Abstract

This paper is related to two others by the same authors, [BOst6], [BOst7]. In the first, we make a systematic study of properties that hold ‘automatically’, for example, automatic continuity. This topic has been extensively studied for Banach algebras, but it has roots in classical real analysis, and that is the setting here. The second paper is on ‘analytic automaticity’, where the test set on which a weaker property is to imply a stronger one is analytic, and ‘thin’.

The roots of the present paper lie in an important (but little cited) paper of Kestelman [Kes], and a subsequent paper of Borwein and Ditor [BoDi]. There the setting is measure-theoretic, and one assumes sets measurable, but one can also proceed topologically, and assume sets to have the Baire property (to be ‘Baire’). In both contexts one has the sequence containment property at generically all points (that
is, avoiding a ‘thin’ exceptional set – which may be null or meagre): typically, a subsequence of deviations from a location in the set, fixed according to a given sequence of deviations, stays within the set. Instead of deviations in this pointwise mode, as in [Kes], [BoDi], [BOst6], here we consider functional deviations, again obtaining generic conclusions; these permit a form of fixed-point property to hold.

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1 The Kestelman-Borwein-Ditor Theorem

We begin with the definitions needed to formulate the Kestelman-Borwein-Ditor Theorem, proved in a stronger form in Section 2.

**Definition 1 (The Kestelman universal and subuniversal class).**

Let $T \subseteq \mathbb{R}$.

(i) We call $T$ universal and write $T \in \mathcal{K}$ (Gothic ‘K’ for Kestelman) if for any null sequence $\{z_n\} \to 0$ in $\mathbb{R}$ there is a co-finite $M \subseteq \omega$, and $t \in \mathbb{R}$ such that

$$\{t + u_n : n \in M\} \subseteq T.$$  \hspace{1cm} (1)

(ii) If (1) holds for some infinite $M$, we call $T$ subuniversal and write $T \in \mathcal{S}$ (Gothic ‘S’ for ‘subsequence’). Clearly

$$\mathcal{K} \subseteq \mathcal{S}.$$

(iii) We will say that $T$ is generically subuniversal and write $T \in \mathcal{S}_{\text{gen}}$ if, for any null sequence of real numbers $z = \{z_n\} \to 0$, there are $t \in T$ and an infinite set $M_t$ such that

$$\{t + z_m : m \in M_t\} \subseteq T.$$

The distinction here is that the ‘translator’ $t$ is now required to be in $T$, so that, in particular, $t$ is a limit point of $T$. This observation forms the basis for a useful connection with (sequential) compactness [BOst8].

The property of $T$ in definition (iii) is typical in the sense captured by the following theorem, due in the measure case in this form to Borwein and Ditor [BoDi], but already known much earlier albeit in somewhat weaker form by Kestelman ([Kes] Th. 3), and rediscovered by Trautner [Trau]. An early generalization due to Harry Miller [MilH] is discussed below; for further generalizations and inter-connections see [BOst10]. We will need a definition.

**Definition 2 (Genericity).** Suppose $\Gamma$ is $\mathcal{L}$ or $\mathcal{B}a$, the class of measurable sets or Baire sets (i.e. sets with the Baire property). We will say that $P \in \Gamma$ holds generically for all $t$ if $\{t : t \notin P\}$ is null or meagre according as $\Gamma$ is $\mathcal{L}$ or $\mathcal{B}a$

**Kestelman-Borwein-Ditor Theorem.** Let $z = \{z_n\} \to 0$ be a null sequence of reals. If $S$ is measurable and non-null/Baire and non-meagre,
then for generically all \( u \in S \) there are points \( t \in S \) arbitrarily close to \( u \) and infinite sets \( M_i \) such that

\[
\{ t + z_m : m \in M_i \} \subset S.
\]

Miller’s generalization replaces the terms \( t + z_n \) by a functional form \( H(t, z_m) \), where \( H \) establishes a differentiable homotopy to the identity \( i(t) := t \) (see Section 2 for a definition). A stronger form still, implying Miller’s Theorem, is derived in Section 2 (the Generic Reflection Theorem), though we do not aim for maximal generality but rather for brevity. See also [BOst3] Section 2.1 Note 3, [BOst4] Section 2.1 Note 1. For proofs see the original papers [Kes], [BoDi] and [MilH]. For a fuller discussion of the status of Miller’s Theorem see [BOst10]. For a topological (Baire category) generalization, not involving differentiability properties, see [BOst11].

**Note.** Observe that, in our conditions, it is enough that our set, \( S \) say, should contain some measurable non-null set, \( T \) say (i.e. that \( S \) have positive inner measure) – \( S \) itself need not be measurable, and similarly in the Baire case; shrinking \( S \) to this measurable/Baire subset \( T \) gives the more general result for any such \( S \). We retain the formulation above for convenience in expressing the conclusion, ‘generically for all \( t \in T \)’.

In [BOst6], while developing ‘automatic property theory’ for subadditive and convex functions, we made explicit two properties of sets linking affine and combinatorial features (which are implicit in earlier work of Csaszar [Csa] and Kominek [KomZ] and are ‘local’ in character). The underlying purpose was to require that functions of a certain class (e.g. additive, subadditive, convex) should be well enough behaved on such sets to elicit strong properties, for instance continuity. The ‘combinatorial-affine’ properties hold generically in relation to Lebesgue non-null/Baire non-meagre sets as a consequence of the Kestelman-Borwein-Ditor Theorem. In Section 2 we identify far richer ‘combinatorial-geometric’ features and a stronger form still of this theorem. We now recall the Averaging Lemma and definitions from [BOst6] that we need.

