Returns to costly pre-bargaining claims : Taking a principled stand

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Abstract

We construct a 'divide the dollar' bargaining game which formalizes Schelling's notion of a 'qualitative commitment'. This requires a substantial capitulation cost to be incurred – discontinuously – if and only if a player accepts a share of an asset below his pre-announced 'claim' on it, no matter how little below. The 'commitment game' opens with an 'announcement round' in which the two players simultaneously announce their claims on the asset, and is followed by a Rubinstein alternating-offers 'negotiation subgame'. We determine the unique subgame-perfect, stationary, pure-strategy equilibrium outcome of the commitment game and find it to be efficient. The main feature of the model is that gains, relative to the game without commitment, do result to the first-mover provided the capitulation cost is above a certain threshold. The more the capitulation cost exceeds the threshold, the greater is the gain. The higher the impatience level of the players, the higher the stakes need to be.

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1. Introduction and Main Theorem

1.1. Motivation and model specification

In his celebrated 'Essay on Bargaining' Schelling (1956) points to one source of bargaining power as the '*power to bind oneself*' and discusses observed pre-bargaining union activities undertaken to create a commitment tactic: '*stirring up excitement and determination in the union*' for the purposes of making clear what the negotiators would not accept. In a word: spin. He argues that *'To be convincing, commitments usually have to be qualitative rather than quantitative'* such as staking or '*pledging a principle*', whereupon the agent persuades his adversary '*that he would accept stalemate rather than capitulate and discredit the principle.*'

This paper proposes a quantitative analysis of Schelling's 'qualitative commitment', here translated into the bargainer sustaining a substantial enough capitulation cost if and only if his claim on the asset under negotiation is not met. Our starting point follows the received literature, see for example Muthoo (1999), and defines a version $\mathcal{G}_{c\delta}$ of the 'divide a dollar' game. Thus in common with other authors, we begin with a **preliminary round** ('round -1') in which two players (simultaneously) announce their commitment not to accept a share of the dollar below their own announced 'claim' on the dollar (respectively z_1, z_2). A subsequent penalty structure and a resolution mechanism are required when conflicting claims arise, i.e. when claim pairs are announced summing above the dollar ($z_1 + z_2 > 1$). For example, Muthoo (1996) uses an implicit negotiation mechanism with outcome characterized by the Nash Bargaining Axioms to arrive at a resolution of the claims. Our contribution is to adopt a different approach: we allow the bargainers to resolve claims by the use of an explicit negotiation scheme as afforded by the 'alternating-offers' infinite game-structure of Rubinstein (1982) and (1987). Having made their announcements, the players are allowed to bid freely (make **proposals** for a division of the dollar) in the ensuing **negotiation subgame** $\mathcal{N}(z_1, z_2) = \mathcal{N}_{c\delta}(z_1, z_2)$ comprising 'round 0', 'round 1' etc.[1] Despite this freedom, the players are influenced by the effect on their utilities of the announced claims resulting from the capitulation cost (whenever incurred). The **capitulation cost** is a known penalty, deducted from the pay-off of a player if revoking his or her commitment. The penalty is paid to a third party and is a function of the **concession** (i.e. the amount by which an agent accepts strictly below the announced claim). We assume the penalty structure is antecedent to the game $\mathcal{G}_{c\delta}$ (created by pre-bargaining activity), is known to the two players and is beyond their control in the game $\mathcal{G}_{c\delta}$.

To formalize the qualitative character of the commitment, we take the view that in accepting a share x below the announced claim, z, (so that $x < z$) the bargainer has '*discredited his principle'*, as staked on z, and suffers the same capitulation cost no matter how small the concession $z - x$. We propose interpreting capitulation cost as a fixed proportion c of the original asset under negotiation[2]. It is in this sense that the penalty may be called a **fixed charge**, by contrast to the alternative of a progressive charge. We assume the players' utility reflects both the cost of

[1] Thus in the negotiation subgame, the players, starting with Player 1, take turns in proposing the division of the asset into a proportion x to be awarded to the player currently bidding and a proportion $1 - x$ to the opponent. The opponent may either agree the division, or refuse it, and in so doing close the current round of bidding, becoming in turn the bidder of the new round.

[2] Choosing a fixed proportion, rather than a fixed (absolute) amount, may be taken to reflect a tacit understanding by the two negotiating parties of the need to employ some kind of release mechanism. In Schelling's words this amounts either to a kind of '*casuistry*' for release from a public expectation of the preordained punishment of a broken commitment, or alternatively to a kind of '*rationalized re-interpretation of the original commitment*' (applicable to the terms, at the very least, of the consequent penalty).

making a concession and the passage of time. As for the latter we assume both players apply the same time discount factor δ (with $0 < \delta < 1$). We thus adopt a pay-off structure that has the players valuing an agreement which in 'round n' of $\mathcal{N}(z_1, z_2)$ gives a proportion x of the asset to Player 1 and $1-x$ to Player 2, with $0 \le x \le 1$, by reference to their the **utility** which is defined as follows:

$$u_1(x, n; z_1) = \delta^n w(x, z_1), \quad u_2(x, n; z_2) = \delta^n w(1-x, z_2), \tag{1.1}$$

where $0 < \delta < 1$ denotes the time discount factor, the amount by which the two players perceive the value to have shrunk at the end of any one round, and[3]

$$w(x, z) = x - c \cdot 1_{[0,z)}(x), \tag{1.2}$$

where $1_A(x)$ is the indicator function of the set A. The utility of **stalemate** (perpetual disagreement) is zero; in particular $w(x, z) < 0$, for $x < c$, so that a Player may well receive negative pay-off if offered too little, and will then '*rather accept stalemate*'.

The utilities in the game $\mathcal{N}(z_1, z_2)$ are time homogenous. It seems natural therefore to seek out a subgame-perfect equilibrium supported by stationary (time homogeneous) strategies whereby each player proposes constantly the same share to Player 1 and accepts a minimum own share always at one and the same level, both levels being freely selected.

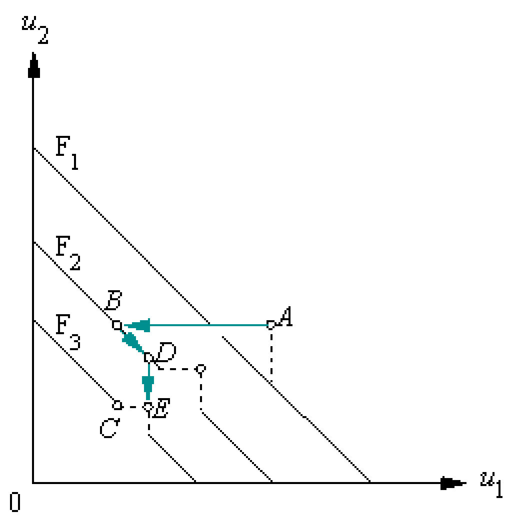

Figure 1. A is a subgame-perfect equilibrium in $\mathcal{N}(z_1, z_2)$

Now consider, for the subgame $\mathcal{N}(z_1, z_2)$, the feasible set of utility-outcome pairs of 'round n', which we denote $\mathcal{F}_n$. Reference to (1.1) and the fact that the pay-off at stalemate is zero shows that $\mathcal{F}_n = \{(u_1, u_2) : u_1, u_2 \ge 0 \text{ and } u_1 = \delta^n w(x, z_1), u_2 = \delta^n w(1-x, z_2) \ge 0$, for some x with $0 \le x \le 1\}$. This set, albeit not in general convex, is star-shaped to the origin. The subgame $\mathcal{N}(z_1, z_2)$ is thus in principle amenable to a variation of the standard geometric arguments for identifying a subgame-perfect equilibrium division arising from the play of a pair of stationary strategies. We take a brief look at the corresponding geometry in order to observe the need for compensatory actions.

Figure 1 shows $\mathcal{F}_1 = \{(z_1, z_2)\} \cup \{(t, 1-c-t) : t > z_1 \text{ or } t < z_1 - c\}$, where[4] $z_1 + z_2 = 1$. The

[3]Thus if $z_1 = z_2 = 0$ we obtain what we term the standard Rubinstein bargaining model with $u_1(x, n; z_1) = \delta^n \cdot x, \quad u_2(x, n; z_2) = \delta^n \cdot (1-x)$.

[4]If $x \ne z_1$we have $u_1(x, 1) + u_2(x, 1)$ equal to $1 - c \cdot [1_{[0,z_1)}(x) + 1_{[0,z_1)}(1-x)]$, i.e. to $1-c$.

set comprises three components: one point and two half-open intervals. For the (appropriately) chosen value of z_1 of this example the point $A = (z_1, z_2)$ is a subgame-perfect outcome of $\mathcal{N}(z_1, z_2)$ supported by stationary (i.e. time independent) strategies.

The geometric justification is the following. The point A is proposed by Player 1. It will be optimal for Player 2 to accept. To verify this she first identifies the point B in $\mathcal{F}_2$ on the horizontal through A; she is indifferent between A and B. But she cannot propose B in 'round 2' without some concession (or side-payment) to Player 1. Why? In the first place the point C vertically below B on $\mathcal{F}_3$ is not even feasible for our selected value of z_1, as C is at the open-end of one of the components of $\mathcal{F}_3$. She must therefore slide down to the point D on $\mathcal{F}_2$ that is vertically above the point of $\mathcal{F}_3$ nearest to C, which is the point Player 1 will propose (under the terms of the stationary strategy) if 'round 3' is reached. That point on $\mathcal{F}_3$ is just the 'twice-delayed' version of A, namely $E = \delta^2 A$ on the radial line OA. Player 1 will propose E in 'round 3' and this proposal will be accepted. Thus (i) D is the contract point of $\mathcal{F}_2$ of least or equal u_1-coordinate to that of E; (ii) A is the contract point on $\mathcal{F}_1$ of least or equal u_2-coordinate to D. Player 2 is in fact better off at D than at E, and at A she is better off still.

We also note that when D is vertically above E (in the limit) Player 2 exhausts the simultaneous possibility of satisfying her own announced claim and compensating Player 1 to the level of utility that he can obtain in 'round 3'. That is, there is a natural upper bound for costs that can be compensated.

It is clear from this analysis that finding sub-game perfect equilibria carries intrinsic combinatorial complications to the standard paradigm as a result of the 'sliding-down' variation. This variation will be formalized in Section 2 in terms of Player i's 'compensated present-values' $d_i(y)$, the contract value from some appropriate division $(x, 1-x)$ in the current round, least or equal to the contract value from $(y, 1-y)$ in the subsequent round; the subgame-perfect equilibria in $\mathcal{N}(z_1, z_2)$ (possibly as many as three) are generated from the fixed points of $d_2 d_1(.)$.We avoid considering complications to the geometric analysis, by favouring an algebraic approach, especially as the claims z_1, z_2 themselves must be checked for mutual optimality; see section 3.3 for some further intuitions.

As a reference point for our work we turn to the game without commitments, the simple Rubinstein negotiation game with $w(x) = x$. This has a unique subgame-perfect equilibrium outcome (which can be interpreted as a limit of terminated versions of the negotiation scheme in analogy to Figure 1) in the form:

$$R(\delta) = (\rho_1, \rho_2), \text{ where } \rho_1 + \rho_2 = 1 \text{ and } \rho_2 = \delta\rho_1,$$

so that $\rho_1(1+\delta) = 1$. Consequently we have

$$\rho_1 = \frac{1}{1+\delta}, \qquad \rho_2 = \frac{\delta}{1+\delta}. \tag{1.3}$$

For later comparison, we note the justification of this formula from the observation that a one-round delay switches the identity of the bidder and, apart from a time shift, the continuation game faced by the new bidder is otherwise identical; hence pay-offs are necessarily transposed and discounted by δ.

