mathcal{F}=\mathcal{A}dd=\text{Additive},\\
\mathfrak{D} &= \mathfrak{A}_{\mathcal{F}}(\text{Bounded below}\Longrightarrow\text{Loc. bounded below}), \qquad \mathcal{F}=\mathcal{A}dd=\text{Additive},\\
\mathfrak{B}_{\mathrm{sub}} &= \mathfrak{A}_{\mathcal{F}}(\text{Bounded above}\Longrightarrow\text{Loc. bounded above}), \qquad \mathcal{F}=\mathcal{S}ub=\text{Subadditive},\\
\mathfrak{U} &= \mathfrak{A}_{\mathcal{F}}(\text{Uniformly convergent}\Longrightarrow\text{Uniformly convergent}), \qquad \mathcal{F}=\mathcal{S}v=\text{Slowly varying}.
\end{aligned}$$

These are respectively: the Ger-Kuczma classes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ ([GerKucz], [Kucz] p. 206), the *Kominek classes* $\mathfrak{A}_{\mathcal{C}}, \mathfrak{B}_{\mathcal{C}}$ and the variants $\mathfrak{A}_{\overline{\mathcal{C}}}, \mathfrak{B}_{\overline{\mathcal{C}}}$ (see [Kom-Kom], [Kucz] p. 227 and 249, but here we have required 'precompactness' in place of continuity), the *Smítal* class $\mathfrak{D}$ (see [Sm], [Kucz] p. 223 and 244). We offer some new results concerning these classes in Section 4. The last two classes are new – $\mathfrak{B}_{\text{sub}}$ is studied in Section 4, and $\mathfrak{U}$, the automatic uniform convergence class, in Section 7 below.

4 Elementary automatic continuity theorems

Theorem (First Automatic Continuity Theorem – additive functions). *For $S \in \mathfrak{S}$, the set S has the property that any additive function bounded above on the set is continuous, i.e. in symbols:*

$$\mathfrak{S} \subseteq \mathfrak{B}.$$

In particular

$$\mathfrak{S}_{\text{gen}} \subseteq \mathfrak{S} \ (= \mathfrak{S}_{\text{ref}} = \mathfrak{S}_{\text{av}}) \subseteq \mathfrak{B}.$$

Proof. Let $T \in \mathfrak{S}_{\text{gen}}$. Suppose that $f \in \mathcal{A}dd(\mathbb{R})$ is bounded on T but not continuous. By Ostrowski's Theorem we may suppose that $f(z_n) \to \infty$ for some $z_n \to 0$. Now, for some $t \in \mathbb{R}$ and for some infinite $\mathbb{M}_t$, we have $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$. So, for $m \in \mathbb{M}_t$, we see that

$$f(z_m) = f(t + z_m) - f(t)$$

is bounded, a contradiction. $\square$

Remarks. For a related result see part (iii) of the next theorem requiring the use of $\mathfrak{S}_{\text{gen}}$. The class $\mathfrak{S}^k$ (see Section 2) thus provides an improvement to the theorem above through thinning out the test sets. Suppose that the additive function f is bounded on a set $S \in \mathfrak{S}^k$. If now, for some $t \in T = S + ... + S$ and some infinite $\mathbb{M}_t$, we have $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$, then, putting $z_m = s_m^1 + ... + s_m^k$ with $s_m^1, ..., s_m^k \in S$, we have, for $m \in \mathbb{M}_t$,

$$f(z_m) = f(s_m^1) + ... + f(s_m^k) - f(t)$$

and so $f(z_m)$ is again bounded.

Recall that a function $f(.)$ is *sublinear* if $f(nx) = nf(x)$ for all x and all non-negative integer n, or equivalently $f(qx) = qf(x)$ for non-negative rational q. Thus a sublinear function is subadditive iff it is convex, as follows from

$$f\left(\frac{x+y}{2}\right) \leq f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

Indeed, it is sufficient for f to be 'doubling', that is, $f(2x) = 2f(x)$ for all x (equivalently $f(x/2) = f(x)/2$).

Also recall the theorem due to M.E. Kuczma ([KuczME]) asserting coincidence of the sets of automatic continuity for convex and additive functions when these are bounded above, in symbols

$$\mathfrak{A} := \mathfrak{A}_{\mathcal{K}}(\mathcal{B}^+, \mathcal{C}) = \mathfrak{B} := \mathfrak{A}_{\mathcal{A}dd}(\mathcal{B}^+, \mathcal{C}).$$

A similar closeness exists between additivity and subadditivity:

Theorem (Second Automatic Continuity Theorem – subadditive functions).

(i) *Let T have the property that a subadditive function, locally bounded above on T, is necessarily locally bounded on $\mathbb{R}$. If $f \in \mathcal{A}dd(\mathbb{R})$ is bounded above on T, then f is continuous on $\mathbb{R}$. In symbols: $\mathfrak{B}_{\text{sub}} \subseteq \mathfrak{B}$.*

(ii) *For $f \in \mathcal{S}ub(\mathbb{R})$, if $T \in \mathfrak{S}$ and f is bounded above on T then f is locally bounded above on $\mathbb{R}$. In symbols: $\mathfrak{S} \subseteq \mathfrak{B}_{\text{sub}}$.*

(iii) *For $T \in \mathfrak{S}_{\text{gen}}$ and $f \in \mathcal{A}dd(\mathbb{R})$, if f is continuous on T then f is continuous on $\mathbb{R}$. In symbols: $\mathfrak{S}_{\text{gen}} \subseteq \mathfrak{B}_{\mathcal{C}}$.*

Proof. (i) Let f be additive and $T \in \mathfrak{B}_{\text{sub}}$. Suppose f is bounded above on T. Then f is locally bounded, so is continuous by Ostrowski's Theorem (as above). So $T \in \mathfrak{B}$.

(ii) For $T \in \mathfrak{S}$ and f subadditive and bounded above by M on T, suppose that f is not locally bounded at 0 and so not locally bounded above (cf. Lemma 1 of [BOst5], or [Kucz] Th 2. p. 404). Then for some null sequence z_n we have $f(z_n) > n$. But, for some t and some infinite $\mathbb{M}$, we have, for $m \in \mathbb{M}$, that

$$f(z_m) = f(z_m + t - t) \leq f(t + z_m) + f(-t) \leq M + f(-t),$$

a contradiction for m large enough.