**Averaging Lemma** ([BOst6], Section 2). A set \( T \) is subuniversal iff it is ‘averaging’, that is, for any null sequence \( \{ z_n \} \to 0 \), any given point \( u \in T \), and with \( u_n := u + z_n \) (thus an arbitrary convergent sequence, but with limit
in \( T \), there are \( w \in \mathbb{R} \) (an averaging translator) and \( \{v_n\} \subseteq T \) such that, for infinitely many \( n \in \omega \), we have:

\[
  u_n = \frac{1}{2}w + \frac{1}{2}v_n.
\]

Equivalently, there are \( w \in \mathbb{R} \) (a reflecting translator) and \( \{v_n\} \subseteq T \) such that, for infinitely many \( n \in \omega \), we have:

\[
  v_n = \frac{1}{2}w + \frac{1}{2}u_n.
\]

The averaging notion appears implicitly in [KomZ] and the reflecting notion in [Csa]. The latter name is suggested by the reflecting property:

\[
  (v_n - u) \approx (v_n - u_n) = -(v_n - w).
\]

The Lemma may thus be summarized symbolically:

\[
  \mathcal{S}_{\text{ref}} = \frac{1}{2} \mathcal{S} = \mathcal{S} = 2 \mathcal{S} = \mathcal{S}_{\text{av}}.
\]

**Definition 3.** Say that a set \( S \) is strongly averaging if some \( T \subseteq S \) (henceforth a strong core of \( S \)) has the following property:

For any null \( \{z_n\} \to 0 \), any bounded sequence \( \{\gamma_n\} \) and any \( \{u_n = u + z_n\} \to u \in T \), there are \( w \in \mathbb{R} \) (a translator) and an increasing sequence \( m(n) \) such that, for \( n \in \omega \), we have

\[
  v_n := \gamma_n w + (1 - \gamma_n)u_{m(n)} \in T.
\]

Equivalently, for any null \( \{z_n\} \to 0 \), and any sequence \( \{\alpha_n\} \) bounded away from zero, there are points \( t_n \in T \) and an increasing sequence \( m(n) \) such that for \( n \in \omega \) the following affine combinations are constant, i.e.

\[
  \alpha_n t_n + (1 - \alpha_n)u_{m(n)} := w \text{ for some } w \in \mathbb{R}.
\]

Thus \( \alpha_n \equiv -1 \) and \( \alpha_n \equiv 2 \) (or \( \gamma_n \equiv \frac{1}{2} \)) yield respectively the reflecting and the averaging case.

A further definition, more general still, may be introduced but with some extra terminology.
Definition 4. Let \( \alpha = \{ \alpha_n \} \) be a sequence bounded away from zero, that is \( |\alpha_n| \geq \eta \), for all \( n \) and some \( \eta > 0 \). We say that the null sequence \( z = \{ z_n \} \) is an \( \alpha \)-uniformly null sequence if, for some positive constant \( K \),

\[
|\alpha_n z_n| \leq K 2^{-n}, \text{ for all } n \in \omega.
\]

If \( \alpha_n \equiv 1 \), we will simply say that \( z \) is a uniformly null sequence, or that \( z \to 0 \) uniformly (for functional versions see the next Definition).

Definition 5 (\( \alpha \)-uniformly null function sequence). Let \( \alpha = \{ \alpha_n \} \) be a sequence bounded away from zero. We say that the sequence of functions \( z(.) = \{ z_n(.) \} \) is an \( \alpha \)-uniformly null function sequence if each \( z_n(.) \) is measurable and, for all \( u \), the Császár condition holds, namely for some constant \( K \)

\[
\max\{|z_n(u)|, |\alpha_n z_n(u)|\} \leq K \cdot 2^{-n}. \quad \text{(Cs)}
\]

If \( \alpha_n \equiv 1 \), we will say that \( z \) is a uniformly null function sequence, or that \( z \to 0 \) uniformly. When each \( z_n(t) \equiv z_n \) is a constant function we speak of a constant null function sequence. In this case it is easy to meet the Császár condition by passing to an appropriate subsequence \( \{ z_{k(n)} \} \).

Definition 6. We say that \( h : \mathbb{R} \to \mathbb{R} \) is bi-Lipschitz (a notion implicit in [Br]) if, for some \( \alpha, \beta \),

\[
0 < \alpha \leq \frac{h(u) - h(v)}{u - v} \leq \beta, \text{ for } u \neq v.
\]

In particular, \( h \) is continuous and strictly increasing, and so is invertible with continuous and strictly increasing inverse, also bi- Lipschitz, and differentiable except possibly for at most countably many points. The bi-Lipschitz functions preserve density points – indeed images and preimages of null/meagre sets are null/meagre (see [Br], or [CL1] and [CL2]).

Convention: In what follows we will abbreviate the point \( u + z_n(u) \) to \( u_n \), for typographical convenience.