The equilibrium point $R(\delta)$ yields a 50:50 split in the limit when δ nears 1. Hence it is plausible in the case of a concave feasible outcome set to resolve conflicting claims by not modelling negotiation, and imposing instead an axiomatic solution. Such an approach, as Muthoo (1996) shows, favours a 50:50 split in equilibrium – ceteris paribus, i.e. when both players have identical linear capitulation costs. The Muthoo (1966) paper remains apparently silent on the second issue raised by Schelling: can one impart a special role to a 'qualitative commitment'. Prima facie, this silence seems to be due to the very approach adopted in Muthoo's paper, by virtue of modelling with continuous penalties. Hence the present paper asks by way of a discontinuity hypothesis: do qualitative commitment tactics overturn the equitable outcome, say in the face of large enough, identical, capitulation costs?

Our answer to this second issue raised by Schelling is in the affirmative. More precisely, relying on the medium of an explicit negotiation structure, we can confirm his tenet by demonstrating a qualitative shift away from the standard 'almost fifty:fifty' outcome. Such are, metaphorically speaking, the wages of spin.

The first main contribution of this paper is to show that, under an explicit negotiation scheme, there is necessarily a unique equilibrium division (arising from stationary strategies), so that by implication, as Muthoo (1996) points out, this uniqueness offers a 'theory of strategic bargaining'. The second main contribution is to show how, in the model, the size of the capitulation cost affects the equilibrium outcome and leads to a division veering well away (when δ nears unity) from the 'almost 50:50' equilibrium division $R(\delta)$ of the negotiation game without commitment.

In the model of the current paper conflicting claims are avoided at equilibrium; but, this need not hold in all models and we refer readers concerned with impasses to Crawford (1982).

1.2. Statement and discussion of main results

This section states and puts into perspective the main results of the paper, unfolding them in four stages. The first is that a unique equilibrium outcome arising from stationary strategies exists, despite the discontinuities in utility. The precise result is this.

Theorem 1 (Uniqueness and Existence Theorem). *For $\frac{2}{3} < \delta < 1$ and $0 < c \leq \bar{C}(\delta)$, the game $\mathcal{G}_{c\delta}$ of Section 1.1 possesses a unique outcome division $S(c,\delta) = (x, 1-x)$ that is supported by a pure-strategy, subgame-perfect, Nash equilibrium, according to which subsequently to the announcement round, Players 1 and 2 play the stationary strategies $\Sigma^1_{xy}, \Sigma^2_{yx}$ with $y = \delta x + c$.*

Uniqueness is proved in Section 4 and existence in Section 5. We should explain that the upper bound placed on c, defined as follows

$$\bar{C}(\delta) := \frac{1-\delta}{2-\delta},$$

which we refer to as the 'limit of capitulation cost', is akin to a self-financing condition, ensuring that players are always able to retain an acceptable payoff while offering compensation (with which to entice the opponent away from a contract one round later, to a contract of equal or better value to the opponent one round earlier). Compare Figure 1. We have thus not studied the game when this self-financing condition fails. The limit of capitulation cost, $\bar{C}(\delta)$, remains bounded by $\frac{1}{4}$ for $\frac{2}{3} < \delta < 1$.

In the statement above Σ^1_{xy}, Σ^2_{yx}, for any real $0 \leq x, y \leq 1$, denote the standard **stationary strategies** in a Rubinstein alternating offers game, as recalled below. The equilibrium existence is not accidental (see section 6).

Definition. *In the negotiation subgame $\mathcal{N}(z_1, z_2)$, let Σ^1_{xy} denote the* **stationary strategy** *for Player 1 according to which his play is: propose a division $(x, 1-x)$ and accept an own share at or above y in all rounds, and let Σ^2_{yx} be the counterpart strategy for Player 2 in which she plays: always propose $(y, 1-y)$ and accept a proposal $(t, 1-t)$ with t at or below x.*

The play of $(\Sigma^1_{xy}, \Sigma^2_{yx})$ thus results in an outcome $(x, 1-x)$ agreed in the first round of of the negotiation subgame.

The second main finding of the paper is that at a subgame-perfect Nash equilibrium of the commitment game there is a regime switch in the stationary strategies of the equilibrium outcome $S(c, \delta)$. This switch compares the capitulation cost c to a **regime switching function** $L(\delta)$ which is defined by

$$L(\delta) =: \frac{(1-\delta)^2}{1+(1-\delta)^2},$$

and which satisfies $L(\delta) < \frac{1}{2}(1-\delta)$. The second result is this.

Theorem 2 (Division Characterization Theorem). *In the setting of Theorem 1 the unique division $S(c, \delta)$ depends on the regime switching value $L(\delta)$ as follows:*

$$\begin{aligned} \text{i) } S(c,\delta) &= (\sigma_1, \sigma_2), \text{ if } 0 \le c \le L(\delta) \text{ (low charge)}, \\ \text{ii) } S(c,\delta) &= (\hat{z}, 1-\hat{z}), \text{ if } L(\delta) < c \le \bar{C}(\delta) \text{ (high charge)}. \end{aligned}$$

Here

$$\sigma_1 = \sigma_1(c,\delta) := (1-c)\cdot\rho_1, \quad \sigma_2 = \sigma_2(c,\delta) = 1-\sigma_1; \qquad \hat{z} = \hat{z}(c,\delta) := \sigma_1 + (1-\delta)\left(\frac{c}{L} - 1\right)\rho_1,$$

where (ρ_1, ρ_2) was defined by (1.3), so that in case (i) Player 2 receives above ρ_2.

Remarks. 1. It follows from the formula for $\hat{z}$ that $\hat{z}(L(\delta), \delta) = \sigma_2(L(\delta), \delta)$, i.e. the strategy switch from (i) to (ii) is continuous.

2. It is interesting that $R(\delta) = (\rho_1, \rho_2)$ does not figure as an equilibrium division when $c > 0$. Player 2 may announce $z_2 = \rho_2$ and thereby hold Player 1 down to an outcome of ρ_1. However, ρ_2 is not her best reply announcement to his announcement $z_1 = \rho_1$.

3. Our findings support and quantify some common-sense expectations. The outcome is efficient. There is a first-mover advantage occurs in the bargaining subgame (gain over $R(\delta)$, the standard outcome of section 1.1), but only when c is above a certain threshold value, determined below in (1.6) (that is, the amount which Player 1 receives rises with c, albeit after an initial fall, as in Figure 2, eventually rising above ρ_1). The more the capitulation cost c exceeds the threshold, the greater is this gain. By contrast, it transpires that low penalty costs move the standard division unfavourably for the first-mover of the bargaining subgame. It would therefore be interesting to study first-mover advantage/disadvantage effects of unequal penalties.

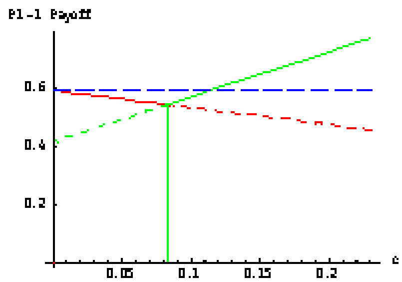

Figure 2. Pay-off to Player 1 against c : σ_1 declining (red-fine) to a regime switch to $\hat{z}$ (green-thick) which is rising and eventually above ρ_1 (blue-dashed). (Here $\delta = 0.7$)

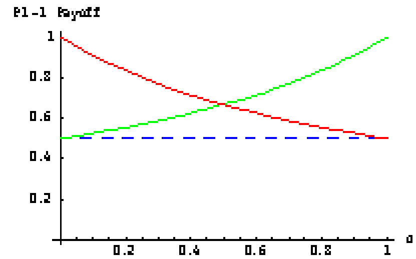

Figure 3. Pay-off to Player 1 against δ : ρ_1 (red-fine) decreases to 0.5 (blue-dashed); $\hat{z}(\bar{C}(\delta))$

(green-thick) increases to unity.

4. We have the explicit formulas:

$$\left.\begin{array}{rr} \sigma_1 = \frac{1-c}{1+\delta}, & \sigma_2 = \frac{c+\delta}{1+\delta}, \\ \hat{z} = \frac{\delta}{1+\delta} + c \cdot \frac{(1-\delta)+\delta^2}{1-\delta^2}, & 1 - \hat{z} = \frac{(1-c)(1-\delta)-c\delta^2}{1-\delta^2} \ . \end{array}\right\} \tag{1.4}$$

5. Note that in the range (i) $\sigma_1 > \sigma_2$. By (1.4) we have $\sigma_2 = \delta\sigma_1 + c$. This last equation may be read as saying that in case (i) a one-round delay switching the identity of the proposer is accompanied by the new proposer facing an outcome equal to the discounted outcome of the original proposer reimbursed for the capitulation cost. Compare this to the analogous remark following equation (1.3) and to the identification of compensations in Figure 1. By contrast, in (ii) since

$$1 - \hat{z} = \delta\hat{z} + c - (1-\delta)\left(\frac{c}{L(\delta)} - 1\right), \tag{1.5}$$

a one-round delay switches the identity of the proposer and yields, as outcome to the new proposer, the discounted outcome of the predecessor, adjusted by a falling compensation from the predecessor. The compensation eventually reverses direction to one in favour of the predecessor. See also below Remark 2 to Theorem 3, and Remark 4 to Theorem 4.

6. The payoff σ_2 to Player 2 in case (i) may be presented as the convex combination

$$\sigma_2 = c + (1-c) \cdot \rho_2$$

(of ρ_2 and unity) and is thus clearly better than ρ_2.

7. We comment that the term $c/(1+\delta)$ arising in (i) represents a compensation shouldered by Player 1. Interpreted as the series $c(1 - \delta + \delta^2 - ...)$, it is seen to be the limiting present value of a stream of compensations incorporated into a back-stepping argument, appropriate to a terminated version of the subgame.

8. The piecewise linear form of $S(c,\delta)$ is a consequence of the affine structure of what we call the compensated present-value function, see (1.8) below, though it is not obvious how many 'pieces' there might be.

The two possible divisions of Theorem 2 are identified in the course of proving 'uniqueness' in Section 4. The existence proof of Section 5 verifies that in the two complementary c ranges equilibrium conditions are satisfied in respect of possible supporting announcement strategies. This coupled with a calculation based on (1.4) that $\hat{z}(c,\delta) \geq \sigma_1(c,\delta)$ iff $c \geq L(\delta)$ completes a proof of Theorem 2, since Player 1 chooses whichever is the larger own share available to him at an SPNE in $\mathcal{N}(z_1, z_2)$.

Our third result describes the announcement structure in equilibrium; note the inclusion of a symmetric announcement (σ_2, σ_2) for $c \leq L(\delta)$.

Theorem 3 (Announcement Characterization Theorem). *In the setting of Theorem 2, the first round announcements (z_1, z_2) that are supported by a subgame-perfect Nash equilibrium of the commitment game $\mathcal{G}_{c\delta}$ are determined as follows:*

$$\begin{aligned} \text{i) } (z_1, z_2) &= (t, \sigma_2)\ , \text{ for } \sigma_2 \leq t \leq \sigma_1, \text{ if } 0 \leq c \leq L(\delta) \text{ (low charge)}, \\ \text{ii) } (z_1, z_2) &= (\hat{z}, 1-\hat{z}), \text{ if } L(\delta) < c \leq \bar{C}(\delta) \text{ (high charge)}. \end{aligned}$$

The announcement characterization is proved in the course of the existence proof of Section 5.

Remarks 1. Recall that $\sigma_2 < \sigma_1$ for $c < L(\delta)$, in fact for $c < (1-\delta)/2$. Thus at the subgame-perfect equilibrium both players' claims are satisfied (possibly Player 1 receives in excess of his claim).