This also proves (iii) because $\{t+z_m : m \in \mathbb{M}\} \subseteq T$ with $t \in T$ implies $t+z_m \to t \in T$ and so $\lim_{\mathbb{M}} f(t+z_m) = f(t)$. Hence, if f is additive, $\lim_{\mathbb{M}} f(z_m) = 0$, a contradiction. $\square$

Remark. The class $\mathfrak{S}^k$ (see Section 2) again provides an improvement to the theorem above through thinning out the test sets. Suppose that the additive function f is bounded on a set $S \in \mathfrak{S}^k$. If now, for some $t \in T = S + ... + S$ and some infinite $\mathbb{M}_t$, we have $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$, then, putting $z_m = s_m^1 + ... + s_m^k$ with $s_m^1, ..., s_m^k \in S$, we have here, for $m \in \mathbb{M}_t$,

$$f(z_m + t - t) \leq f(s_m^1) + ... + f(s_m^k) - f(t) \leq kM + f(-t),$$

again leading to a contradiction.

5 Two New Automatic Analyticity Theorems

We will need some definitions.

Definition 1.

(i) Let $\tau + T := \{\tau + t : t \in T\}$. Say that $\mathcal{F}$ *preserves* $\mathcal{Q}$ *under shifts* if $\mathcal{Q}(\tau + T) \subseteq \mathcal{Q}(T)$, for all sets $T-$ more precisely, for all $f \in \mathcal{F}$ and for all τ, the condition $f|(\tau + T) \in \mathcal{Q}|(\tau + T)$ implies $f|T \in \mathcal{Q}|T$.

(ii) Say that $\mathcal{F}$ *preserves* $\mathcal{Q}$ *under vector addition and subtraction on bounded/compact domains* if for $f \in \mathcal{F}$ and S, T bounded/compact, if $f|S \in \mathcal{F}(S)$ and $f|T \in \mathcal{F}(T)$ then $f|(S \pm T) \in \mathcal{F}(S \pm T)$.

Notes.

1. For $\mathcal{Q}$ sequential/completely sequential the definitions of (ii) may be expressed equivalently in terms of bounded/convergent sequences.

2. The definition is similar in spirit, but different in detail, to notions such as the $\mathfrak{A}_{\mathcal{F}}(\mathcal{Q}, \mathcal{P})$-*conservative* set operations of Ger and Kuczma (compare [Kucz], Ch. IX), in that our approach refers only to $\mathcal{Q}$.

We record:

Lemma. If $\mathcal{F}$ *preserves* $\mathcal{Q}$ *under shift,* then $T \in \mathfrak{A}_{\mathcal{F}}(\mathcal{Q}, \mathcal{P})$ iff $t + T \in \mathfrak{A}_{\mathcal{F}}(\mathcal{Q}, \mathcal{P})$.

Examples.

Example 1. The additive class preserves $\mathcal{C}$ and $\overline{\mathcal{C}}$ under shift.

Example 2. Similarly, the additive class preserves $\mathcal{B}$ under shift.

Example 3a. $\mathcal{A}dd$ preserves $\mathcal{C}$, the continuous functions, under vector sums and differences on compact domains, i.e. for $f \in \mathcal{A}dd$ and S, T compact, if $f|S \in \mathcal{C}(S)$ and $f|T \in \mathcal{C}(T)$ then $f|S \pm T \in \mathcal{C}(S \pm T)$. Indeed let $u_n = s_n \pm t_n \in S \pm T$. Then $\{s_n : n \in \omega\}$ and $\{t_n : n \in \omega\}$ are precompact sets. By compactness of S and T, without loss of generality we may assume that $s_n \to s \in S$ and $t_n \to t \in T$. Then by additivity, $\lim f(s_n \pm t_n) = \lim[f(s_n) \pm f(t_n)]$, and by continuity $\lim[f(s_n) \pm f(t_n)] = f(s) \pm f(t)$. Thus f is continuous on $S \pm T$.

Example 3b. $\mathcal{A}dd$ preserves $\mathcal{B}_{\text{loc}}$, the locally bounded functions, under vector sums and differences on compact domains, i.e. for $f \in \mathcal{A}dd$ and S, T compact, if $f|S \in \mathcal{B}_{\text{loc}}(S)$ and $f|T \in \overline{\mathcal{C}}(T)$ then $f|S \pm T \in \mathcal{B}_{\text{loc}}(S \pm T)$, for a similar reason.

Example 3c. $\mathcal{S}ub$ preserves $\mathcal{B}_{\text{loc}}$, the locally bounded functions, under shift and vector sums on compact domains, i.e. for $f \in \mathcal{S}ub$ and S, T compact, if $f|S \in \mathcal{B}_{\text{loc}}(S)$ and $f|T \in \mathcal{B}_{\text{loc}}(T)$ then $f|S + T \in \mathcal{B}_{\text{loc}}(S + T)$, for much the same reasons. Indeed, $f(\tau + u_n) \leq f(\tau) + f(u_n)$ implies that $\{f(\tau + u_n)\}$ is bounded if $\{f(u_n)\}$ is, and $f(u_n + v_n) \leq f(u_n) + f(v_n)$ implies that $\{f(u_n + v_n)\}$ is bounded if both $\{f(u_n)\}$ and $\{f(v_n)\}$ are.

As preliminaries to two of our main results, Theorems 1, 2 below, we quote below three results, Theorems A-C, from [BOst7]. Recall the comments in the introduction regarding Darboux's theorem for additive functions (on local boundedness on $\mathbb{R}$ implying continuity), namely that Ostrowski's Theorem may be regarded as *thinning out* the weak property ('local boundedness') from holding on an interval to holding on a set of positive measure, and that Jones's theorem is a further thinning out. The main theorem of [BOst7] and two variants identify circumstances when a weak property that has been given 'analytic thinning out' still implies the strong property. The theorem calls for three ingredients: an initial 'weak implies strong (on $\mathbb{R}$)' hypothesis, sequential character of the weak property (Definition 2 of Section 3), and a modicum of additive structure (given by the theorem, see below for a definition of additive structure). The canonical 'weak implies strong' hypothesis is Darboux's theorem, a reversal of $\mathcal{C}$ implies $\mathcal{B}_{\text{loc}}$ to

$$(\forall f \in \mathcal{A}dd)[f \in \mathcal{B}_{\text{loc}} \Longrightarrow f \in \mathcal{C}].$$

Theorem A (Analytic Automaticity Theorem). *Suppose that func-*

tions of $\mathcal{F}$ having $\mathcal{Q}$ on $\mathbb{R}^d$ have $\mathcal{P}$ on $\mathbb{R}^d$, where $\mathcal{Q}$ is a property of functions from $\mathbb{R}^d$ to $\mathbb{R}$ that is completely sequential on $\mathbb{R}^d$.