Genericity in its local context is captured by the Kestelman-Borwein-Ditor Theorem. In its uniform context it is illustrated by the following result, which is a special case of a theorem to be proved in Section 2.
Theorem (On generic uniform behaviour). Let \( S \) be measurable/Baire, \( \alpha = \{ \alpha_n \} \) be any sequence bounded away from zero, \( z \to 0 \) be a uniformly null function sequence and \( t \to t + z_n(t) \) be locally bi-Lipschitz.

(i) **Generic Inclusion**: For generically all \( u \in S \), there is some infinite set \( \mathcal{M}_u \) such that \( \{ u_m : m \in \mathcal{M}_u \} \subseteq S \).

(ii) **Generic Averaging**: For generically all \( u \in S \), there are points \( t_m \in S \) and an infinite set \( \mathcal{M}_u \) such that \( \frac{1}{2} t_m + \frac{1}{2} u = u_m \).

(iii) **Generic Reflection**: For generically all \( u \in S \), there are points \( t_m \in S \) and an infinite set \( \mathcal{M}_u \) such that \( t_m = \frac{1}{2} u + \frac{1}{2} u_m \).

(iv) **Generic Centering**: For generically all \( u \in S \), there are points \( t_m \in S \) and an infinite set \( \mathcal{M}_u \) such that \( u = \frac{1}{2} t_m + \frac{1}{2} u_m \).

(v) **Strong (Császár) Generic Reflection**: For generically all \( u \in S \), there are points \( t_n \in S \) and increasing sequence \( m(n) \) such that, for \( n \in \omega \),

\[
    u = \alpha_n t_n + (1 - \alpha_n) u_{k(n)}. \tag{3}
\]

(vi) **Generic Bilateral Centering**: For generically all \( u \in S \), and any uniform \( z \to 0 \), there are points \( t_m \in S \) and an infinite set \( \mathcal{M}_u \) such that \( u = \frac{1}{2} t_m + \frac{1}{2} u_m \), where \( u_m \in S \).

**Note.** Observe the distinction between (3) in the Theorem and (2) in Definition 3. Given a \( u \) the \( w \) in the latter is guaranteed to exist by the KBD Theorem as some point in \( R \). In the uniform setting, we both start and finish with \( u \) (a ‘fixed point’ result).

Returning to the theme of the present section (local genericity), our investigations on the measure side make use of metric density, so we begin by developing some associated notation and the definitions.$z$-limits. We recall some standard items first.

**Definition 7 (Metric density).** Put \( I_\delta(u) = (u - \delta, u + \delta) \).

(i) In this definition we do not distinguish between a set \( T \) and the property \( t \in T \). We say that the property \( T \) holds in some neighbourhood of \( u \) quasi always if, for some \( \delta > 0 \), the set \( I_\delta(u) \setminus T \) is meagre (of first category).

(ii) For \( T \) measurable, we say that \( u \) is a density point of \( T \) if the metric density at \( u \), defined by

\[
    d(T, u) := \lim_{\delta \to 0} \frac{|T \cap I_\delta(u)|}{2\delta},
\]
exists and is 1. Otherwise the point is said to be an exceptional point of $T$.
(The definitions do not require $u$ to be in $T$.)

(iii) If $u$ is a density point of $T$ then, for any $\varepsilon > 0$, there is $\delta > 0$ so that
\[
\frac{|T \cap I_\delta(u)|}{2\delta} \geq (1 - \varepsilon);
\]
we then say that the property $T$ holds in some neighbourhood of $u$ ‘nearly always’.

Note. The set $U = \{x \in T : (\exists \delta > 0)|T \cap I_\delta(x)| = 0\}$ has measure zero on $T$. By the Lebesgue Density Theorem ([Hal] Section 61, [Oxt] Th. 3.20), there is a sequence of measurable functions converging in measure almost everywhere to unity outside the set $U$. Hence the density points are a measurable set.$^1$ (It also follows that, for almost all points of $\mathbb{R}$, the metric density of $T$ exists and is zero or 1; see [Goff].)

Definition 8. We call a uniformly null sequence $z(.) = \{z_n(.)\}$ bi-Lipschitz if the mappings $u \to u_n$ are bi-Lipschitz uniformly in $n$, i.e. for some $\alpha, \beta$ and all $n$ we have
\[
0 < \alpha \leq 1 + \frac{z_n(u) - z_n(v)}{u - v} \leq \beta, \text{ for } u \neq v.
\]
In particular $z'_n$ is bounded away from $-1$, except perhaps at countably many points.

Definition 9 (z-limit set – unilateral, bilateral). If $z = \{z_n\} \to 0$ is any null sequence of reals, the unilateral and bilateral z-limit sets are defined for $T$ a fixed subset of $\mathbb{R}$) by:
\[
z(T) := \bigcap_{k \in \omega} \bigcup_{n \geq k} (T - z_n) \quad \text{and} \quad z^+(T) := \bigcap_{k \in \omega} \bigcup_{n \geq k} (T - z_n) \cap (T + z_n).
\]
The next definition is just the corresponding functional form of the above.