2. Note that in the case (i) of the Theorem 3 when the announcement is (σ_1, σ_2) the claim z_2 agrees with $y = \delta z_1 + c$. By contrast, in (ii) $z_2 < y = \delta z_1 + c$, since as noted in (1.5) the claim $z_2 = 1 - \hat{z}$ falls progressively below $y = \delta\hat{z} + c$ as the capitulation cost rises. See also Remark 4 to Theorem 4 below.

Our final main result, a corollary of Theorem 2, identifies how significantly the equilibrium outcome $S(c,\delta)$ characterized by Theorem 2 may move away from the standard Rubinstein equilibrium $R(\delta) = (\rho_1, \rho_2)$ of section 1.1. To formulate our result we need to define the **threshold value** $H(\delta)$ by:

$$H(\delta) := \frac{(1-\delta)^2}{\delta + (1-\delta)^2}, \tag{1.6}$$

and now the qualitative shift can be quantified as follows.

Theorem 4 (Comparison Theorem). *For $\frac{2}{3} < \delta < 1$, it is the case that $\hat{z}(c,\delta)$ is increasing in c on the interval $[0, \bar{C}(\delta)]$ and satisfies*

$$\hat{z}(c,\delta) = \rho_1 + (1-\delta)\left(\frac{c}{H(\delta)} - 1\right) \cdot \rho_1, \text{ so that } \hat{z}(c,\delta) > \rho_1 \text{ iff } c > H(\delta).$$

Thus at the limit of capitulation cost $\bar{C}(\delta)$ the payoff to the first-mover is given by

$$\hat{z}(\bar{C}(\delta), \delta) = \frac{1}{2-\delta} = \frac{\bar{C}(\delta)}{1-\delta},$$

and this, as a function of δ, increases from $\frac{1}{2}$ to unity.

Example. When $\delta = 0.7$ we have $\bar{C} = 0.23$ and $\hat{z} = .77$ whereas $\rho_1 = 0.588$.

The proof of Theorem 4 is a routine deduction and is omitted.

Remarks. 1. Note that although the capitulation cost at its upper bound, i.e. for $c = \bar{C}(\delta)$, has a very significant qualitative impact on the equilibrium, it itself remains limited being, as noted earlier, less than 0.25 of the asset, and in fact is vanishingly small, as δ nears unity.

2. Note that for $0 < \delta < 1$ the threshold value $H(\delta)$ lies above the regime switching value $L(\delta)$ and is likewise decreasing with δ. Moreover it is the case that

$$\frac{1}{2}H(\delta) < L(\delta) < H(\delta) < \frac{1}{2}(1-\delta) < \bar{C}(\delta) < \frac{1}{4}. \tag{1.7}$$

3. Since $H(\delta)$ decreases from unity to zero, compare Figure 3, lower values of δ (reflecting higher impatience on the part of the players) correspond to a higher threshold value; so the higher the impatience level of the players, the higher the stakes need to be (as measured by the threshold value).

4. We note in addition to the identity (1.5) we also have

$$1 - \hat{z} = \delta\hat{z} - (1-\delta)\left(\frac{c}{H(\delta)} - 1\right),$$

so, as noted above, the outcome to Player 2 not only falls progressively as the capitulation cost rises, but in fact for $c > H(\delta)$ we have $z_2 < \delta z_1$. This means that after a one-period delay the compensation, accompanying the switch in identity of the new proposer, actually reverses direction in the compensation flow which is now in favour of the predecessor.

Proof strategy. Our main results are established in Sections 4 and 5 after some preparatory work. We outline the main ideas of the proof of the theorem and its corollary. A critical device of the Rubinstein bargaining model is to bring forward in time the value of a share x of the asset, if offered to a Player later, by substitution of an earlier offer y of a share at least equivalent in utility (to that Player) taking into account the time-discount factor. Our argument necessarily hinges on the form of this kind of present-value calculation in the context of a penalty, and on the need to consider a 'penalty compensation' to induce acceptance of a share below the announced claim z. As a consequence of interpreting penalty as loss of a fixed *proportion* of the asset under negotiation, the assumed identical penalty structure gives rise to two time-homogeneous **compensated present-value** functions derived from one common function $y = f(x, z)$, attributing value to a share x of asset accepted one round later, where the function $f(x, z)$ is defined in a piecewise affine format, depending on the announced claim z, as follows:

$$f(x, z) = f_i(x, z) = a_i x + b_i(z) \tag{1.8}$$

for x in intervals of the form $u_i(z) \leq x < v_i(z)$, for $i = 1, ..., n$. It is monotone in x. Thus once (z_1, z_2) has been announced the players have individual present-value functions defined from f as follows:

$$d_1(x) = f(x, z_1), \qquad d_2(x) = 1 - f(1-x, z_2). \tag{1.9}$$

The possible stationary strategy pairs $(\Sigma^1_{xy}, \Sigma^2_{yx})$ in the subgame $\mathcal{N}(z_1, z_2)$ which follows the announcement (z_1, z_2) are determined by the fixed points of $F(x) = d_2(d_1(x))$. However, there may be several fixed points (actually as many as three such), each offering a different allocation x to the first-mover in $\mathcal{N}(z_1, z_2)$. Thus to find the subgame-perfect Nash equilibrium involves identifying the largest fixed-point $x(z_1, z_2)$, since that is Player 1's optimal choice of x in the subgame. This largest fixed-point may be regarded as defining the payoff pair for a commitment game $\mathcal{G}'_{c\delta}$ starting from the 'round 0' simultaneous announcement (z_1, z_2), and ending on a first round agreement with payoff $(w(x(z_1, z_2), z_1), w(1 - x(z_1, z_2), z_2))$. The argument now turns on finding an announcement (z_1, z_2) which gives a Nash equilibrium in $\mathcal{G}'_{c\delta}$. It turns out that in a subgame-perfect Nash equilibrium of $\mathcal{G}_{c\delta}$ one will have $z_1 = x(z_1, z_2)$ and that $y = d_1(x) = \delta x + c$. The details are worked out in Section 4, but for an intuitive grasp consult Section 3.3.

The proof of the Theorem investigates fixed-points by comparing cases. A large enough value for δ ensures that a multitude of parameters, dependent only on c and δ, associated with the equilibrium divisions, obey unchanging comparative inequalities when c, δ vary (see the Lemmas in Section 3). Exploiting this 'comparison-persistence' one can also investigate the model by computer simulation without loss in generality.

Organization of the paper. In Section 2 we derive the compensated present-value functions $d_i(x, z_i)$ of the two players and give conditions in the Verification Theorem for the fixed points of the composition value-function $F(x) = d_2(d_1(x))$ to generate a subgame-perfect equilibrium in $\mathcal{N}(z_1, z_2)$. In Section 3.1 we identify fixed points of $F(x)$ as functions of (z_1, z_2); as these generate stationary strategies in the subgame, we intuitively interpret in 3.2 which from among these are likely to yield the equilibria of the commitment game. In 3.4 we compare various relevant parameters. In Section 4, using these comparisons, we narrow the search for equilibrium outcomes down to two. In Section 5 we verify that these two are indeed equilibrium divisions in complementary cost-ranges. A discussion of the justifiability of stationary equilibria in this distinctive context of discontinuity is offered in the concluding Section 6.

2. Preliminaries

In this section we define the compensated present-value functions $d_i(x; z_i)$, we prove a verification condition for the corresponding fixed-points to generate subgame-perfect equilibria, then we compute the present-value functions for the case of our commitment game, and validate the verification condition.

The functions $d_i(x; z_i)$ are defined by

$$
\begin{aligned}
u_1(d_1(x), n) &= \inf_y \{u_1(y, n) \geq u_1(x, n+1)\}, \qquad (n = 0, 1, ...), \\
u_2(d_2(x), n) &= \sup_y \{u_2(y, n) \leq u_2(x, n+1)\}, \qquad (n = 0, 1, ...).
\end{aligned}
\tag{2.1}
$$

Thus if $y = d_1(x; z_1)$, then y is the least such that

$$w(y, z_1) \geq \delta w(x, z_1),$$

where w is defined by (1.2); similarly, if $y = d_2(x, z_2)$, then y is the greatest such that $w(1-y, z_2) \leq \delta w(1-x, z_2)$. In the standard Rubinstein game without commitments, where $w(x) = x$, we have of course that $d_1(x) = \delta x$ and $d_2(x) = 1 - \delta(1-x)$. More generally, define

$$f(x, z) = d_1(x; z),$$

then, since the subgame starting with Player 2's proposal is identical to the original negotiation subgame up to player transposition, we have (1.9).

These definitions enable us now to gives sufficient conditions for the stationary strategy pair $(\Sigma^1_{xy}, \Sigma^2_{yx})$ of Section 1.2 to constitute a subgame-perfect Nash equilibrium.

Proposition 2.1 (Verification Theorem). *Let x, y satisfy*

$$y = d_1(x), \quad x = d_2(y), \tag{2.2}$$

and assume $d_1(d_1(x)) \leq x$ and $d_2(d_2(y)) \leq 1-y$. Then the pair $(\Sigma^1_{xy}, \Sigma^2_{yx})$ constitutes a subgame-perfect Nash equilibrium with outcome $(x, 1-x)$. In these circumstances a fixed point of the function $F(x) = d_2(d_1(x))$ generates such an equilibrium.

Proof. In any round if Player 1 is the bidder and bids for more than x, then according to τ_{yx} Player 2 refuses. In the next round agreement gives Player 1 at most y, the compensated present value of which is $d_1(y) = d_1(d_1(x)) \leq x$, he therefore will not gain from delaying agreement. On the other hand, if Player 2 is the bidder and awards herself more than $1-y$ (offering the opponent less than y), then according to Σ^1_{xy} Player 1 refuses, so she can expect at best x in the next round, whose compensated present value to her is $d_2(x) = d_2(d_2(y)) \leq 1-y$. Thus Player 2 cannot improve her pay-off by deviating from the stationary bid and delaying agreement.■

To compute $f(x,z) = d_1(x;z)$, a more general approach is helpful. The utility $w(x) = u_1(x,0)$ takes the form

$$w_i(x) = A_i x + B_i \text{ when } k_i \leq x < l_i \text{ for } i = 1,2.$$

Indeed, by (1.2) we have

$$w_1(x) = x - c, \text{ for } 0 \leq x < z, \; w_2(x) = x, \text{ for } z \leq x \leq 1.$$

Thus the range of w excludes $[z-c, z)$. Hence the contract offering x 'tomorrow' with $k_i \leq x < l_i$ is 'today' worth to Player 1 y if, for some i and j

$$w_j(y) = \delta w_i(x); \tag{2.3}$$

however, if δw_i lies in $[z-c, z)$, the least inducement y making the Player prefer acceptance one round earlier is $y = z$. Solving the equation (2.3) when

$$A_j k_j + B_j \leq \delta(A_i x + B_i) < A_j l_j + B_j$$

we obtain the formula

$$y_{ij} = \frac{\delta(A_i x + B_i) - B_j}{A_j}, \tag{2.4}$$

$$\text{for } x \text{ with } \max\left(\frac{A_j k_j + B_j - \delta B_i}{\delta A_i}, k_i\right) \leq x < \min\left(l_i, \frac{A_j l_j + B_j - \delta B_i}{\delta A_i}\right).$$

We can now compute $f(x,z) = d_1(x,z)$.