Suppose that $\mathcal{F}$ preserves $\mathcal{Q}$ under shift and also under vector addition and subtraction of compact sets, that is:

(i) *functions of $\mathcal{F}$ having $\mathcal{Q}$ on any T have $\mathcal{Q}$ on $t+T$ for any t;*

(ii)*, for compact sets S and T, functions of $\mathcal{F}$ having $\mathcal{Q}$ on S and T have $\mathcal{Q}$ on $S \pm T$.*

Then, for any analytic set T spanning $\mathbb{R}$ as a vector space over $\mathbb{Q}$ (e.g. containing a Hamel basis), functions of $\mathcal{F}$ having $\mathcal{Q}$ on T have $\mathcal{P}$ on $\mathbb{R}$, i.e. in symbols:

$$T \in \mathfrak{A}_{\mathcal{F}}(\mathcal{Q}, \mathcal{P}).$$

Theorem B. (Symmetric Analytic Automaticity Theorem). *Suppose that functions of $\mathcal{F}$ having $\mathcal{Q}$ on $\mathbb{R}^d$ have $\mathcal{P}$ on $\mathbb{R}^d$, where $\mathcal{Q}$ is a property of functions from $\mathbb{R}^d$ to $\mathbb{R}$ that is completely sequential on $\mathbb{R}^d$.*

Suppose that $\mathcal{F}$ preserves $\mathcal{Q}$ under shift and under vector addition on compact sets, that is:

(i) *functions of $\mathcal{F}$ having $\mathcal{Q}$ on any $T \subseteq \mathbb{R}^d$ have $\mathcal{Q}$ on $\tau + T := \{\tau + t : t \in T\}$, for any $\tau \in \mathbb{R}^d$;*

(ii)*, for compact sets S and T, functions of $\mathcal{F}$ having $\mathcal{Q}$ on S and T have $\mathcal{Q}$ on $S+T$.*

Then, for any analytic set T spanning $\mathbb{R}^d$ as a vector space over $\mathbb{Q}$ (e.g. containing a Hamel basis) such that $S = \tau + T$ is, for some τ, symmetric (i.e. $S = -S$), functions of $\mathcal{F}$ having $\mathcal{Q}$ on T have $\mathcal{P}$ on $\mathbb{R}^d$.

As to the the third variant note that as the property $\mathcal{Q}$ of the Theorem A is completely sequential, the requirements (i) and (ii) of the theorem need to hold only for compact sets that are convergent sequences. The following definition from [BOst7] formalizes the three properties of convergent sequences needed in Theorem A:

Definition 2. A set $\mathcal{G}$ of convergent sequences $\mathbf{u} = \{u_n\}$ in $\mathbb{R}^d$ is a *sequential additive structure* if it is closed under shift, addition, subtraction and subsequence formation, that is:

(i) $\mathbf{u} \in \mathcal{G}$ implies $t + \mathbf{u} = \{t + u_n\} \in \mathcal{G}$, for each t in $\mathbb{R}^d$,

(ii) $\mathbf{u}, \mathbf{v} \in \mathcal{Q}$ implies that $\mathbf{u} \pm \mathbf{v} \in \mathcal{G}$,

(iii) $\mathbf{u} \in \mathcal{G}$ implies that $\mathbf{u}_{\mathbb{M}} = \{u_m : m \in \mathbb{M}\} \in \mathcal{G}$ for every infinite $\mathbb{M}$.

Say that a sequence $\mathbf{u} = \{u_n\}$ is $\mathcal{Q}$-good for h if

$$h|\{u_n\} \in \mathcal{Q}|\{u_n\}.$$

If $\mathcal{Q}$ is completely sequential then $\mathbf{u}$ is $\mathcal{Q}$-good for the function h iff every subsequence of $\mathbf{u}$ is $\mathcal{Q}$-good for h. Thus letting $\mathcal{G}_{h\mathcal{Q}}$ be the set of sequences that are $\mathcal{Q}$-good for the function h, one has:

Lemma. *If $\mathcal{Q}$ is completely sequential and $\mathcal{F}$ preserves $\mathcal{Q}$ under shift and under vector addition and subtraction on compact sets, then $\mathcal{G}_{h\mathcal{Q}}$ is, for $h \in \mathcal{F}$, a sequential additive structure.*

One thus also has:

Theorem C (Analytic Automaticity Theorem – additive structure).

Suppose that functions of $\mathcal{F}$ having $\mathcal{Q}$ on $\mathbb{R}$ have $\mathcal{P}$ on $\mathbb{R}$, where $\mathcal{Q}$ is a property of functions from $\mathbb{R}$ to $\mathbb{R}$ that is sequential on $\mathbb{R}$.

For $h \in \mathcal{F}$, suppose that $\mathcal{G}_{h\mathcal{Q}}$, the family of $\mathcal{Q}$-good sequences good for h, is an additive structure (closed under shift, vector addition and subtraction and subsequence formation). Then, for any analytic set T spanning $\mathbb{R}$ as a vector space over $\mathbb{Q}$ (e.g. containing a Hamel basis), functions of $\mathcal{F}$ having $\mathcal{Q}$ on T have $\mathcal{P}$ on $\mathbb{R}$, i.e. in symbols:

$$T \in \mathfrak{A}_{\mathcal{F}}(\mathcal{Q}, \mathcal{P}).$$

Notes

1. Theorem C is stated for the one-dimensional context, as the application in view concerns regular variation. See [BGT], [BOst4] for the real case, [BGT] Appendix A1.4, [BOst1] Section 5 for higher-dimensional aspects.

2. The examples 1-3 quoted at the beginning of this Section demonstrate that this theorem implies the two known instances of the Analytic Automaticity Theorem:

(i) Jones's Theorem ([Jones], or [Kucz] p. 227): *For any analytic T containing a Hamel basis, $T \in \mathfrak{B}_{\mathcal{C}}$*; the weaker result with $\mathfrak{B}_{\mathcal{C}}$ replaced by $\mathfrak{B}_{\bar{\mathcal{C}}}$ also holds by Example 3b.

(ii) Kominek's Theorem, ([KomZ], or [Kucz] p. 214): *For any analytic T containing a Hamel basis, $T \in \mathfrak{C}$.*

3. Kominek's theorem implies Ostrowski's Theorem when the latter is restricted to analytic sets T of positive measure on which an additive function is bounded; indeed then T spans $\mathbb{R}$ over $\mathbb{Q}$, as $T+T$ contains an interval.

We are now able to give the first of three new results on analytic automaticity mentioned in [BOst7].

Theorem 1 (Symmetric analytic automaticity – subadditive functions). *For any analytic set T spanning $\mathbb{R}^d$ as a vector space over $\mathbb{Q}$ (e.g. containing a Hamel basis) such that $S = \tau + T$ is, for some τ, symmetric (i.e. $S = -S$), functions of $\mathcal{S}ub$ that are locally bounded on T are locally bounded (on $\mathbb{R}^d$). In symbols:*

$$T \in \mathfrak{B}_{\text{sub}} = \mathfrak{A}_{\mathcal{S}ub}(\mathcal{B}^{+}_{\text{loc}}, \mathcal{B}^{+}_{\text{loc}}).$$

Proof. Apply Theorem B taking $\mathcal{P} = \mathcal{Q} = \mathcal{B}^{+}_{\text{loc}}$. □

The next theorem links analytic automaticity with convexity. We include it here because of its affinity with the result above, rather than in the next section, where other results on convexity appear. In it, we obtain a thinned-out version of Ostrowski's theorem for convex functions, though at the cost of a condition of symmetry.