Definition 10 (functional z-limit set – unilateral, bilateral). For $z(.) = \{z_n(.)\}$ an $\alpha$-uniformly null function sequence, put $i_n^\pm(t) = t \pm z_n(t)$. The corresponding z-limit sets are defined by:
\[
z(T) := \bigcap_{k \in \omega} \bigcup_{n \geq k} i_n^-(T) \quad \text{and} \quad z^\pm(T) := \bigcap_{k \in \omega} \bigcup_{n \geq k} i_n^-(T) \cap i_n^+(T).
\]

$^1$Alternatively, one can deduce this result by showing that, for $H$ a $G_\delta$ set, the density points of $H$ form an analytic set (thus a measurable set).
Evidently $i^+_\varepsilon(T) = \{ t \pm z_n(t) : t \in T \}$. Thus one has:

**Lemma.** For any null sequence $z = \{z_n\} \to 0$, $t \in z(T)$ (resp. $z^\pm(T)$) iff there is infinite $\mathcal{M}_t \subseteq \omega$ such that

\[ \{ t + z_m : m \in \mathcal{M}_t \} \subseteq T, \quad \text{resp.} \quad \{ t + z_m, t - z_m : m \in \mathcal{M}_t \} \subseteq T. \]

Similarly, for $z(.) = \{z_n(.)\}$ an $\alpha$-uniformly null function sequence, $t \in z(T)$ (resp. $z^\pm(T)$) iff there is infinite $\mathcal{M}_t \subseteq \omega$ such that

\[ \{ t + z_m(t) : m \in \mathcal{M}_t \} \subseteq T, \quad \text{resp.} \quad \{ t + z_m(t), t - z_m(t) : m \in \mathcal{M}_t \} \subseteq T. \]

We denote the set of density points of $T$ by $\Phi_N(T)$ (cf. [CLO], or [Wil1] and [Wil2] for background). The notation refers to the $\sigma$-ideal $N$ of Lebesgue null sets. (See [CL1] for a discussion of density topologies generated by $\sigma$-ideals.) By the Lebesgue Density Theorem almost all points of a measurable set are density points and so $\Phi_N(\Phi_N(T)) = \Phi_N(T)$. See [Szenes] for a study of the exceptional points $E_N(T)$.

If $u$ is density point of both $S$ and $T$ it follows that $u$ is a density point of $S \cap T$. This justifies the definition of the $d$-topology, the ‘density topology’, on $\mathbb{R}$ given by

\[ T_N = \{ T \in L : T \subseteq \Phi_N(T) \}, \]

introduced in [GoWa] (see also [HauPau]) and studied also in [GNN] (cf. [CL2], and for a textbook treatment [Kech]).

**Lemma (Fundamental Lemma on Genericity).** Let $z = \{z_n\} \to 0$ be any null sequence. Suppose that every point of $T$ is a density point, i.e. $T \in T_N$. Then, for any $u \in T$, $u$ is a density point of $z(T)$ and of $z^\pm(T)$, or in symbols:

\[ u \in \Phi_N(z(T)) \quad \text{and} \quad u \in \Phi_N(z^\pm(T)). \]

**Proof.** We consider the weaker result first. Let $u \in T$. As $T \in T_N$, $u$ is a density point of $T$ and hence for any $\varepsilon > 0$ there is $\delta$ such that

\[ \frac{|T \cap I_\delta(u)|}{2\delta} \geq (1 - \varepsilon), \]

i.e. nearly all of $I_\delta(u)$ has $T$ (is in $T$). Let $\eta = 2\delta\varepsilon$. Now, for $n > N$, $|z_n| < \eta$ and so putting $T_n = T \cap (T - z_n)$

\[ |T_n \cap I_\delta(u)| \geq (1 - \varepsilon)2\delta - 2\eta, \]

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i.e. nearly all of $I_\delta(u)$ has (is in) $T - z_n$. Similarly, for $k > N$, 
\[ \left| \bigcup_{n \geq k} T_n \cap I_\delta(u) \right| \geq (1 - \varepsilon)2\delta - 2\eta. \]

Hence we have (cf. [EKR])
\[ |z(T) \cap I_\delta(u)| = \left| \bigcap_{k \in \omega} \bigcup_{n \geq k} T_n \cap I_\delta(u) \right| \geq (1 - 2\varepsilon)2\delta, \]
i.e. nearly all of $I_\delta(u)$ is covered by $z(T)$. Thus $u$ is a density point of $z(T)$ and of course $z(T) \subset T$.

For $z^\pm$ the corresponding property may be proved similarly, noting that, for $\delta$ small enough, nearly all of $I_\delta(u)$ has both of $T \pm z_n$ (for all large enough $n$). □

**Corollary.** For $u \in T$, there is arbitrarily close to $u$ a point $t \in z(T)$ such that $t \in T$ and \( \{ t + z_m : m \in M_t \} \subset T \), for some infinite $M_t$.

We turn now to the proof of the Kestelman-Borwein-Ditor Theorem.