Proposition 2.2. *For the fixed-charge penalty regime, assuming $c < 1-\delta$ and $c \leq z \leq 1$, we have, for $0 \leq x \leq 1$, that $f(x,z) = f_i(x,z)$, where :*

$$\begin{aligned}
f_1(x,z) &= \delta x + c(1-\delta), \text{ if } x < z, \\
f_2(x,z) &= \delta x + c, \text{ if } z \leq x < (z-c)/\delta, \\
f_3(x,z) &= z, \text{ if } \max\{z, (z-c)/\delta\} \leq x < z/\delta, \\
f_4(x,z) &= \delta x, \text{ if } z/\delta \leq x < 1.
\end{aligned}$$

Comments. A direct interpretation of these formulas bears on the type of equilibrium that may be achieved. See section 3.2.

Note that $c/(1-\delta) < c+\delta$. Thus if $z < c+\delta$, case 2 arises iff $z < (z-c)/\delta$, iff $c/(1-\delta) < z$. Note that if $\delta z + c < z$, then the present value of a future claim z falls below the current claim even after compensation. Evidently $c/(1-\delta) < 1$. If case 2 arises, then case 3 occurs for $(z-c)/\delta \leq x < z/\delta$. Also if $1 \geq z \geq c+\delta$, then cases 3 and 4 do not arise; but if $z < c+\delta$ and case 2 fails, then case 3 occurs for $z \leq x < z/\delta$.

Proof of Proposition 2.2. The equation (2.3) fails iff δw_2 falls in the interval $[z-c, z)$ and this occurs iff $x \geq z$ (so that $x \in dom(w_2)$) and $z - c \leq \delta x < z$. To see this consider the graph of w, or note that for $0 \leq x < z$ we have $\delta w_1(x) = \delta(z-c) < z-c < z$. Thus indeed $f_3 = z$, as shown, since z is the least inducement to an acceptance leaving the player no worse off than in the next round. The remaining cases are routine applications of (2.4).■

We finally verify that the conditions of Proposition 2.1 hold.

Proposition 2.3. *For the fixed-charge penalty regime, $d_1(d_1(x) \leq x$ holds for $x \geq c$, and hence $d_2(d_2(y)) \leq 1-y$ holds for $y \leq 1-c$.*

Proof. We recall that $d_1(x) = f(x, z_1)$ and $d_2(x) = 1 - f(1-x, z_2)$. Fix $z > 0$. Note that $f_1(0, z) = c(1-\delta) < c$. The increasing function $f(x, z)$ has either one fixed point only at $x = c$ when $c/(1-\delta) < z$, or otherwise it has one fixed point at $x = c$, and a second fixed point at $x = z$, and none others. In the first case we have for $x > c$ that$c < f(x, z) < x$, and so

$$f(f(x,z),z) \leq f(x,z) < x.$$

In the second case, for $c \leq x \leq z$, we have $c \leq f(x,z) < x \leq z$, so $f(f(x,z),z) < x$; also for $z \leq x$ we have $z \leq f(x,z) \leq x$, hence

$$f(f(x,z),z) \leq x.$$

It follows from $f(x,z) \leq x$ that $f(1-y,z) \leq 1-y$ provided $1-y > c$, so that $d_2(y) = 1 - f(1-y,z) \geq y$ provided $y < 1-c$. Consequently, since f is increasing, $f(1-f(1-y,z),z) \geq f(y,z) \geq y$ and so $d_2(d_2(y)) = 1 - f(1-f(1-y,z)) \leq 1-y$.■

3. Fixed points of $F(x)$, interpretation, and comparisons

In the previous section Propositions 2.1 and 2.3 reduced the task of finding subgame-perfect Nash equilibria of $\mathcal{N}(z_1, z_2)$ to computing the fixed points of

$$F(x) = d_2(d_1(x)). \tag{3.1}$$

In this section we tabulate these fixed points, interpret some of them as likely equilibrium candidates, and then rank all the fixed points by size.

3.1. Tabulation

We begin by identifying an algorithm for computing the fixed points of (3.1), noting that importantly they fall into three distinct types. Then we tabulate the computed fixed points and offer an example graph.

Definition and algorithm. Recalling the notation of Proposition 2.2 that $f(x,z) = f_i(x,z) = a_i x + b_i(z)$ when $u_i(z) \leq x < v_i(z)$, define the functions

$$x_{ij}(z_1, z_2) = \frac{1 - a_i - b_i(z_2) + a_i b_j(z_1)}{1 - a_i a_j}. \tag{3.2}$$

This formula validly provides all the fixed points of F provided $u_i(z_2) \leq 1 - d_1(x_{ij}) < v_i(z_2)$ and $u_j(z_1) \leq x_{ij} < v_j(z_1)$. The conditions on x_{ij} require to be rewritten as inequalities to be satisfied by z_1 and z_2. The algorithmic output is tabulated below, but the routine calculations are omitted. Evidently, the formula (3.2) solves explicitly the fixed-point equation:

$$x = d_2(d_1(x)) = 1 - f_i(1 - f_j(x, z_1), z_2) = (1 - a_i - b_i(z_2) + a_i b_j(z_1)) + a_i a_j x.$$

Fixed-point types. Equation (3.2) indicates three types of fixed points, as follows. (a) Nine fixed points with value independent of the claims z_1 or z_2, correspond to $i \neq 3$ and $j \neq 3$ (cf. Proposition 2.2). (b) Three depending on z_1 alone which correspond to $i \neq 3$ and $j = 3$. (c) Four with value $x_{ij} = 1 - z_2$ corresponding to $i = 3$. The nine 'constants' are compared in a lemma to follow.

In the table x_{ij} refers ambiguously to the function defined by (3.2) and to its value. Reference is made to $\hat{z} = \hat{z}(c, \delta)$, to $\zeta = \sigma_2 = 1 - x_{22}$, (with the x_{22} shown below) as defined by (1.4) and to the additional parameters:

$$\xi = \frac{1 - c - \delta}{1 - \delta}, \qquad \eta = \frac{c + \delta - \delta^2}{1 - \delta^2} = 1 - x_{24}.$$

Table 3.1

Case $j = 1$

$x_{41} =$	$\frac{1+c\delta}{1+\delta}$	$x_{41} < z_1$	$z_2 \leq \delta x_{22}$
$x_{31} =$	$1 - z_2$	$1 - z_2 < z_1$	$\delta x_{22} \leq z_2 \leq \hat{z}$
$x_{21} =$	$1 - \hat{z}$	$1 - \hat{z} < z_1$	$\hat{z} \leq z_2 \leq x_{12}$
$x_{11} =$	$\frac{1-c+c\delta}{1+\delta}$	$x_{11} < z_1$	$x_{11} < z_2$

Note that $\delta x_{22} < x_{11} < x_{41} < x_{12}$ and that $\delta x_{22} < \hat{z} < x_{12}$. It is significant that the z_2 range of case 11 is half-open.

Case $j = 2$

$x_{42} =$	$\frac{1-\delta+c\delta}{1-\delta^2}$	$\eta \leq z_1 \leq x_{42}$	$z_2 \leq \delta x_{24}$
$x_{32} =$	$1 - z_2$	$c + \delta(1 - z_2) \leq z_1 \leq 1 - z_2$	$\delta x_{24} \leq z_2 \leq \zeta$ and $z_2 \leq x_{24}$
$x_{22} =$	$\frac{1-c}{1+\delta}$	$\zeta \leq z_1 \leq 1 - \zeta$	$\sigma_2 = \zeta \leq z_2 \leq 1 - \zeta = \sigma_1$
$x_{12} =$	$\frac{1-c-\delta+2c\delta}{1-\delta^2}$	$\hat{z} \leq z_1 \leq x_{12}$	$1 - \hat{z} < z_2$

Notes: (i) $\zeta \leq \xi$ iff $c \leq \frac{1}{2}(1 - \delta)$; (ii) $1 - \hat{z} \leq \xi$; (iii) $1 - \hat{z} < 1 - \zeta$ and $\delta x_{24} < \zeta$ and $\hat{z} < x_{12} < x_{42}$. It is significant that the z_2 range of case 12 is half-open.

Case $j = 3$

$x_{43} =$	$(1 - \delta) + \delta z_1$	$\delta x_{44} \leq z_1 \leq \eta$	$z_2 \leq \delta(1 - z_1)$
$x_{33} =$	$1 - z_2$	$\delta(1 - z_2) < z_1 \leq c + \delta(1 - z_2)$ and $z_1 + z_2 \leq 1$	$\delta(1 - z_1) \leq z_2 \leq c + \delta(1 - z_1)$
$x_{23} =$	$\delta z_1 + (1 - \delta - c)$	$\delta x_{24} \leq z_1 \leq \zeta = 1 - x_{22}$ and $z_1 \leq \xi = \frac{1-c-\delta}{1-\delta}$	$c + \delta(1 - z_1) \leq z_2 \leq 1 - z_1$
$x_{13} =$	$(1 - c)(1 - \delta) + \delta z_1$	$\delta x_{22} \leq z_1 \leq \hat{z}$	$1 - z_1 < z_2$

Notes: (i) $\delta x_{24} < \delta x_{22} < \delta x_{44} < \hat{z} < \eta$; (ii) Case 23 holds iff $z_1 < \xi$ and if $z_1 = \zeta$ we have $c + \delta(1 - z_1) = \zeta$; (iii) Case 33 fails unless $\delta x_{24} \leq z_1 \leq \eta$. It is significant that the z_2 range of case 13 is half-open.

$$\begin{array}{rccc} & & \text{Case } j=4 & \\ x_{44}= & \frac{1}{1+\delta} & z_1 \leq \delta x_{44} & z_2 \leq \delta x_{44} \\ x_{34}= & 1-z_2 & z_1 \leq \delta(1-z_2) & \delta x_{44} \leq z_2 \leq \eta \\ x_{24}= & \frac{1-c-\delta}{1-\delta^2} & z_1 \leq \delta x_{24} & \eta \leq z_2 \leq x_{42} \\ x_{14}= & \frac{1-c}{1+\delta} & z_1 \leq \delta x_{22} & z_2 > x_{41} \end{array}$$

Note: $\delta x_{24} < \delta x_{22} < \delta x_{44} < \hat{z} < \eta < x_{41} < x_{42}$.

3.2. Example graph

We close this section with an example graph showing fixed-points for a fixed z_1 against z_2 varying. Player 2 seeks to minimize Player 1's proposed division so identifies the least value on the descending graph of the payoff to Player 1 $1 - z_1$ not lying under any horizontal line.

In Figures 4a and 4b below note the presence on an 'elbow' at $z_2 = \hat{z}$ where 31 and 21 are contiguous. In 4a cases represented from left to right: 43,41,32,31,21,11,12 so that the optimal reply is $z_2 = x_{11}$.(Note the three fixed points.) In 4b cases represented from left to right: 43,33,32,31,21,11,13=12, so that the optimal reply is $z_2 = 1 - \hat{z}$.

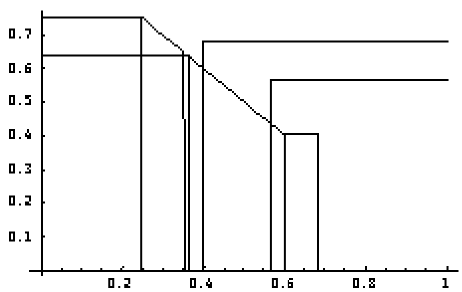

Figure 4a. Fixed-points $x_{ij}(\eta, z_2)$ as functions of z_2.

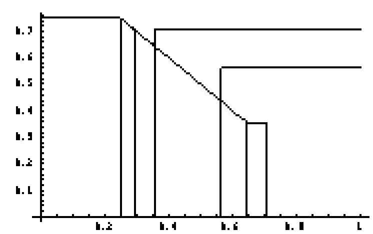

Figure 4b. Fixed-points $x_{ij}(\hat{z}, z_2)$ as functions of z_2.