Theorem 2 (Symmetric analytic automaticity – convex functions). *For any analytic set T spanning $\mathbb{R}^d$ as a vector space over $\mathbb{Q}$ (e.g. containing a Hamel basis) such that $S = \tau + T$ is, for some τ, symmetric (i.e. $S = -S$), functions of $\mathcal{K}$ that are locally bounded on T are continuous (on $\mathbb{R}^d$). In symbols:*

$$T \in \mathfrak{A} = \mathfrak{A}_{\mathcal{K}}(\mathcal{B}^{+}_{\text{loc}}, \mathcal{C}).$$

Proof. We take $\mathcal{P} = \mathcal{C}$ and $\mathcal{Q} = \mathcal{B}^{+}_{\text{loc}}$ and consider a function f convex on $\mathbb{R}^d$. As shifting preserves both the analyticity and spanning properties, we may without loss of generality assume that T is symmetric. Argue as in the proof of the Main Theorem of [BOst7] to deduce that, for some compact subset K of T and some integer j, the j-fold sum $S = K + ... + K$ has positive measure. Let u be a density point of S. We appeal to part (iv) of the Portmanteau Theorem for convex functions of Section 6 below; if f is not continuous, then f is not locally bounded above at u and so we may choose $u_n \to u$ with $\{f(u_n)\}$ unbounded. We now make use of an averaging property much as in the proof ([Kucz] p. 223) of Császár's First Theorem (again, see Section 6). Taking $\alpha = 1/2j$, then for some $s_n \in S$, some $w \in \mathbb{R}$ and some infinite $\mathbb{M}_w$ we have, for $m \in \mathbb{M}_w$, that

$$\alpha s_m + (1-\alpha)w = u_m.$$

Now, write $s_m = k_m^1 + ... + k_m^j$ with $k_m^i \in K$. We recall that Jensen's inequality applies to mid-point convex functions, provided rational convex combinations are taken (see e.g. [HLP] Section 3.6 p. 72, or [Kucz] Th 5.3.5 and Lemma 5.3.2 p. 125-6). Thus we deduce from

$$\frac{1}{2j}k_m^1 + ... + \frac{1}{2j}k_m^j + \frac{1}{2}\left(\frac{2j-1}{j}w\right) = u_m,$$

that

$$f(u_m) \le \frac{1}{2}f\left(\frac{2j-1}{j}w\right) + \frac{1}{2j}f(k_m^1) + ... + \frac{1}{2j}f(k_m^n),$$

for $m \in \mathbb{M}_w$. But the right-hand side is bounded for $m \in \mathbb{M}$, since f is bounded on T and so on K. This contradiction shows that in fact f is continuous. □

We delay the third new theorem on analytic automaticity to Section 7 as its context is regular variation.

6 Convexity

In the interest of transparency, in this section we work in $\mathbb{R}$ rather than $\mathbb{R}^d$. All the results are, however, generalizable.

Definition 1. Here f is called *convex* if it is mid-point convex, i.e.

$$f\left(\frac{1}{2}x + \frac{1}{2}y\right) \le \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

see [HLP] Section 3.5 p.71. This is to be contrasted with the stronger condition

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \qquad (\forall \lambda \in [0,1]),$$

which for finite-valued f implies local boundedness and so yields continuity, by way of the Bernstein-Doetsch Theorem (see below and the Introduction). We note there is one further, the Linear Minorant, definition, for which see [HLP] section 3.19 p. 95.

Remarks. 1. We refer to [Kucz] for the case where the domain of definition of f is restricted to convex open sets. We note that the current theory applies to such a context, ultimately because the Kestelman-Borwein Theorem, as stated in Section 2, holds when relativized to an open subset.

2. Similarly, we refer to [Rock] for the case where one uses ordinary rather than mid-point convexity, but allows f to take values in the extended real line.

Definition 2. The lower hull $m_f(x)$ is defined by

$$m_f(x) = \lim_{\delta \to 0+} \inf \{f(t) : |t - x| < \delta\}.$$

We begin by recalling some classical theorems on convex functions.

Császár Convexity Theorem *For convex f, either $m_f \equiv -\infty$, or m_f is finite, convex and continuous* ([Kucz] p. 141).

Portmanteau Theorem for Convex Functions *For convex f :*

(i) *If f is locally bounded above at some point, then f is locally bounded above at all points* ([Kucz] p. 138).

(ii) *If f is locally bounded below at some point, then f is locally bounded below at all points* ([Kucz] p. 139).

(iii) *If f is locally bounded above at some point, then it is everywhere locally bounded* ([Kucz] p. 140).

(iv) *If $f(x) \neq m_f(x)$ for some x, then f is not locally bounded at x* ([Kucz] p. 144).

The common feature here is that the sequence witnessing bad behaviour at one point yields by translation a sequence witnessing bad behaviour at any desired point.

The final two classical results may be viewed as early examples of automatic properties. So too are the Propositions following them.

Bernstein-Doetsch Theorem ([BeDoe], [Kucz] p. 145) *For convex f, if f is locally bounded above at some point, then f is continuous.*

Dichotomy Theorem for convex functions ([Kucz] p. 147) *For convex f (so in particular for additive f) either f is continuous everywhere, or it is discontinuous everywhere.*

Proposition 2. *If f is convex and bounded below on a reflecting (that is, subuniversal) set S, then f is locally bounded below.*

Proof. Suppose not. Let T be a reflecting core of S. Let K be a lower bound on T. If f is not locally bounded from below, then at any point u in $\bar{T}$ there is a sequence $\{u_n\} \to u$ with $\{f(u_n)\} \to -\infty$. For some $w \in \mathbb{R}$, we have $v_n = \frac{1}{2}w + \frac{1}{2}u_n \in T$, for infinitely many n. Then

$$K \leq f(v_n) \leq \frac{1}{2}f(w) + \frac{1}{2}f(u_n), \text{ or } 2K - f(w) \leq f(u_n),$$

i.e. $f(u_n)$ is bounded from below, a contradiction. □

Remark. Here again we may thin out the hypothesis. For example, suppose that f is bounded on S and $T = S + S \in \mathfrak{S}^2$. Then for some $w \in \mathbb{R}$ there are points $v_n \in T$ such that

$$w = \frac{3}{4}u_n + \frac{1}{4}v_n$$

We may now write

$$w = \frac{1}{2}\left(\frac{3}{2}u_n\right) + \frac{1}{4}s_n^1 + \frac{1}{4}s_n^2,$$

so that

$$f(w) \leq \frac{1}{2}\left(\frac{3}{2}u_n\right) + \frac{1}{4}f(s_n^1) + \frac{1}{4}f(s_n^2),$$

and this again leads to a contradiction.