**Proof of the Kestelman-Borwein-Ditor Theorem.** In the measure case, denote the set of density points of $S$ by $T$ and apply the Fundamental Lemma on Genericity. □ (Measure)

In the Baire case, if $S = I \setminus M \cup M'$, where $I$ is an interval and $M, M'$ are meagre, take $T = I \setminus M$. We will show that for any \( \{ u_n \} \to u \in T \), there are $v \in T$ and an infinite $M_v$ such that
\[ \{ v + u_m : m \in M_v \} \subset S. \]

Select $\delta > 0$ so that $J = (u - \delta, u + \delta) \subset I$. We wish to pick $v, v_n \notin M$, where we will later put $v_n := v + u_n$. This means we require in particular that $v + u_n \notin M$. So pick $v$ in $J$ to avoid the meagre set
\[ M \cup \bigcup_{n \in \omega} M - u_n. \]

Now, for $n$ large enough, $u_n \in J$. By choice $v, v_n \notin M$. Hence, for large enough $n$, we have $v, v_n \in T$, as required. □ (Category)

**Note.** With the following modifications, the proof just given establishes also the Bilateral Genericity Theorem of Section 6 of [BOst6]. In the measure
case one needs only replace \( z(T) \) by \( z^\pm(T) \); in the category case one needs to replace the sequence \( u_n \) by the null sequence \( z_n \) and then to replace \( M - u_n \) by \((M + z_n) \cup (M - z_n)\).

**Corollary (Functional Form).** Let \( \{z_n(t)\} \to 0 \) be any \( \alpha \)-uniformly null function sequence such that, for generically all \( u \), each of the functions \( i_n^\pm(t) = t \pm z_n(t) \) is a density preserving map on a neighbourhood of \( u \). Let \( T \in T_N \). Then, as before, for any \( u \in T \),

\[
u \in \Phi_N(z(T)) \quad \text{and} \quad u \in \Phi_N(z^\pm(T)) .
\]

**Proof.** As before consider the weaker result. It is enough to replace \( T_n \) by

\[
T_n = i_n^-(T) = \{ t - z_n(t) : t \in T \},
\]

and to note that (i) density points of \( T \) are transformed to density points of \( T_n \) and (ii) for any \( t \in T \cap I_\delta(u) \), we have \( t \pm z_n(t) \in I_{(1+\varepsilon)\delta}(u) \), for all \( n \) large enough. \( \square \)

## 2 Genericity: The Generic Reflection Theorem

Throughout this subsection \( f : \mathbb{R} \to \mathbb{R} \) will always denote a continuously differentiable function. For all small enough \( z \),

\[
f_z(u) := f(u + z) + u - f(u)
\]

is for all \( u \) invertible locally at \( u \) (as a function of \( u \)) and, in the measurable case, locally Lipschitz. Thus pre-images of meagre/null sets under \( f_z(.) \) are meagre/null.

**Example.** The linear case \( f(t) = \alpha t \) is of particular interest. Here

\[
f_z(u) = u + \alpha z .
\]

Taking matters one step further, for \( \{z_n\} \) a null sequence of reals, let

\[
f_n(u) = f(u + z_n) + u - f(u).
\]
Assuming \( f \) is differentiable at \( u \), for large enough \( n \), we have

\[
f'_n(u) = 1 + [f'(u + z_n) - f'(u)]
\]

and so \( f_n \) is locally a homeomorphism, and hence pre-images of meagre sets are meagre. Furthermore, since

\[
f_n(u) - f_n(v) = [f(u + z_n) - f(v + z_n)] + [u - v] - [f(u) - f(v)],
\]

\( f_n \) is locally Lipschitz if \( f \) is. Under such circumstances images under \( f \) of null sets are null. The latter property continues to hold when the real sequence \( z_n \) is replaced by a uniformly null bi-Lipschitz sequence \( z_n(.) \) since

\[
|f(u + z_n(u)) - f(v + z_n(u))| \leq M\{||u - v|| + ||z_n(u) - z_n(v)||\},
\]

where \( M \) is a Lipschitz constant for \( f \).

We also have, at any point \( t \) where \( z_n \) is differentiable, that

\[
f'_n(t) = 1 + f'(t + z_n(t))(1 + z'_n(t)) - f'(t)
\]

\[
= [1 + f'(t)z'_n(t)] + [f'(t + z_n(t)) - f'(t)] [1 + z'_n(t)].
\]

Let \{\( z_n(.) \)\} be bi-Lipschitz, so that \( z'_n(t) \) is bounded (in both \( t \) and \( n \)). Consider \( u \) with \( f'(u) \neq 0 \). Then the function \( f_n(t) \) is increasing in some neighbourhood \( U \) of \( u \), for all \( n \) large enough iff, for all large enough \( n \),

\[
1 + f'(t)z'_n(t) > 0, \text{ for all } t \in U \text{ with at most countably many exceptions.}
\]

**Example.** The linear case \( f(t) = \alpha t \) is again of particular interest. Here

\[
f_n(u) = u + \alpha z_n(u).
\]

For \( \alpha > 0 \), the derivative condition for \( f_n \) to be increasing reads

\[
1 + \alpha z'_n(t) \geq 0, \text{ or } z'_n(t) \geq -\frac{1}{\alpha}.
\]

So, if the null function sequence is constant (as in Kestelman-Borwein-Ditor Theorem), with \( z_n(t) \equiv z_n \), the condition is satisfied, as it reduces simply to

\[
0 \geq -\frac{1}{\alpha}.
\]

Our next definition capitalizes on the above observations.
**Definition.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $\{z_n(t)\}$ is a null function sequence. Define the $f$-conjugate sequence $\{\bar{z}_n(t)\}$ by

$$\bar{z}_n(u) = f(u + z_n(u)) - f(u).$$

Thus

$$u + \bar{z}_n(u) = f(u + z_n(u)) + u - f(u) = f_{z_n(u)}(u).$$

**Example.** For the linear case $f(t) = \alpha t$, we have $\bar{z}_n(u) = \alpha z_n(u)$ is manifestly bi-Lipschitz if $\{z_n(.)\}$ is.