3.3. Interpretation

A propos Proposition 2.2 case 2, the two equilibria of the Theorem satisfy

$$1 - \xi < \hat{z} \text{ iff } c < \bar{C}(\delta), \text{ and } 1 - \xi < x_{22} \text{ iff } c < \frac{1}{2}(1 - \delta). \tag{3.3}$$

We now offer an intuitive argument, which needs proper substantiation. (A rigorous treatment is taken up in section 4.) Invoking the Coase Principle, see Coase (1960), for the purposes only of this section, we make the simplifying assumption that in equilibrium both players aim to make demands that can be, and are, met by the opponent. We will eventually see that this need not hold in relation to all announcements in equilibrium; compare also Crawford (1982), and Anderlini and Felli (1998). Our assumption needs to be coupled to the 'Principle of Indifference' according to which, in equilibrium, the responding player in any subgame of $\mathcal{N}(z_1, z_2)$ is either indifferent to a one-round delay, or prefers not to delay.

The simplest scenario is when the delay requires each player to compensate the shrinking of the asset with a side payment of c. Working back from the first-mover's satisfied claim z_1, a one-round delay in meeting the claim z_1 yields an uncompensated value $\delta z_1 < z_1$, so to induce an acceptance the bidder offers compensation by c, and hence might offer $\delta z_1 + c$. If $\delta z_1 + c < z_1$, this is advantageous for the bidder, permitting her in that intermediate round to keep more asset and still satisfy the counter-party. This suggests an equilibrium of type 22 based on $y = \delta z_1 + c = d_1(z_1)$. (Here $z_1 > c/(1-\delta) = 1-\xi$, so this agrees with case 2 of Proposition 2.2). This is in keeping with the ideas expressed in connection with (1.3) in that an efficient subgame-perfect equilibrium division could plausibly be of the form $(\sigma, \delta\sigma + c)$. This yields $\sigma(1+\delta) + c = 1$, so that $\sigma = \sigma_1 = (1-c)/(1+\delta)$.

The preceding argument holds only provided the compensation is not too large. It is possible (for large c) that a payment of c leads to over-compensation, in the sense that $\delta z_1 + c > z_1$. In this circumstance the only option to the bidder is again to meet the claim z_1 in the current round, and this corresponds to case 3 of the Proposition 2.2 where $d_1(z_1) = z_1$.

On the above argument (repeated mutatis mutandis for Player 2) any equilibrium must be covered by one of the cases 22,23,32,33. Case 22 yields the equilibrium of Theorem 2 referred to under (i)[5]. The pay-off in case 23 is increasing in z_1 up to x_{22} (see section 4.1), so is unlikely as an equilibrium value; the other two cases yield pay-off z_2 to Player 2 provided no other cases dominate this pay-off. Comparison of the case 32 against 12 where $z_1 \geq \hat{z}$ and $z_2 > 1-\hat{z}$ leads to consideration of the validity or otherwise of the inequality $1-\hat{z} \leq \zeta = \sigma_2$.This turns out to be true iff $c \geq L$. This in turn implies implies that for $c \geq L$ the payoffto Player 1 in case 12 dominates that of case 32 for $z_2 > 1-\hat{z}$; thus Player 2 maximizes her pay-off under 32 with $z_2 = 1-\hat{z}$. That is the attainable upper limit of undominated play by Player 2 of z_2 in the case 32 is for $z_2 = 1-\hat{z}$. See Figure 4b above. This goes some way towards explaining where the other subgame-perfect equilibrium division arises. Note that in either case we arrive at $x = z_1$ and $y = d_1(z_1) = \delta z_1 + c$. The second equilibrium vector is not actually immediate since one must tediously investigate (exhaustively) dominances between Player 1's pay-offs arising from potential multiplicities of fixed points given an announcement (z_1, z_2).

3.4. Comparison: some inequalities

In the lemmas which follow we rank the various parameters and the various pay-offs occurring in Table 3.1 by size in order to verify constraints, and to locate the position of the all important pay-offs $\hat{z}$ and $1-\hat{z}$. Lemma-b and Lemma-c help to determine optimal behaviour of the players. The proofs are routine and so are omitted.

Many of the inequalities between parameters are true for all c below the limit of capitulation cost $\bar{C}(\delta)$. These we term persistent. Others depend on what side of a single, threshold value (depending on δ) the value c stands. To identify the dependence of comparisons on c we will attach various qualifiers to the comparison sign. Thus: $g <^{\text{Property}(c)} t$ is to mean: $g < t$ iff Property(c) holds. In particular $<^{-}$ is to mean 'iff $c < \frac{1}{2}(1-\delta)$' holds. Specifically we will need to refer to the c ranges defined by the following thresholds, shown in ascending order (for $\delta > 1/2$) :

$$\frac{1}{2}H(\delta) < G(\delta) < L(\delta) < T(\delta) < M(\delta) < H(\delta) < \frac{1}{2}(1-\delta), \tag{3.4}$$

an extension of (1.7), where

$$G(\delta) := \frac{(1-\delta)^2}{(2-\delta)}, \qquad T(\delta) := (1-\delta)^2, \qquad M(\delta) := \frac{(1-\delta)^2}{(2-3\delta+2\delta^2)}.$$

[5] $z_1 = d_2(d_1(z_1)) = 1 - \delta(1-\delta z_1 - c) - c$ yields $(1-\delta^2)z_1 = (1-c)(1-\delta)$.

Referring to the fixed-point types identified in section 3.1, we first establish a linear ranking for the type (a) 'constant pay-offs'. Then we locate the two parameters ξ and η defined at the top of Table 3.1. Next we follow up the location of ζ and of $\hat{z}$ which were defined in (1.4), and also of $1-\hat{z}$. Subsequently, we consider pay-offs of type (b), i.e. those which depend on z_1 only. Lastly we consider a continuity feature of pay-offs of type (c), i.e. those which depend only on z_2. Henceforth we make the following.

Blanket Assumption.

$$2/3 < \delta < 1 \text{ and } 0 < c \leq \bar{C}(\delta) = \frac{1-\delta}{2-\delta}. \tag{3.5}$$

As regards the 'constant pay-offs', the three x_{22}, x_{24} and $1-\hat{z}$, require special treatment; the remaining six satisfy persistent inequalities. We consider these six first.

Lemma-P (Persistent inequalities). *Assuming* (3.5) *we have:*

$$\delta x_{22} < \delta x_{44} < x_{22} = x_{14} < x_{11} < x_{44} < x_{41} < x_{12} < x_{42}. \tag{3.6}$$

The location of the constant payoff x_{24} is bounded by x_{22} and, depending on c, can drop below δx_{22}.

Lemma-24 (Properties of x_{24}) *Assuming* (3.5) *we have:*

$$\delta x_{44} <^{c<T} x_{24} < x_{22} < x_{44} \text{ and } \delta x_{22} <^{c<H} x_{24}.$$

A delimiter of cases 32 and 23 is given by ξ. Recall that $1-\xi = c/(1-\delta)$ is a delimiter of case 3 in Proposition 2.2. (See the subsequent comments.)

Lemma-ξ (Properties of ξ): *Assuming* (3.5), *we have transitively that:*

$$1-\xi <^{-} \zeta <^{-} x_{22} = 1-\zeta <^{-} \xi < 1-c.$$

The next lemma gives the only persistent inequalities concerning η. All other inequalities depend on the location of c, e.g. $x_{41} <^{c>H} \eta$. We need the following.

Lemma-η (Properties of η): *Assuming* (3.5) *we have transitively:*

$$\delta x_{24} < \delta x_{22} < \delta x_{44} < \zeta < \eta <^{-} x_{12} < x_{42}, \text{ and } x_{22} <^{c>G} \eta <^{c<L} x_{11} \text{ and } x_{41} <^{c>H} \eta$$

The parameter ζ identifies the incidence of the all important case 22.

Lemma-ζ (Properties of ζ): *Assuming* (3.5) *we have:*

$$\delta x_{44} < \zeta < x_{11} < x_{12}, \text{ and } x_{24} <^{c>G} \zeta <^{-} x_{22} = 1-\zeta < x_{11}.$$

The next two lemmas locate the pay-offs $\hat{z}$ and $1-\hat{z}$.

Lemma-Z (Properties of $\hat{z}$): *Assuming* (3.5) *we have:*

$$\frac{c}{1-\delta} = 1-\xi < \hat{z} < \eta, \text{ and } \delta x_{22} < x_{41} <^{*} \hat{z} < x_{12},$$

implying $\eta > \hat{z} > \zeta$ etc., with the second inequality requiring $c > (1-\delta)^2/(1-2\delta+2\delta^2)$ and moreover

$$\frac{\delta}{1+\delta} = \delta x_{44} < \zeta < x_{11} <^{c>M} \hat{z} < x_{12},$$

implying $\hat{z} > \zeta$ etc. Furthermore

$$\frac{1}{2} <^{c>H/2} \hat{z} \text{ equivalently } 1-\hat{z} <^{c>H/2} \hat{z} \ .$$

Lemma-1-Z (Properties of $1-\hat{z}$): *Assuming* (3.5) *we have:*

$$\delta x_{24} < x_{24} < 1-\hat{z} = x_{21} < x_{22} < x_{11} \text{ and } 1-\hat{z} < \xi,$$

and moreover

$$\hat{z} <^{c<L} x_{22} \text{ equivalently } \zeta <^{c<L} 1-\hat{z} \ . \tag{3.7}$$

The next lemma identifies extreme values taken by the payoff-functions depending on z_1. These will help to find improved pay-offs for Player 1.

Observation. The case 22 in fact occurs iff $c < L(\delta)$ in view of (3.7). If this case subsists it has the effect of preventing $(\hat{z}, 1-\hat{z})$ from being an equilibrium, because $x_{22} > x_{32} = 1 - z_2 = \hat{z}$.

Lemma-b (Ascending Elbows): *For the functions $x_{i3}(z_1, z_2)$ defined by* (3.2) *regarded as functions of z_1 it is the case, for any z_2, that*

$$\begin{aligned} x_{14} \le x_{13}(z_1, z_2) \le x_{12}, &\quad \text{for } \delta x_{14} \le z_1 \le \hat{z}, \\ x_{44} \le x_{43}(z_1, z_2) \le x_{42}, &\quad \text{for } \delta x_{44} \le z_1 \le \eta, \\ x_{24} \le x_{23}(z_1, z_2) \le x_{22}, &\quad \text{for } \delta x_{24} \le z_1 \le \zeta, \end{aligned}$$

with equality occurring only at the end-points.

A continuity feature implies an 'elbow' shape also for the payoff x_{3j} regarded as a function of z_2, so identifies a local minimum in z_2. The information helps to find optimal pay-offs for Player 2.

Lemma-c (Descending Elbows): *For the functions $x_{3j}(z_1, z_2) = 1 - z_2$ regarded as a function of z_2 it is the case that*

$$\begin{array}{lcc} x_{31} = x_{21} & \text{for } z_2 = \hat{z} & \\ x_{32} = x_{22} & \text{for } z_2 = \zeta & \text{provided } \zeta \le \xi \text{ and } \zeta \le z_1 \le 1-\zeta \\ x_{33} = x_{23} & \text{for } z_2 = c + \delta(1-z_1) & \text{provided } z_1 < \xi \text{ and } \delta x_{24} \le z_1 \le \eta. \\ x_{34} = x_{24} & \text{for } z_2 = \eta & \text{provided } z_1 < \delta x_{24}. \end{array}$$

Note that $\delta x_{24} < \eta$ (and $\eta > \xi$ for large enough c).

We are now able to make an immediate and useful deduction.