Proposition 3 (cf. [Meh] Th. 3). *If f is convex and bounded above on an averaging (that is, subuniversal) set S, then f is continuous.*

Proof. Let T be an averaging core of S. Suppose that f is not continuous, but is bounded above on T by K. Then f is not locally bounded above at some point of $u \in \bar{T}$. Then there is a null sequence $z_n \to 0$ with $f(u_n) \to \infty$, where $u_n = u + z_n$. Select $\{v_n\}$ and w in $\mathbb{R}$ so that, for infinitely many n, we have

$$u_n = \frac{1}{2}w + \frac{1}{2}v_n.$$

But for such n,we have

$$f(u_n) \leq \frac{1}{2}f(w) + \frac{1}{2}f(v_n) \leq \frac{1}{2}f(w) + \frac{1}{2}K,$$

contradicting the unboundedness of $f(u_n)$. $\square$

Remarks.

1. Here again we may thin-out the hypothesis. For example, suppose again that f is bounded on S and $S+S \in \mathfrak{S}^2$. Then we may write

$$u_n = \frac{3}{4}w + \frac{1}{4}(s_n^1 + s_n^2) = \frac{1}{2}\left(\frac{3}{2}w\right) + \frac{1}{4}s_n^1 + \frac{1}{4}s_n^2,$$

for some $w \in \mathbb{R}$. Thus

$$f(u_n) \leq \frac{1}{2}f(\frac{3}{2}w) + \frac{1}{4}f(s_n^1) + \frac{1}{4}f(s_n^2) \leq \frac{1}{2}f(\frac{3}{2}w) + \frac{1}{2}K,$$

and again we contradict unboundedness.

2. Specialization of the Proposition above yields again the First Automatic Continuity Theorem.

Corollary. *Any additive function bounded above on an averaging set is continuous, i.e. in symbols*

$$\mathfrak{S}_{\text{gen}} \subseteq \mathfrak{S}\ (= \mathfrak{S}_{\text{ref}} = \mathfrak{S}_{\text{av}}) \subseteq \mathfrak{B}.$$

Proposition 3, together with subuniversality and the Kestelman-Borwein-Ditor Theorem of Section 2, implies the classical result below, another early automaticity theorem.

Császár-Ostrowski Theorem ([Csa], [Kucz] p. 210). *A convex function $f : \mathbb{R} \to \mathbb{R}$ bounded above on a set of positive measure/non-meagre set is continuous.*

Our next result locates the reflecting and averaging sets within the established automatic property classes.

Theorem (Hukuhara Theorem, [Huk]**,** or [Kucz] p. 226**)**

$$\text{(i) } \mathfrak{S}\ (= \mathfrak{S}_{\text{ref}}) \subseteq \mathfrak{D}, \text{ and (ii) } \mathfrak{S}\ (= \mathfrak{S}_{\text{av}}) \subseteq \mathfrak{A}_{\mathcal{C}}.$$

Proof. Immediate: (i) follows from Proposition 2, while (ii) follows from the argument of Proposition 3. $\square$

Note that the measure and Baire version of (ii) appear in [Kom-Kom] and [KomZ] resp.(cf. [Kucz] p. 229-230). Note that $\mathfrak{A}_{\mathcal{C}} \subseteq \mathfrak{B}_{\mathcal{C}}$.

We now need to recall further combinatorial definitions from [BOst4] (for use only in the rest of this section). The notation **NT,** referring to *No Trumps*, follows the card-suit notation used by set-theorists for various combinatorial principles (see [BOst3] for motivation and for their relation with the condensation principles of set-theory).

Definition 3 (No Trumps). For a family $\{T_k : k \in \omega\}$ of subsets $\mathbb{R}$, $\mathbf{NT}(\{T_k : k \in \omega\})$ means that:
for any convergent sequence $\{u_n\}$ in $\mathbb{R}$ there is $k \in \omega$, an infinite $\mathbb{M} \subseteq \omega$, and $t \in \mathbb{R}^d$ such that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq T_k.$$

In words: for every convergent sequence $\{u_n\}$ in $\mathbb{R}$, some T_k contains a translate of a subsequence of $\{u_n\}$.

In applications the family $\{T_k : k \in \omega\}$ will be a stratification induced by some function $h : \mathbb{R} \to \mathbb{R}$ with the substrata either the *symmetric* lower **level sets**, or just the *upper* or *lower* level sets (as in Definition 5 below)

$$H_k = \{t \in \mathbb{R}^d : |h(t)| < k\} \text{ for } k \in \omega.$$

If $T_k \equiv T$, $\mathbf{NT}(T)$ reduces to subuniversality of T.

Definition 4. Say that the function f is **Weak No Trumps** (f is **Weak NT**, $f \in$**WNT),** if $\mathbf{NT}(\{H_k : k \in \omega\})$ holds, with H_k as above.

Theorem (Weak No Trumps Theorem). *If f is Baire/measurable, then f is* **WNT**.

Proof. For completeness, we quote the proof from[BOst3] Section 4, cf. [BOst4] Section 2.2. In the Baire case, as

$$\mathbb{R} = \bigcup_{k \in \omega} H_k,$$

some set H_k is non-meagre, and so is averaging. Similar considerations apply to a measurable function f. □

As the terminology implies, the Weak No Trumps Theorem is weaker than the No Trumps Theorem ([BOst1] Section 1.2), which in turn is weaker than the Strong No Trumps Theorem ([BOst4] Section 2).

The following result generalizes to the convex case a theorem proved in [BOst3] in the additive case.

Theorem. *For f convex (in particular, for f additive), f is continuous iff f is* **WNT**.

This is immediate from Definition 4 by the Proposition 3. □

The last two theorems imply the following classical result due to Sierpiński [Sier], again an early automaticity theorem. This is slightly weaker than the Császár-Ostrowski Theorem above; cf. [BGT] p. 5.

Corollary (**Sierpiński's Theorem** [Sier], [Kucz] p. 218). *A measurable/Baire convex function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

We will soon need the following strengthening of a classical result (for completeness we include the proof, as it is short).

Theorem (Császár's First Theorem) ([Kucz] p. 223.) *Suppose f is convex and bounded below by K on a strongly averaging set S. Then m_f is bounded below by K on the closure of any strong core set of S.*

Proof. Suppose otherwise. Let $\gamma_n \to 0$ rational (e.g. $\gamma_n = 2^{-n}$), let f be convex and let T be a strong core of S. For some $u \in \bar{T}$ there is a sequence $u_n \to u$ with $f(u_n) \to L < K$. By assumption there is w and $m(n)$ such that $v_n := \gamma_n w + (1 - \gamma_n) u_{m(n)} \in T$. Hence

$$K \leq f(v_n) \leq \gamma_n f(w) + (1 - \gamma_n) f(u_{m(n)}).$$

Passing to the limit we obtain the contradiction $K \leq L$. □

Definition 5. For a function $f : \mathbb{R}^N \to \mathbb{R}$ define its *upper and lower level sets* U_r, L_r by

$$U_r = \{x : r < f(x)\}, \qquad L_r = \{x : f(x) < r\}.$$

We now give the counterpart for functions of our previous definition of 'strongly averaging' for sets.