We record the following Lemma, whose routine proof we omit.

**Lemma.** The $f$-conjugate sequence is null: $\{\bar{z}_n(u)\} \to 0$. If further, $f$ is continuously differentiable near $t = u$ with non-zero derivative at $t = u$, then $\{\bar{z}_n(u)\}$ satisfies the Császár condition (Csa). Moreover, if the derivative $f'(t)$ is bounded away from 0 and infinity in a neighbourhood of $t = u$ then $\{\bar{z}_n(u)\}$ is bi-Lipschitz.

**Theorem (Generic Reflection Theorem).** Let $T$ be measurable/Baire. Let $f(.)$ be continuously differentiable and non-stationary at generically all points. Let $\{z_n(.)\} \to 0$ be a uniformly null sequence that is bi-Lipschitz with

$$1 + f'(t)z_n(t) > 0, \text{ for all } n,$$

for generically all $t \in T$. Then, for generically all $u \in T$, there is an infinite set $M_u$ such that

$$f(u_n) + u - f(u) \in T, \text{ for all } n \in M_u. \quad (7)$$

In particular, if $f$ is linear and $f(t) = \alpha t$ with $\alpha \neq 0$, then for generically all $u \in T$, there is an infinite set $M_u$ such that

$$\alpha u_n + (1 - \alpha)u \in T \text{ for all } n \in M_u, \text{ where } u_n = u + z_n(u). \quad (8)$$

**Proof.** (i) In the Baire case, we may as well assume that $T = I \setminus M$, where $I$ is an interval on which $f$ is non-stationary and $M$ is meagre. Consider any uniformly null sequence $\{z_n(.)\} \to 0$. Let $u \in T$. Put

$$g_u(z) = f(u + z) + u - f(u) \text{ and } v_n = g_u(z_n(u)).$$
Choose $\delta > 0$ so that $I_\delta(u) \subseteq I$. As $g_n(.)$ is continuous and $g_n(0) = u$, for $n$ large enough we have $v_n \in I_\delta(u)$. Let

$$f_n(u) := u + f(u + z_n(u)) - f(u),$$

so that $v_n = f_n(u)$.

Since $v_n \in I$, we require further that

$$v_n \notin M, \text{ i.e. } f_n(u) \notin M.$$

We may thus satisfy (7) by restricting $u$ to the complement of the meagre set

$$M' := \bigcup_{n \in \omega} f_n^{-1}(M).$$

Here we are using the derivative condition (6) to guarantee that $f_n(.)$ is invertible (and we refer to (4) and (5) to see that $f_n$ is Lipschitz). This establishes the Baire case. (Baire) \(\Box\)

(ii) In the measure case, let $f$ be continuously differentiable and non-stationary in a neighbourhood $I_\delta(u_0)$ of $u_0$. Put

$$f_n(t) = f(t + z_n(t)) + t - f(t),$$

so that

$$f_n(t) = t + \bar{z}_n(t),$$

where $\bar{z}_n(t)$ is the $f$-conjugate null sequence. Working in the neighbourhood $I_\delta(u_0)$, let $u = g_n(v)$ be a local inverse of $v = f_n(u)$ whose existence is guaranteed by (6). So, for fixed $u$ and $n$,

$$f_n(u) = u + f'(\tilde{u}_n)z_n(u),$$

for some $\tilde{u}_n := \tilde{u}_n(u)$ between $u$ and $u + z_n(u)$. Let $\varepsilon > 0$ be given. If $n$ is large enough so that $z_n(u)$ is (uniformly) small enough, then we have $|f'(\tilde{u}_n)| < M$, say. Thus

$$|f_n(u) - u| < M|z_n(u)|.$$

Hence, on substituting $g_n(v)$ for $u$, we have

$$|v - g_n(v)| < M|z_n(g_n(v))|,$$

which is arbitrarily small for $\delta$ small enough and $n$ large enough. Put

$$w_n(t) := g_n(t) - t,$$
so that $t + w_n(t)$ is density-preserving (since $g_n$ is so). Note that $w_n(f_n(u)) = -\bar{z}_n(u)$ or $w_n(v) = -\bar{z}_n(g_n(v))$, since
\[-w_n(v) = v - g_n(v) = f_n(u) - u = f(u + z_n(u)) - f(u) = \bar{z}_n(u).\]

Put
\[T_n = g_n(T) \quad \text{and} \quad W = \bigcap_{k \in \omega} \bigcup_{n \geq k} T_n.\]

Then
\[W = w(T).\]