Corollary 1 (Consistency). *All z_1 (resp. z_2) intervals in Tables 3.1 which are defined independently of the value of z_2 (resp. z_1), i.e. only by expressions involving c, δ, are non-empty, with the exception of Case 22 unless additionally $c < \frac{1}{2}(1-\delta)$.*

Proof. We omit the routine verification. But note that

$$\frac{1}{2}(1-\delta) < \bar{C}(\delta) = \frac{1-\delta}{2-\delta} < 1-\delta.$$

Hence $c < 1-\delta$, as required by cases 42 and 24; however, the case 22 requires that $c < \frac{1}{2}(1-\delta)$. This is a stronger restriction than we assume in the Theorem.

Proposition 3.1 *All fixed points in the fixed-charge penalty model generate subgame-perfect equilibria, provided $c \le \bar{C}(\delta)$.*

Proof. This is mostly routine and omitted. (Actually, the condition $c \le (1-\delta)/(2-\delta^2)$ arises when requiring x_{23} to be above c for z_1 at the left endpoint of the z_1-domain. But in fact, we are concerned only with the right end of the domain, where the fixed-point is above c for $c < \bar{C}(\delta)$.) The Proposition thus follows from Propositions 2.1 and 2.3.■

4. Proof of Main Results: uniqueness

This section is organized as follows. In Section 4.1, as a first step towards uniqueness, starting at Section 4.1.1 we begin narrowing down the choice of announced claim pairs to ultimately just three cases and note that these give rise to two possible division outcomes. The argument examines Player 1's choice of a fixed point x of $d_2(d_1(x))$, subsequent to the announcement round, and we find that for two of the possible announcement vectors Player 1 selects a fixed point (his proposal in the bargaining subgame) equal to his announced claim. We leave to Section 5 the check that the three possible announcement vectors satisfy equilibrium conditions in two complementary sets of cost values c as described in the Main Theorem of section 1.1. This last step is tedious and relies on lemmas detailing case incidence. Before proceeding, we note a direct consequence of the Verification Theorem (Proposition 2.1) for the bargaining strategies $\Sigma^1_{xy}, \Sigma^2_{yx}$. In the two cases where Player 1 employs Σ^1_{xy} with $x = z_1$ and $y = d_1(x)$ in the negotiation subgame, the value z_1 satisfies $c/(1-\delta) < z_1$, so a reference to (3.3) and case 2 of Proposition 2.2 yields from (2.2) that $y = \delta x + c$.

4.1. Exclusion Lemmas: Narrowing down the search for subgame-perfect equilibria

This section shows that at a subgame-perfect equilibrium in the commitment game Player 1 receives one of x_{22} or $\hat{z}$, and that for $c \geq L$ his announcement satisfies $z_1 = x_{32}(z_1, z_2) = \hat{z}$. We will later show that for $c < L$ he announces $z_1 \in (\zeta, x_{22})$. The search-argument falls naturally into three parts according to the three types of fixed point x_{ij} identified in section 3.1, with each type considered in a separate subsection, starting at 4.1.1. Its purpose is to identify announcements (z_1, z_2) which are potentially supported by an SPNE and to dismiss the majority of these. Before we start the search we clarify the method employed and the kind of technicalities that need to be confronted to prove that certain announcements are not supported by an SPNE.

In each of the sixteen cases (i, j) we will look at pairs (z_1, z_2) permitted by the case, and test whether unilateral deviation from (z_1, z_2) can improve a Player's payoff. One criterion, referring to Player 1's optimal behaviour, is that if we identify a case (k, l) and a vector (z_1', z_2) in its domain such that $x_{ij}(z_1, z_2) < x_{kl}(z_1', z_2)$, then (z_1, z_2) is excluded from being an announcement in a subgame-perfect equilibrium.

The analogous criterion for Player 2 is more involved. To show that (z_1, z_2) in the domain of case (i, j) is excluded, we must identify (k, l) and z_2' so that $x_{kl}(z_1, z_2') < x_{ij}(z_1, z_2)$ and also check that there is no '**alternative dominance**' for some (m, n) of the form $x_{kl}(z_1, z_2') < x_{mn}(z_1, z_2')$, as Player 1 would prefer to propose such $x_{mn}(z_1, z_2')$ in the negotiation subgame. The procedure here is to refer to an instance of Lemma-c identifying a local minimum of $1 - z_2$ and checking constraints imposed by end-points of cases 11,12,13,14. Lemma 10 provides an example.

We now list our Exclusion Lemmas which are used in an exhaustive search for the at most two divisions achieved in a subgame-perfect equilibrium.

Lemma 4.1 6. *Any announcement with $z_2 = \delta/(1+\delta)$; in particular $R(\delta) = (1, \delta)/(1+\delta)$ is not supported by a subgame-perfect equilibrium.*

Proof. The play $z_2 = \delta/(1+\delta)$ either yields an identical payoff of $1/(1+\delta)$ to Player 1 for $c < H$ under all cases that subsist, or, if $c > H$, offers Player 1 a better reply of $z_1 = \hat{z}$. In the former case to avoid penalty Player 1 must select $z_1 \leq 1/(1+\delta)$. But for $z_1 \leq 1/(1+\delta)$ reference to case 31 shows that against such play, Player 2 can receive at least $z_2 > 1 - z_1 \geq \delta/(1+\delta)$, since dominance occurs from cases 12 or 11 only.

Lemma 4.2 7. *Any announcement $(t, 1-t)$ with $t < \hat{z}$ is not supported by a subgame-perfect equilibrium.*

Proof. The reply $z_1 = \hat{z}$ to $z_2 = 1 - t$ places (z_1, z_2) in case 12 and gives a payoff to Player 1 of $x_{12} = \hat{z} > t$, so is a better response by Player 1.

Lemma 4.3 8. *The announcement $(\xi, 1-\xi)$ is not supported by a subgame-perfect equilibrium.*

Proof. With $z_2 = 1 - \xi$ we refer to case 13. Note that $x_{13} > \xi$ for $z_1 > \xi$. Thus $1 - z_2$ is not a best reply for Player 1 to z_2. Finally note that $\delta x_{22} < \xi$ by Lemma ξ.

Lemma 4.4 9. *The announcement $(1-\xi, \xi)$ is not supported by a subgame-perfect equilibrium.*

Proof. One checks that $z_1 = 1 - \xi = c/(1-\delta)$ is not a best response to $z_2 = \xi$. An inspection of case 12 shows that $z_2 > 1 - \hat{z}$ holds (see note(ii) to Table 3.1 for case $j = 2$); and $x_{12} > \hat{z} > 1 - z_2$, so that $z_1 = x_{12}$ is in fact the best response.

Lemma 4.5 10. *If $\delta x_{24} < z < \zeta$ and $z_1 = 1 - z$, then the fixed-point proposals x over which Player 1 maximizes in correspondence to z_2 are as tabulated below.*

x_{42}	for $z_2 \leq \delta x_{24}$;	$x_{21} = 1 - \hat{z}$	for : $\hat{z} \leq z_2 \leq x_{12}$;
$x_{32} = 1 - z_2$	for $\delta x_{24} \leq z_2 \leq \zeta$;	x_{12}	for $1 - \hat{z} < z_2$;
$x_{31} = 1 - z_2$	for $\delta x_{22} < z_2 \leq \hat{z}$;	x_{11}	for : $x_{11} < z_2$

Thus since $\hat{z} < x_{11} < x_{12}$ a best reply for Player 2 is $z_2 = \hat{z}$.

Proof Cases $j = 4$, $j = 3$, and the case 22 fail. Note that $x_{42} \geq x_{32} \geq x_{31} \geq x_{21}$.

Lemma 4.6 11. *None of the following announcements, including $(1-\zeta, \zeta)$ for $c > L$, and $(\hat{z}, 1-\hat{z})$ when $c < L$, is supported by a subgame-perfect equilibrium with outcome equal to $(1-z, z)$:*

$$(1-z, z) \text{ with } \delta x_{24} = \delta(1-\delta-c)/(1-\delta^2) < z \leq \min\{\xi, \zeta\}. \tag{4.1}$$

Proof. To understand the claim here, note that $c > L$ iff $1 - \hat{z} < \zeta$, and that $\zeta \leq \xi$ iff $c \leq \frac{1}{2}(1-\delta)$. We will need the comparisons of various thresholds in (3.4) of section 3.3. Finally note that $1 - \hat{z} < \xi$ by Lemma P-Z, and that if $L < c < \frac{1}{2}(1-\delta)$, then $1 - \hat{z} < \zeta$. Thus we have $1 - \hat{z} < \min\{\xi, \zeta\}$ for $c > L$.

First we consider the z range $1 - \hat{z} < z \leq \xi$. We refer to case 12 with $z_2 = z$ where for $z_1 = \hat{z}$ we obtain a superior outcome to $1 - z$, as $x_{12} > \hat{z} > 1 - z$. This observation includes the case $z = \zeta$ when $L < c < \frac{1}{2}(1-\delta)$, so this is when $(1-\zeta, \zeta)$ is indeed not an SPNE (but then $(1-\hat{z}, \hat{z})$ is).

Now we consider the range $\delta x_{24} < z < 1 - \hat{z}$ and we seek a best reply[6] to any fixed $z_1 = 1 - z > \hat{z}$. According as $x_{11} \geq \hat{z}$ or $x_{11} < \hat{z}$ our argument refers to one of the cases 21 and 31 which are contiguous at $z_2 = \hat{z}$ (where the payoff to Player 1 is $1 - \hat{z}$). So recall from Lemma Z that $x_{11} < \hat{z}$ iff $c > M(\delta)$, and that $M(\delta) > L(\delta)$.

Suppose first that $c > L$. Thus if $c \leq M(\delta)$, then $x_{11} \geq \hat{z}$, and so against z_1 the choice $z_2 = \hat{z}$ gives Player 2 a best payoff of $\hat{z}$, since $x_{21} = 1 - \hat{z}$. But$1 - z_1 < 1 - \hat{z} < \hat{z}$, as $c > L > \frac{1}{2}H(\delta)$. Thus the response $z_2 = 1 - z_1$ with assumed outcome $1 - z_1$ yields an inferior payoff to that achieved for $z_2 = \hat{z}$. In conclusion $z_2 = 1 - z_1$ is not a best reply to z_1.

Now for $c > M(\delta)$, we have $x_{11} < \hat{z}$, and the best response is $z_2 = x_{11}$. Here Player 1 receives $1 - x_{11}$, i.e. Player 2 gets x_{11}, and moreover by Lemma P-Z $1 - z_1 < 1 - \hat{z} < x_{11}$.

Finally consider the case $c < \frac{1}{2}H < L < M$. By Lemma1-Z $1 - \hat{z} > \hat{z} > \zeta > \delta x_{24}$. Consider z with $\delta x_{24} < z < \zeta$ and $z_1 = 1 - z$. Here $z_2 = \hat{z}$ satisfies $z_2 > z = 1 - z_1$ and so case 31 holds for

[6] If $L < c < \frac{1}{2}(1-\delta)$ then $1 - \hat{z} < \zeta$, and here for $z_2 < 1 - \hat{z}$ it seems that best play by Player-1against z_2 is with 43 where $z_2 = \delta(1 - z_1)$ yielding $1 - z_2$ to Player-1.

the pair $z_1 = 1 - z, z_2 = \hat{z}$. Moreover we again have $c \le M(\delta)$ and so against z_1 the choice $z_2 = \hat{z}$ gives Player 2 a best payoff of $\hat{z}$.

Armed with the exclusion lemmas we proceed to an exhaustive search for subgame-perfect equilibria among the fixed points of Table 3.1.