Definition 6. Say that a function $f : \mathbb{R}^N \to \mathbb{R}$ is *strongly averaging* if for any family of open sets $\{G_q : q \in \mathbb{Q}\}$ covering $\mathbb{R}^N$ there are r, q such that the set

$$U_r \cap G_q \text{ is strongly averaging.}$$

Theorem. *If f is measurable/Baire, then f is strongly averaging.*
Proof. We have

$$\mathbb{R}^N = \bigcup_{r \in \mathbb{Q}} U_r \cap \bigcup_{q \in \mathbb{Q}} G_q = \bigcup_{r,q \in \mathbb{Q}} U_r \cap G_q.$$

In the measurable case, we have for some r, q that $U_r \cap G_q$ has positive measure and so is strongly averaging. Likewise in the Baire case we have that for some r, q that $U_r \cap G_q$ is second category. $\square$

Theorem (Császár's Non-separation Theorem, [Kucz] p. 226). *For f convex, there is no strongly averaging g with*

$$m_f(x) < g(x) \leq f(x), \text{ for all } x.$$

Proof. Take $U_r = \{x : r < g(x)\}$ and note that by Császár's convexity Theorem ([Kucz] p. 140) if f is locally bounded from below then m_f is convex and continuous but otherwise $m_f(x) \equiv -\infty$. In the latter case take $G_q = \mathbb{R}$. In the former case, by continuity, the sets $G_q = \{x : m_f(x) < q\}$ constitute an open cover. In either case, if $S = U_r \cap G_q$ is strongly averaging then on S we have

$$m_f(x) < q < g(x) \leq f(x).$$

Thus $f > q$ on a strongly averaging set, so by Császár's First Theorem $m_f(x)$ is bounded below by q on the closure of a core set of S, whereas it is in fact bounded above by q. This is a contradiction. $\square$

Note. Taking g measurable, we obtain the result as Császár gave it.

Comment. Much of the proof structure in this section has relied on sequential combinatorics, so it is worth noting the presence here of a hierarchy of combinatorial properties, or more properly of various forms of additive compactness. The key here is the idea that a set T contains a point t approachable (from within the set T) in a specified way, along a null sequence $\mathbf{z}$.

(i) Subuniversality: for a given null sequence $\mathbf{z}$, there is a translator $t \in T$ such that: $t = \lim_{\mathbb{M}}(t + z_m)$ through T, that is, we regard $\mathbf{z}$ as specifying a mode of approach through T and so t is a $\mathbf{z}$-limit from within T.

The equivalent affine formulation is:

(i′) Affine compactness with fixed weight γ: for a given convergent sequence $\mathbf{u}$, there is an 'affine translator' $w \in \mathbb{R}$ that is a $\mathbf{w}$-limit through T, where $w_n = (1-\gamma)z_n$. That is, $\gamma w + (1-\gamma)u$ is the limit $\lim_{\mathbb{M}}(\gamma w + (1-\gamma)u + (1-\gamma)z_m)$ through T.

In this light strong averaging may be restated as follows.

(ii) Affine compactness with varying/converging weights $\{\gamma_n\}$: For converging weights $\gamma_n \to \gamma$ and for a given convergent sequence $\mathbf{u} = \{u + z_n\}$, there is an 'affinizing translator' $w \in R$, i.e., a $\mathbf{w}$-limit through T, where $w_n = (1-\gamma_n)z_{m(n)}$. That is, $\gamma w + (1-\gamma)u$ is the limit $\lim(\gamma_n w + (1-\gamma_n)u + (1-\gamma_n)z_{m(n)})$ through T.

The case (i′) offers only a sequence of affine translators w_k, each term depending on the weighted term $(1-\gamma_k)z_n$, whereas case (ii) creates a diagonalization of translators to a 'one fits all weights'. In this context, the 'folklore' theorem below, concerned with 'one fits two', is illuminating.

Definition 7. Say that a set S is *bilateral* if there is $T \subseteq S$ such that, for any $\{z_n\} \to 0$, there is $t \in T$ (a bilateral translator) and infinite $\mathbb{M}_t$ such that

$$\{x_m = t + z_m, y_m = t - z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Thus $t = \frac{1}{2}x_m + \frac{1}{2}y_m$. The defining clause implies that $\mathbf{NT}(\lambda T)$ holds for $\lambda = \pm 1$, as

$$t + z_m \in T \text{ and } -t + z_m = -y_m \in -T.$$

Theorem (Bilateral Genericity Theorem). *If S is non-meagre/non-null, then S is bilateral. In fact generically all $t \in S$ are bilateral translators.*

The proof is in [BOst9]. Compare Kemperman [Kem], [Kucz] Lemma 2, p. 70.

7 Automatic Uniform Convergence

We return here to the Uniform Convergence Theorem (UCT) for slowly varying functions, the subject of [BOst1], from our present perspective.

Definitions 1. Let $h : \mathbb{R} \to \mathbb{R}$ be slowly varying.

(i) UCT_h is the assertion that for any $\{x_n\} \to \infty$, the convergence

$$\lim_{n\to\infty} |h(u + x_n) - h(x_n)| = 0$$

is uniform for u restricted to compact sets.

(ii) Say that a convergent sequence $\mathbf{u} = \{u_n\}$ is *good* (for h) if, for all $\{x_n\} \to \infty$,

$$\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0.$$

Say that a convergent sequence $\mathbf{u} = \{u_n\}$ is ε-good if, for all $\{x_n\} \to \infty$,

$$|h(u_n + x_n) - h(x_n)| < \varepsilon,$$

for large enough n.

Bounded Equivalence Principle ([BOst1]). UCT_h *is equivalent to the assertion that all convergent sequences are good for* h.

Recalling the definition of $\mathfrak{U}$ in Section 3, we note that a set T is in $\mathfrak{U}$ provided that, for any slowly varying function h, if all subsequences of T are good for h, then UCT_h holds. That is, if all subsequences of T are good, then all sequences are good.

Lemma 1. (i) $\mathbf{u}$ is good iff it is good for each $\varepsilon > 0$.

(ii) if $\mathbf{u} = \{u_n : n \in \omega\}$ is good then so is $\{u_n : n \in \mathbb{M}\}$ for any infinite $\mathbb{M}$.

Proof. Immediate from the definition. $\square$

The next two lemmas identify the additive structure of good sequences.