Indeed $W$ can be obtained from $z(T)$ by replacing $z_n(t)$ by $w_n(t)$ and here the displacement $w_n(v)$ is at most $M|z_n(u)|$, i.e. arbitrarily small for $n$ large enough. Referring to Section 1 for the definition of the $z$-limit (Definition 5) and the Corollary to the Fundamental Lemma, we have for $u$ a density point of $T$ that $u$ is a density point of $W$. Hence, for $u \in T \setminus E(W)$ (i.e. for $u$ avoiding the null set of exceptional points of $W$), we have $u \in W$. Thus, for some infinite $M_u$ and some $v_n \in T_n$, we have
\[u = g_n(v_n), \quad \text{for } n \in M_u.\]

Equivalently, for almost all $u \in T$, we have
\[v_n = f_n(u) = f(u_n) + u - f(u), \quad \text{for } m \in M_u, \quad \text{where } u_n = u + z_n(u),\]
as asserted. (Measure) $\square$

**Comments and further generalization**

1. As a matter of convenience, one can avoid the assumption that $f$ is generically non-stationary, by relativizing the theorem to an open set $U \subseteq \mathbb{R}$ on which the derivative $f'$ is of constant sign, say $f'(u) > 0$ for $u \in U$.

2. For the quadratic case $f(t) = \alpha t^2$, the generic reflection condition (7) reads:
\[\alpha u_n^2 + u - \alpha u^2 \in T, \quad \text{for all } n \in M_u,\]

which goes beyond the affine combination result (8) obtained for $f(.)$ linear.

3. The argument can take in the Császár-type reflection condition. With $\alpha_n$ a sequence bounded away from zero, the theorem applies to the more general format
\[v = f_n(u) = u + \alpha_n f(u + z_n) - \alpha_n f(u),\]
provided the Császár condition (Csa) is met, i.e. for some subsequence \( k(n) \),
\[
|\alpha_n z_{k(n)}| \to 0.
\]
Indeed, with \( w_n(u) \) between \( u \) and \( u + z_n \) as before, we have
\[
\alpha_n f(u + z_n) - \alpha_n f(u) = \alpha_n z_n f'(w_n(u)).
\]
4. Suppose that \( \phi \) is a density-preserving homeomorphism. Then the
theorem applies to the more general format
\[
v = f_n(u) = f(u + z_n) + \phi(u) - f(u),
\]
where \( z_n \) is a null sequence. Indeed, let \( \psi = \phi^{-1} \) and put \( w = \psi(u) \). We may
now re-write the proof employing the definition:
\[
v = f_n(u) = f(\psi(u) + z_n) + u - f(\psi(u)).
\]
We shall still have, as before with \( w_n := w_n(u) \), that
\[
f_n(u) = u + f'(w_n) z_n(u).
\]
5. We turn now to Miller’s Theorem, and show how it follows from ours
First we need a definition (the terminology is ours and localizes the one in
[MilH]).

Definition (Miller homotopy, cf. [MilH]). Let \( U \) be open and let \( I \)
be a non-degenerate interval (possibly infinite, or semi-infinite). We call a
function \( H : U \times I \to \mathbb{R} \) a Miller homotopy acting on \( U \) with distinguished
point \( z_0 \) if:
(i) \( H(u, z_0) \equiv u \), for all \( u \in U \),
(ii) \( H \) has continuous first-order partial derivatives \( H_1 \) and \( H_2 \), and
(iii) \( H_2(u, z_0) > 0 \), for all \( u \in U \).

Note. As the function \( H \) is differentiable, and hence jointly continuous, it
is natural to regard it as establishing a homotopy to the identity (albeit utilizing
a distinguished point \( z_0 \) other than 0, and some interval about \( z_0 \) instead
of the customary unit interval). Condition (iii) is only a non-stationarity
requirement (map \( z \to -z, z_0 \to -z_0 \), if \( H_2(u, z_0) < 0 \)).

Miller’s Homotopy Theorem. Let \( H \) be a Miller homotopy acting on
an open set \( U \) with distinguished point \( z_0 \). Let \( z_n \to z_0 \) be a null sequence
and let $T \subseteq U$ be measurable and non-null/Baire and non-meagre. Then, for generically all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that

$$\{H(t, z_m) : m \in \mathbb{M}_t\} \subseteq T.$$  

It is shown in [BOst10] that the functions

$$z_n(t) := H(t, z_n) - t,$$

satisfy the assumptions of the Generic Reflection Theorem locally. They also allow a functional reinterpretation of Miller’s result: viewing the sequence of functions $\{z_n(t)\} \to 0$ as the datum, the conclusion of Miller’s theorem now reads

$$\{t + z_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$  

Miller’s result thus follows by taking $f(u) = u$.

Postscript. David Borwein (born in 1924 in Lithuania) took his PhD at University College, London in 1950, where he was a pupil of Stephen Bosanquet (1903-1984), and a contemporary of Ambrose Rogers (1920-2005). His lifelong principal interest, arising from his work under Bosanquet, is in summability theory; [BoDi] arose from answering a question of Erdős, but is relevant also to [Kes] by Hyman Kestelman (1908-1983) of UCL. The first author’s link with David is through a shared interest in summability theory, particularly the Tauberian aspects; the second author (ancestry: Hardy-Bosanquet-Rogers) is linked with him, as a pupil of Bosanquet, Kestelman and Rogers, through UCL (where the authors met at seminars in the early 1970s). It is thus a pleasure for both authors to dedicate this paper to David on the occasion of his 85th birthday.