4.1.1. Nine functions x_{ij} that are independent of z_1 and z_2

The aim in this subsection is to show that at equilibrium Player 2 selects z_2 so that $\delta x_{24} < z_2 \le 1 - \hat{z}$. This will come after our first step, which is to inspect the nine functions of the current heading. We will then discover that in consequence one only of these nine, namely x_{22}, gives a possible subgame-perfect equilibrium division.

The first six, of the nine functions, to be considered in our first step satisfy $x_{ij} = z_1$ at an end-point of their domain, though three achieve equality only in the limit. By inspection of Table 3.1 these six turn out to be $x_{21}, x_{22}, x_{11}, x_{41}, x_{12}, x_{42}$. For all c these six have values in the same ascending order, namely:

$$1 - \hat{z} = x_{21} < x_{22} < x_{11} < x_{41} < x_{12} < x_{42}. \tag{4.2}$$

Note that *subject to different domain restrictions on* z_1 the six functions satisfy

$$x_{21}, x_{11}, x_{41} < z_1 \le x_{22}, x_{12}, x_{42}.$$

The first three are thus unlikely candidates, as we shall confirm. (One expects intuitively at equilibrium to have $x_{ij} \ge z_1$.) The remaining three functions of this group, ranked according to size, are :

$$x_{24} < x_{14} = x_{22} < x_{44} <^{c>H} \hat{z}, \tag{4.3}$$

with the restricted inequality requiring $c > H(\delta)$. They all satisfy $x_{ij} > z_1$ with strict inequality, even in the limit at the edge of their domains. We will see that the three cases indicated by the subscript are also excluded at equilibrium (case 14 despite the fact that value-wise $x_{14} = x_{22}$).

As a second step, we consider the observation that the values x_{42} and x_{12} are the two highest possible payoff values independent of z_2 as per (4.2). In choosing a best reply at equilibrium, Player 2 may prevent the former as an outcome by selecting $z_2 > \delta x_{24}$ and the latter by selecting $z_2 \le 1 - \hat{z}$. Note that $\delta x_{24} < 1 - \hat{z}$. This leaves only one case with $x_{ij}(z_1, z_2) = z_1$, namely 22 as initially suggested.

Comment. Observe that the restriction just derived, namely,

$$1 - x_{42} = \delta x_{24} < z_2 \le 1 - \hat{z} \tag{4.4}$$

opens the possibility of Player 1's pay-offs $x_{3j} = 1 - z_2$ falling in the range $\hat{z} \le 1 - z_2 < x_{42}$.

Having excluded two outcomes, the rest of this section is dedicated to excluding six others in consequence of the range restriction (4.4) on z_2.

First consider the three cases with $x_{ij} > z_1$ as noted in (4.3)

(i) **Case 24.** Begin by noting that if $1 - \hat{z} < \eta$ then the case is excluded. So suppose $\eta \le 1 - \hat{z} < x_{42}$ (see Lemma P-Z and Lemma P). Refer to the ascending elbows of Lemma-b: the right end-point for z_1 in case 24 , namely $z_1 = \delta x_{24}$, coincides with the left end-point for z_1 in case 23. But at $z_1 = \delta x_{24}$ the the same restrictions on z_2 arise under case 24 and 23. It follows that for the z_2 restriction being considered, the payoff x_{24} is dominated by the increasing payoff function x_{23} (see Lemma b).

(ii) **Case 14** is excluded since $z_2 \le 1 - \hat{z} < x_{41}$ (see Lemma P and 1-Z).

(iii) **Case 44.** If $z_2 < \delta/(1+\delta)$, the contiguous case 43 (Ascending Elbow) offers Player 1 a superior payoff. According to Lemma 4.1, the case $z_2 = \delta/(1+\delta)$ cannot arise.

Finally we consider the three cases with $x_{ij} < z_1$.

Case 11 is ruled out since $z_2 \le 1 - \hat{z} < x_{22} < x_{11}$ by Lemmas 1 and 1-Z.

Case 41. For $z_2 < \delta x_{22}$ we have $x_{41} < x_{32}$. For $z_2 = \delta x_{22} = 1 - x_{41}$, one checks that the best payoff to Player 1 is x_{41}, but the claim $z_1 = x_{41}$ is excluded by this case.

Case 21 is ruled out if $1 - \hat{z} < \hat{z}$. So suppose $\hat{z} \le 1 - \hat{z}$ and note that $1 - \hat{z} < x_{22} < x_{12}$. So under case 21 we are limited to $\hat{z} \le z_2 \le 1 - \hat{z}$. But we have $\zeta < \hat{z} \le 1 - \hat{z} < 1 - \zeta$ so for our z_2 range, case 22 offers more to Player 1 (as $1 - \hat{z} < x_{22} = 1 - \zeta$, by Lemma Z).

Of the nine case we have thus ruled out all but one, namely the case 22.

4.1.2. Exclusion of the three outcomes linearly depending on z_1 only

As these cases have positive z_1-slope δ, they cannot offer an equilibrium, except possibly at the right end-points of their z_1-domains. We consider these end-points and find under case 23 a possible subgame-perfect equilibrium announcement (ζ, ζ). Being an endpoint, it is already subsumed under the contiguous case 22.

(i) **Case 13.** Since we have already shown in (4.4) that $z_2 \le 1 - \hat{z}$, the requirements $\hat{z} \le 1 - z_2 < z_1 \le \hat{z}$ yield a contradiction. The case does not arise.

(ii) **Case 43.** We have either (a) $z_1 = \eta$, or (b) $\delta z_1 = \delta - z_2$.

(a) Consider $z_1 = \eta$. Suppose first $1 - \eta < \hat{z}$. Thus $c > (1-\delta)^2/(2 - \delta + \delta^2)$ and this lower bound is below $\frac{1}{2}(1-\delta)$. The case 31 holds undominatedly for $z_2 < 1 - \hat{z}$, so that the best reply is $z_2 = 1 - \hat{z}$. (Note that there is an elbow at $z_2 = \hat{z}$ where 31 and 21 are contiguous, and $x_{21} < x_{22}$.) But we show in section 4.3 that the best replies to $z_2 = 1 - \hat{z}$ occur for $z_1 \le \hat{z} < \eta$ (see Lemma Z), a contradiction. Next suppose only that $c < \frac{1}{2}(1-\delta)$. Refer now to case 22 which is dominated by x_{11} only for $z_2 > x_{11}$. But $\zeta < x_{11}$, so the response $z_2 = x_{11}$ is best. However $z_1 = \eta$ under case 43 gives Player 1 $x_{43} = 1 - \delta(1-\eta) = x_{42}$ by Lemma b. But $x_{42} > x_{22}$ (Lemma P) so that with $z_2 = x_{11}$ Player 2 receives $1 - x_{22} > 1 - x_{42}$.

When (b) arises, we have $\delta z_1 = (\delta - z_2)/\delta \le \eta$, so that $\delta x_{24} \le z_2 \le \delta/(1+\delta)$.

Consider the best play against any z_1 in $[\delta x_{44}, \eta]$. Now $\delta x_{22} < \delta x_{44} < \hat{z} < \eta$. If $z_1 \le \hat{z}$ then case 13 starts at $z_2 = 1 - z_1$ making this the best reply (from case 31); or note that $x_{13} < x_{43}$ and case 13 gives better replies than for $z_2 \le \delta(1 - z_1)$.

If $z_1 > \hat{z}$, (which includes $z_1 = \eta$) note that case 11 occurs if $c > M(\delta)$, giving then a best reply of $z_2 = x_{11}$. Otherwise Case 12 is available for blocking case 31, giving a best reply of $z_2 = 1 - \hat{z}$. Case 31 requires $1 - \hat{z} \le z_1$, which holds iff $\hat{z} \ge 1 - \hat{z}$ iff $c \ge \frac{1}{2}H$. Otherwise, we have $\zeta < \hat{z} < z_1 < 1 - \hat{z} < 1 - \zeta$ and case 22 arises, and here best replies are up to $z_2 = 1 - \hat{z}$, by case 12: note that $\eta <^{-} x_{12}$ by Lemma η.

(iii) **Case 23.** For the z_2-domain to be non-empty it is necessary and sufficient that $z_1 \le \xi$. Assume so. There are three possibilities: $z_1 = \zeta, z_1 = \xi$, $z_1 = 1 - z_2 < \zeta, \xi$.

(a) Assume first that the right end-point of the z_1-domain is $z_1 = \zeta$ and that $\zeta < \xi$. Thus $c < \frac{1}{2}(1-\delta)$ by Lemma ξ, so $\zeta < 1 - \zeta$ by Lemma ζ, and correspondingly in case 23 we have $\zeta \le z_2 \le 1 - \zeta$.

Suppose first $c > L$. By (3.7) we have $1 - \hat{z} < \zeta$, and so case 23 cannot subsist under the restriction already derived above, namely that $z_2 \le 1 - \hat{z}$.

Now suppose $c \le L$. We have $\zeta \le 1 - \hat{z} < 1 - \zeta$ (by Lemma Z). Thus for $z_1 = \zeta$ we have in case 23 that $\zeta \le z_2 \le 1 - \zeta$ and the payoff to Player 1 is $x_{22} = 1 - \zeta$ (see Lemma b), so

Player 2 optimizes his payoff by taking $z_2 = \zeta$ (any larger choice incurs penalty). As the choice for $(z_1, z_2) = (\zeta, \zeta)$ is in the domain of case 22 we can confirm that it offers an equilibrium (see section 4.2.1). Since $\zeta < 1 - \zeta$, the payoff to the players is $(x_{22}, 1 - x_{22}) = (1 - \zeta, \zeta)$.

Conclusion: For $c < L$ the case yields an announcement (ζ, ζ) which is part of a SPNE but with outcome $(1 - \zeta, \zeta)$.

(b) Now assume that $z_1 = \xi$ is the right end-point. Then the z_2-domain collapses to the one point $z_2 = c/(1 - \delta) = 1 - \xi$. But $(z_1, z_2) = (\xi, 1 - \xi)$ is not an equilibrium by Lemma 4.3.

(c) Finally assume that $z_1 = z$ is the right end-point with $z = 1 - z_2 < \zeta$. For $z_2 = 1 - z$ consider case 13. Taking any $z_1 > \zeta$ with $\zeta < z_1 < \hat{z}$ we have $z_1 > z = 1 - z_2$ and

$$x_{23} = \delta z + (1 - \delta - c) < \delta z_1 + 1 - c - \delta + c\delta = x_{13}.$$

This concludes our consideration of outcomes linearly dependent on z_1.

4.1.3. Four cases depending on z_2 with $x_{ij}(z_1, z_2) = 1 - z_2$

We argue with z_1 fixed throughout these four cases. As $1 - x_{ij}$ is increasing in z_2, optimal behaviour on the part of Player 2 requires that z_2 be as large as the domain of definition permits. The cases 32 and 33 thus require that $z_1 + z_2 = 1$ at equilibrium.

(i) **Case 31.** Payoff maximization for Player 2 implies $z_2 = \hat{z}$, assuming $1 - z_1 < \hat{z}$ (for otherwise the case fails). Suppose first that $c > \frac{1}{2}H$. Here case 12 applies since $1 - \hat{z} < \hat{z}$. But $x_{12} > 1 - \hat{z} = x_{31}$ (for all c, see Lemma Z), so in this case there is no equilibrium outcome. Next suppose $c < \frac{1}{2}H$. As $c < L$ we have $\zeta < 1 - \hat{z}$, so $\zeta < \hat{z} < 1 - \zeta$, i.e. $z_2 = \hat{z}$ satisfies a defining condition of case 22. This offers Player 1 an interval of replies yielding a payoff $1 - \zeta > 1 - \hat{z}$ (since $\zeta < \hat{z}$), implying again that the case 31 does not yield a subgame-perfect equilibrium.