Lemma 2 (Shift Lemma). (i) If $\mathbf{u}$ is ε-good, then $t + \mathbf{u} = \{t + u_n\}$ is 2ε-good, and hence

(ii) $\mathbf{u}$ is good, iff $t + \mathbf{u} = \{t + u_n\}$ is good.

Proof. (i) Notice that if $y_n = x_n + t$ and $x \to \infty$, then also $y_n \to \infty$. So ultimately

$$|h(u_n + y_n) - h(y_n)| < \varepsilon.$$

But, as h is slowly varying, pointwise convergence at t yields

$$|h(t+x_n) - h(x_n)\}| < \varepsilon, \text{ for } n > M_t,$$

for some M_t. For such n we have

$$\begin{aligned} |h(t+u_n+x_n) - h(x_n)| &\leq |h(u_n+x_n+t) - h(x_n+t)| + |h(t+x_n) - h(x_n)| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

(ii) If $\mathbf{u}$ is good, then $\mathbf{u}$ is ε-good for all $\varepsilon > 0$. Thus $t + \mathbf{u} = \{t + u_n\}$ is 2ε-good, for all $\varepsilon > 0$. Thus $t + \mathbf{u} = \{t + u_n\}$ is good. Obviously if $t + \mathbf{u}$ is good, then $\mathbf{u} = -t + (t + \mathbf{u})$ is good. □

Thus whether or not UCT_h holds is determined by behaviour local to the origin.

Lemma 3 (Addition and subtraction lemma). (i) If $\mathbf{u}, \mathbf{v}$ are ε-good, then $\mathbf{u} + \mathbf{v} = \{u_n + v_n\}$, and $\mathbf{u} - \mathbf{v} = \{u_n - v_n\}$ are 2ε-good. Hence

(i) If $\mathbf{u}, \mathbf{v}$ are good, then $\mathbf{u} \pm \mathbf{v}$ is good.

Proof. (i) Notice that if $y_n = x_n \pm v_n$ and $x \to \infty$, then also $y_n \to \infty$. So

$$|h(u_n + y_n) - h(y_n)\}| < \varepsilon, \text{ for } n > M_y,$$

and

$$|h(u_n + x_n) - h(x_n)\}| < \varepsilon, \text{ for } n > M_x.$$

Consequently,

$$\begin{aligned} |h(u_n + v_n + x_n) - h(x_n)| &\leq |h(u_n + (x_n + v_n) - h(x_n + v_n)| + \\ &\quad +|h(v_n + x_n) - h(x_n)| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

as asserted. Analogously, writing $z_n = x_n - v_n$ and noting that for some M_z

$$\begin{aligned} |h(u_n + z_n) - h(z_n)\}| &< \varepsilon, \text{ for } n > M_z \text{ and} \\ |h(v_n + z_n) - h(z_n)\}| &< \varepsilon, \text{ for } n > M_z \end{aligned}$$

we ultimately have

$$\begin{aligned}
|h(u_n - v_n + x_n) - h(x_n)| &\leq |h(u_n + (x_n - v_n)) - h(x_n - v_n)| \\
&\quad + |h(x_n - v_n) - h(x_n)| \\
&= |h(u_n + z_n) - h(z_n)| \\
&\quad + |h(z_n) - h(v_n + z_n)| \\
&\leq \varepsilon + \varepsilon = 2\varepsilon. \qquad \square
\end{aligned}$$

Corollary. If $\mathbf{u}$ is good, then $\frac{1}{2}\mathbf{u}$ is good iff $-\frac{1}{2}\mathbf{u}$ is good.
Proof. Immediate, as $\frac{1}{2}\mathbf{u} - \mathbf{u} = -\frac{1}{2}\mathbf{u}$.$\square$

Theorem (Third Automatic Continuity Theorem – slowly varying functions).
Let T be subuniversal. For any slowly varying function h, if

$$\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0, \text{ for all } \{x_n\} \to \infty,$$

for all sequences $\{u_n\}$ lying in T, then UCT_h *holds. In symbols:* $\qquad \mathfrak{S} \subseteq \mathfrak{U}.$

Proof. Suppose that $T \in \mathfrak{S}$, but $T \notin \mathfrak{U}$. Then there are a slowly varying h and a sequence $\{u + z_n\} \to u$ which is not good (for h). Hence $\{z_n\} \to 0$ is not good. But, for some w and infinite $\mathbb{M}$, we have $\{w + z_m : m \in \mathbb{M}\} \subseteq T$. Now by assumption we have UCT on the subsequence $\{w + z_m : m \in \mathbb{M}\}$ of T. So $\{w + z_m : m \in \mathbb{M}\}$ is good, as is $\{z_m : m \in \mathbb{M}\}$, contrary to Lemma 1 (ii) after all. $\square$

For h slowly varying let $\mathcal{G}_h$ be the set of convergent sequences $u = \{u_n\}$ good for h. Then, by Lemmas 1-3, $\mathcal{G}_h$ is a sequential additive structure (Definition 2, Section 5). Applying the Analytic Automaticity Theorem of [BOst7] (see Section 3.1) we obtain a 'thinned out' version of the last theorem.

Theorem (Analytic Automaticity Theorem – uniform convergence). *If T is an analytic set spanning $\mathbb{R}$ as a vector space over $\mathbb{Q}$ (e.g. if T contains a Hamel basis) and*

$$\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0, \text{ for all } \{x_n\} \to \infty,$$

for all sequences $\{u_n\}$ lying in T, then UCT_h *holds, that is, $T \in \mathfrak{U}$.*

Postscript. The reader will have observed how extensive a debt we owe here to the work of Marek Kuczma, both in his paper [GerKucz] that initiated the Ger-Kuczma programme, and in his book [Kucz], unfortunately not widely accessible outside Poland. Kuczma is widely acknowledged as the father of the subject of functional equations, and as the founder of a flourishing school. Functional equations lie at the heart of the subject of regular variation, our motivation in [BOst1] and [BOst4]. It is a pleasure to dedicate this paper to Marek Kuczma's memory.

References

[AD] J. Aczél and J. Dhombres, *Functional equations in several variables,* Encycl. Math. Appl. 31, Cambridge University Press, Cambridge, 1989.

[BeDoe] F. Bernstein and G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Ann. 76 (1915), 514-526.

[BGT] N. H. Bingham, C.M. Goldie, J.L. Teugels, *Regular variation,* Encycl. Math. Appl. 27, Cambridge University Press, Cambridge, 1987.

[BOst1] N. H. Bingham and A. J. Ostaszewski, *Infinite combinatorics and the foundations of regular variation*, CDAM Research Report Series, LSE-CDAM-2006-22 (revised).

[BOst3] N. H. Bingham and A. J. Ostaszewski, *Infinite combinatorics and the theorems of Steinhaus and Ostrowski*, LSE-CDAM Report, LSE-CDAM-2007-15 (revised).