References


3 Appendix: The Császár Genericity Theorem

Theorem (Császár’s Genericity Theorem, [Csa], or [Kucz] p 223). Let $\alpha_n$ be a sequence bounded away from zero. Let $z = \{z_n\} \to 0$ be any $\alpha$-uniformly null sequence. Let $S$ be measurable, non-null/Baire, non-meagre. There is non-null/non-meagre $T \subseteq S$ such that, for generically any $u \in T$, there is an $x \in \mathbb{R}$ and points $t_n \in T$ such that, for some sequence $k(n)$,

$$x = \alpha_n t_n + (1 - \alpha_n) u_{k(n)}, \text{ for } n \in \omega, \text{ where } u_n = u + z_n.$$

**Proof.** Let $\alpha = \{\alpha_n\}$ be bounded away from zero, and say that $\alpha_n \geq \eta$, for all $n$.

(i) In the Baire case we may assume that $S = I \setminus M \cup M'$ for $I$ an interval and some meagre $M, M'$. Take $T = I \setminus M$. Pick $\delta > 0$ so that $I_{2\delta}(0) \subseteq I - u$. Since $\alpha_n \geq \eta > 0$ we have

$$\alpha_n(I - u) \supseteq \eta I_{2\delta}(0) = I_{2\delta\eta}(0).$$

For $n$ large enough we may as well assume that $|z_n(1 - \alpha_n)| < \delta\eta$, for $n > N$, say. Otherwise, replace $z_n$ by $\hat{z}_n = z_{k(n)}$, for some subsequence. Put $\check{z}_n = (\alpha_n - 1) \hat{z}_n$. Then for $n > N$, we have

$$\alpha_n(I - u) - \check{z}_n \supseteq I_{\delta\eta}(0),$$

and hence also

$$\alpha_n T_u - \check{z}_n = \alpha_n [(I - u) \setminus (M - u)] - \check{z}_n \supseteq I_{\delta\eta}(0) \setminus (M - u).$$

Thus

$$\check{z}(\{\alpha_n T_u\}) := \bigcap_{k \in \omega} \bigcup_{n \geq k} (\alpha_n T_u - \check{z}_n) \supseteq I_{\delta\eta}(0) \setminus (M - u).$$

Since $M - u$ is meagre we may select $w \in I_{\delta\eta}(0) \setminus (M - u)$. By the Lemma there are points $t_n \in T$ so that for some infinite $M_w$ we have

$$w = \alpha_n (t_m - u) + (1 - \alpha_m) z_{k(m)}, \text{ for } m \in M_w.$$

Hence

$$w + u = \alpha_m t_m + (1 - \alpha_m)(u + z_{k(m)}), \text{ for } m \in M_w.$$
Putting \( x = w + u \) and re-writing, with \( \mathbb{M}_x \) for \( \mathbb{M}_w \), establishes the theorem. (Baire) \( \square \)

(ii) We consider the measure case. Suppose that \( \alpha_n \geq \eta \) for all \( n \). Let \( T \) consist of the points of \( S \) that are density points of \( S \). Suppose that \( u \in T \). Then 0 is a density point of \( T - u \). Let \( \varepsilon > 0 \) be given. There is \( \delta > 0 \) so that

\[
|I_\delta(0) \cap (T - u)| \geq 2\delta(1 - \varepsilon).
\]

Put

\[
T_n = \alpha_n(T - u) + (1 - \alpha_n)z_n \quad \text{and} \quad T^k = \bigcup_{n \geq k} T_n.
\]

Replacing \( z_n \) by a suitable subsequence \( \tilde{z}_n = z_{k(n)} \) we may assume that \( \tilde{z}_n := (\alpha_n - 1)\tilde{z}_n \to 0 \). Hence, for some \( N \), we have \(|(1 - \alpha_n)\tilde{z}_n| \leq \varepsilon \eta \delta \), for \( n > N \). Evidently \(|I_\delta(0) \cap \alpha_n(T - u)| \geq \eta |I_\delta(0) \cap (T - u)| \).

Thus \( I_\delta(0) \cap T^k \) is a bounded decreasing sequence, with measure bounded away from zero since

\[
|T^k \cap I_\delta(0)| \geq 2\eta \delta(1 - 2\varepsilon).
\]

Hence writing

\[
\bar{z}(\{\alpha_n(T - u)\}) := \bigcap_{k \in \omega} \bigcup_{n \geq k} T_n,
\]

we have \(|\bar{z}(\{\alpha_n(T - u)\}) \cap I_\delta(0)| \geq 2\eta \delta(1 - 2\varepsilon) \) (see for instance [EKR]). Thus 0 is a density point of the set \( \bar{z}(\{\alpha_n(T - u)\}) \), which is therefore non-empty. Let \( w \) be a member. Then there are \( t_n \in T \) and an infinite \( \mathbb{M}_w \) such that, for \( m \in \mathbb{M}_w \),

\[
w = \alpha_n(t_n - u) + (1 - \alpha_n)\tilde{z}_n.
\]

Putting \( x = w + u \) and writing \( \mathbb{M}_x = \mathbb{M}_w \) we have, for \( m \in \mathbb{M}_w \), that

\[
x = \alpha_n t_n + (1 - \alpha_n)u_{k(m)},
\]

as asserted. \( \square \)