[BOst4] N. H. Bingham and A. J. Ostaszewski, *Beyond Lebesgue and Baire: generic regular variation*, Colloquium Mathematicum, to appear (LSE-CDAM Report, LSE-CDAM-2007-16).

[BOst5] N. H. Bingham and A. J. Ostaszewski, *Generic subadditive functions*, Proceedings Amer. Math. Soc. 136 (2008), 4257-4266 (LSE-CDAM Report, LSE-CDAM-2007-17).

[BOst7] N. H. Bingham and A. J. Ostaszewski, *Analytic automaticity: the theorems of Jones and Kominek*, LSE-CDAM Report, LSE-CDAM-2007-24.

[BOst8] N. H. Bingham and A. J. Ostaszewski, *The Converse Ostrowski Theorem: aspects of compactness*, LSE-CDAM Report, LSE-CDAM-2007-25 (revised).

[BOst9] N. H. Bingham and A. J. Ostaszewski, *Infinite combinatorics in function spaces*, LSE-CDAM Report, LSE-CDAM-2007-26 (revised).

[BOst10] N. H. Bingham and A. J. Ostaszewski, *Homotopy and the Kestelman-Borwein-Ditor Theorem*, LSE-CDAM Report, Canadian Math. Bull., to appear (LSE-CDAM-2007-27).

[BOst11] N. H. Bingham and A. J. Ostaszewski, *Bitopology and measure-category duality*, LSE-CDAM Report, LSE-CDAM-2007-29 (revised).

[BoDi] D. Borwein and S. Z. Ditor, *Translates of sequences in sets of positive measure*, Canadian Mathematical Bulletin 21 (1978) 497-498.

[Bou] N. Bourbaki, *Elements of Mathematics, General Topology*, Part 1, Addison-Wesley, 1966.

[Broo] R.M. Brooks,.*On locally m-convex $*$-algebras*, Pacific J. Math. 23 (1967), 5–23.

[CGM] M. Crnjac, B. Guljaš and H.I. Miller, *On some questions of Ger, Grubb and Kraljević*, Acta Math. Hungar. 57 (1991), nos. 3-3, 253-257.

[Csa] A. Császár, *Konvex halmazokról és fuggvényegyenletekröl (Sur les ensembles et les fonctions convexes)*, Mat. Lapok 9 (1958), 273-282.

[Dal1] H. G. Dales, *Automatic continuity: a survey*, Bull. London Math. Soc. 10 (1978), 129-183.

[Dal2] H. G. Dales, *Banach algebras and automatic continuity,* London Math. Soc. Monographs 24, Oxford University Press, Oxford, 2000.

[Dar] G. Darboux, *Sur la composition des forces en statiques*, Bull. des Sci. math. (1) 9 (1875) 281-288.

[Eng] R. Engelking, *General Topology,* Heldermann Verlag, Berlin 1989.

[Fal] K.J. Falconer, *The geometry of fractal sets,* Cambridge Tracts in Math. 85, Cambridge University Press, 1985.

[GerKucz] R. Ger and M. Kuczma, *On the boundedness and continuity of convex functions and additive functions,* Aeq. Math. 4 (1970), 157-162.

[Ham] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung:* $f(x+y)=f(x)+f(y)$, Math. Ann. 60 (1905), 459-462.

[HLP] G. H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Reprint of the 1952 edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.

[Huk] M. Hukuhara, *Sur la fonction convexe*, Proc. Japan. Acad. 30 (1954), 683-685.

[IbLi] I. A. Ibragimov and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff, Groningen, 1971.

[Jo] B.E. Johnson, *Continuity of derivations on commutative algebras*, Amer. J. Math. 91 (1969), 1-10.0.

[Jones] F. B. Jones, *Measure and other properties of a Hamel basis*, Bull. Amer. Math. Soc. 48, (1942). 472–481.

[Kech] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics 156, Springer, 1995.

[Kem] J. H. B. Kemperman, *A general functional equation*, Trans. Amer. Math. Soc. 86 (1957), 28-56.

[Kes] H. Kestelman, *The convergent sequences belonging to a set*, J. London Math. Soc. 22 (1947), 130-136.

[Kom-Kom] B. Kominek and Z. Kominek, *On some set classes connected with the continuity of additive and Q-convex functions*, Prace Naukowe Uniw. Śląsk. 218 , Prace Mat. 8 (1978), 60-63.

[KomZ] Z. Kominek, *On the continuity of Q-convex and additive functions*, Aeq. Math. 23 (1981), 146-150.

[Kucz] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's functional equation and Jensen's inequality,* PWN, Warsaw, 1985.

[KuczME] M.E. Kuczma, *On discontinuous additive function,* Fund. Math. 66(1970), 383-392.

[Lit] J.E. Littlewood, *Lectures on the theory of functions,* Oxford University Press, 1944.

[Mil] A.W. Miller, *Infinite combinatorics and definability*, Ann. of Pure and Applied Mathematical Logic 41(1989), 179-203 (see also updated web version at : http://www.math.wisc.edu/~miller/res/).

[MilH] H. I. Miller, *Generalization of a result of Borwein and Ditor*, Proc. Amer. Math. Soc. 105 (1989), no. 4, 889–893.

[Meh] M.R. Mehdi, *On convex functions*, J. London Math. Soc. 39 (1964), 321-326.

[Ostr] A. Ostrowski, *Mathematische Miszellen XIV: Über die Funktionalgleichung der Exponentialfunktion und verwandte Funktionalgleichungen*, Jahresb. Deutsch. Math. Ver. 38 (1929) 54-62 (reprinted in *Collected papers of Alexander Ostrowski*, Vol. 4, 49-57, Birkhäuser, Basel, 1984).

[Oxt] J. C. Oxtoby, *Measure and category,* 2nd ed., Grad. Texts Math. 2, Springer, New York, 1980.

[Rock] R. T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1997.

[Rog] C. A. Rogers, J. Jayne, C. Dellacherie, F. Topsøe, J. Hoffmann-Jørgensen, D. A. Martin, A. S. Kechris, A. H. Stone, *Analytic sets,* Academic Press, 1980.

[Roy] H.L. Royden, *Real Analysis*, Prentice-Hall, 1988 (3rd ed.)

[Sier] W. Sierpiński, *Sur les fonctions convex mesurable*, Fund. Math. 1 (1920), 125-129 (http://matwbn.icm.edu.pl/).

[Sm] J. Smítal, *On convex functions bounded below*, Aeq. Math 14 (1976), 345-350.

[THJ] F. Topsøe, J. Hoffmann-Jørgensen, *Analytic spaces and their applications*, Part 3 of [Rog].

[Trau] R. Trautner, *A covering principle in real analysis* ,Q. J. Math. Oxford, (2), 38 (1987), 127-130.