(ii) **Case 34**. Here we have $\delta/(1+\delta) \leq z_2 \leq \eta$ and in this range therefore $(c+\delta-z_2)/\delta \geq \delta x_{24}$. This together with $z_1 \leq \delta(1 - z_2) < (1 - z_2)$ shows that case 23 arises. Here Player 1 receives $x_{23} > 1 - z_2$ (by Lemma b) except when $z_2 = \eta$. Now suppose $z_2 = \eta$. If (z_1, z_2) is supported by a subgame-perfect equilibrium, we must in fact have $z_1 = \delta(1 - z_2)$, so $z_1 = \delta(1 - \eta) = \delta x_{24}$. This value for z_1 coincides with the left end-point of case 23 when $z_2 = \eta$. Thus better replies for Player 1, with a payoff x_{23}, which is superior to $x_{34} = 1 - \eta$, and is increasing in z_1, are available throughout the z_1-domain of case 23.

(iii) **Case 32.** Since here at a subgame-perfect equilibrium $z_1 + z_2 = 1$ the condition $z_1 + \delta z_2 \geq c + \delta$ implies that we have $z_1 \geq c/(1 - \delta)$, i.e. $z_2 \leq \xi$. In addition we have $z_2 \leq \zeta$. The case thus leads to the set of possible equilibrium outcomes (4.1), all of which are excluded by Lemma 4.6 with the exception of the pair $(\hat{z}, 1 - \hat{z})$ in the case $c \geq L$.

(iv) **Case 33.** Payoff maximization for Player 2 implies $z_2 = (c+\delta-z_1)/\delta$, or $z_2 = c+\delta(1-z_1)$. Using the condition $z_1 + z_2 = 1$ we find correspondingly the two possibilities for a subgame-perfect equilibrium settled by Lemmas 8 and 9.

As promised, the case considered here offer three possibilities for z_1, namely x_{22}, ζ and $\hat{z}$, but two negotiation subgame proposals (and outcomes) of x_{22} and $\hat{z}$.

We next turn to a detailed study of the cases 22 and

5. Proof of the Main Results: Existence

In this section we will first show that for $c \geq L(\delta)$ the announcement $S(c, \delta) = (\hat{z}, 1 - \hat{z})$ followed by the proposals $x = \hat{z}$ and $y = \delta x + c$ gives a subgame-perfect equilibrium. Then in section 5.2 for

$c < L(\delta)$ we consider the case 22 which offers the division $S(c,\delta) = (x_{22}, 1-x_{22})$. Here there is an interval of first round announcements for Player 1 to consider, and we show all of these together with the proposals $x = x_{22}$ and $y = \delta x + c$ form a subgame-perfect equilibrium of the commitment game $\mathcal{G}_{c\delta}$. For the former division $S(c,\delta)$ sections 5.1.1 and 5.1.2 below identify which cases ij of Table 3.1 may occur when $c \geq L(\delta)$ when one of the Players is assumed to announce his component of the vector S as his announced claim. Inspection of these cases follows a natural structure. Identify sequences of cases with one index, for instance i, fixed and the other j running through consecutive values: these correspond to contiguous intervals of possible announcements by the other player and exhibit monotonicity of pay-offs. This simplifies the identification of a best reply. For the latter division we are mostly concerned with identifying 'elbows' based on the case 22 (see Lemma-c).

5.1. The case $c \geq L(\delta)$

In this section we work under the assumption that $c \geq L(\delta)$.

5.1.1. Incidence of cases when $z_1 = \hat{z}$ for $c \geq L(\delta)$.

One checks that the only cases to occur are: 43,41,33,32,31,13,12. All of this follows from Lemma P and we omit the proof.■

We now consider these cases. The cases: 33,32,31 offer Player 1 an outcome $1 - z_2$. The identical outcome in the two cases 13 and 12 occurs on the domain $z_2 > 1 - \hat{z}$ and offers Player 1 more than $1 - z_2$, so Player 2 maximizes his payoff z_2 at the right end-point of case 32, and thus plays $z_2 = 1 - \hat{z}$. The cases 43,33,32,13 are contiguous (33 ends on $z_2 = (c + \delta - \hat{z})/\delta$ where 32 begins, since $(c + \delta - \hat{z})/\delta < (c + \delta - \delta\hat{z})$ for $c > L$, and $z_1 + z_2 < 1$ for this z_2), and so lead to a minimum payoff to Player 1 of value $\hat{z}$ on these cases. Note that case 32 has a domain constraint of $z_2 \leq 1 - \hat{z}$ and $z_2 \leq \zeta$ so the inequality (3.7) is critical to this case subsisting. Note that $x_{11} < x_{12} = x_{13}$ (Lemma b) and the case 11 is excluded by dominance considerations.

The cases 41,31,21 are also contiguous (obvious from Table 3.1 case $j = 1$) though the case 41 subsists iff $c > (1-\delta)^2/(1 - 2 + 2\delta^2)$. As long as 41 subsists, it is the case that $x_{41} < \hat{z}$ (Lemma b), so all these cases have lower payoff to Player 1 and are excluded by dominance considerations. If the case 41 fails, then the case 31 is contiguous to 32 and so has payoff below x_{13}; so is excluded by dominance considerations.

5.1.2. Incidence of cases when $z_2 = 1 - \hat{z}$ for $c \geq L(\delta)$.

One checks that the only cases to occur are: 44, 43, 41, 33, 32,31,13. All of this follows from Lemma P and we omit the proof.■

Conclusions. The cases 44,43,33,32 are contiguous and end on $z_1 = \hat{z}$. (Cases 44 and 43 subsist iff $c > H(\delta)$. If it subsists, case 43 'ends', i.e. has z_1 right-end point, where $z_2 \leq \delta(1 - z_1)$, as this inequality is sharper than $z_1 \leq \eta$ and here case 33 'begins'. If the case 43 fails, case 33 begins at $z_1 = \delta\hat{z}$. Either way the case 33 'ends' on $z_1 = c + \delta\hat{z}$ for $c > L(\delta)$ by (3.3).) The pay-offs are weakly increasing along this sequence with case 32 offering $1 - z_2 = \hat{z}$. The maximum on this sequence of cases occurs at $z_1 = \hat{z}$. The payoff x_{41} is below $\hat{z}$ iff $c > (1-\delta)^2/(1 - 2\delta + 2\delta^2)$, by Lemma Z, i.e., whenever the case 41 subsists. The domain of case 31 has $z_1 > \hat{z}$ and offers $\hat{z}$, i.e. below the demand z_1.We conclude that $z_1 = \hat{z}$ is a best reply.

5.2. The case $c < L(\delta)$: the case 22

In this section we work under the assumption that $c < L(\delta)$. Our uniqueness analysis of Section 4 showed that in the negotiation subgame case 22 may be supported by a subgame-perfect equilibrium. Since in this case the payoff is $(x_{22}, 1 - x_{22}) = (\sigma_1, \sigma_2) = (1 - \zeta, \zeta)$, Player 2 will not wish to make an announcement higher than ζ. We begin by considering the best response by Player 1 to $z_2 = \sigma_2 = \zeta$.

5.2.1. Incidence of the Case when $z_2 = \sigma_2 = \zeta$ for $c < L(\delta)$.

One checks that the only cases to occur are: 34,33,32,31,22,14,23 with domain of 23 a singleton.

All of these case return the same payoff of x_{22} so that in particular all the replies in the interval $(\zeta, 1 - \zeta)$ are best replies for Player 1.■

We now consider the best response by Player 2 to the two special cases $z_1 = 1 - \zeta$and $z_1 = \zeta$ and then to the general case $\zeta < z_1 < 1 - \zeta$.

5.2.2. Incidence of Case when $z_1 = x_{22} = \sigma_1 = 1 - \zeta$ for $c < L(\delta)$.

Here one shows that the only cases to occur are: 31,32,21,12,22,42 with 42 and 32 contiguous.

The case 22 has left endpoint $z_2 = \zeta$ to the left of case 12, which begins at $z_2 = 1 - \hat{z}$ as $c < L(\delta)$. We note that $x_{12} < x_{42}$. Clearly the case 22 offers Player 1 $x_{22} < x_{12}$, so the largest value z_2 under case 32 consistent with z_2 lying outside the case 22 is $z_2 = \zeta$. That is thus the best reply. Note that $x_{21} < x_{12}$ so the case 21 is dominated.

5.2.3. Incidence of Case when $z_1 = \sigma_1 = 1 - \zeta$ for $c < L(\delta)$.

Here one shows that the only cases to occur are: 43,34,32,31,22,23,13.

An 'elbow' occurs at $z_2 = \zeta$ as 31 has z_2-domain $\delta x_{22} < z_2 < \hat{z}$ (note$1 - \zeta = x_{22}$) contiguous with case 22 which has $\zeta \leq z_2 \leq x_{22}$; at $z_2 = \zeta$ we have $x_{31} = 1 - \zeta = x_{22}$. The case 32 is to the left of $z_2 = \zeta$. The case 34 has $z_2 > \delta x_{44} > \delta x_{22}$ and, as $\zeta < \eta$, is subsumed by cases 31 and 22. Case 13 has $z_2 > 1 - \zeta$, a domain contiguous to the right of case 22, and one checks that $x_{13} > x_{22}$. As for 23 since $x_{23} = \zeta < 1 - \zeta$ no dominance occurs. Case 43 has $z_2 \leq \delta x_{22}$.and $x_{43} = x_{41} > x_{22}$.

5.2.4. Incidence of Case when $\zeta = \sigma_2 < z_1 < \sigma_1 = 1 - \zeta$ for $c < L(\delta)$.

Here one shows that the only cases to occur are: 42,43,34, possibly 33,32,31,22, possibly 21,12,13.

Begin by noting that case 32 is contiguous with 22 at $z_2 = \zeta$.Case 13 has left end-point at $z_2 = 1 - z_1 > \zeta$.We have $\zeta < 1 - z_1 < 1 - \zeta$ and the case 31 falls below 22. cases 34 falls below case 22; case33 falls below 22 and 13. Cases 42 and 43 if they occur are to the left of case 22. Case 21 if it occurs is for $z_2 \geq \hat{z} > \zeta$. Thus $z_2 = \zeta$ is a best reply.

6. Concluding discussion

There are a number of consequences of a discontinuous penalty structure. Foremost is this: if one terminates the negotiation subgame after a finite number of steps in order to use a backwards induction on the commitment game, one needs to adapt the usual arguments and definitions to include discontinuous 'compensated present value functions'. This involves consideration of 'pseudo-fixed' points, i.e. values x such that $F(x) = F(x+) = \lim_{t \searrow x} F(t)$ or $F(x) = F(x-) = \lim_{t \nearrow x} F(t)$, which give rise to limiting equilibria. These are ε-equilibria, but of a specific form, which we term pseudo-equilibria (see web-site version of this paper). Thanks to Tarski's Fixed Point Theorem (see Tarski (1955) and Davies, Hayes, Rousseau (1971)), games with the natural property of a monotone 'present value' (as is the case here) still yield Nash equilibria. It is the case that the largest and smallest subgame-perfect pseudo-equilibrium division is, just as in our model, a true 'fixed-point' in the continuation game – this can be shown by a refinement of the argument of Shaked and Sutton (1984).

We have not attempted to verify whether there are subgame-perfect equilibrium divisions which are not fixed-points. However, one can argue that *for practical purposes,* the pseudo-fixed points include all the pseudo-equilibrium divisions arising from monotone pure strategy pairs (σ, τ) in the negotiation subgame, in the following sense. Going down the negotiation tree one may derive a relation between Player 1's equilibrium offers conditional on the past to obtain the indifference equations

$$x_i = d_2(d_1(x_{i+2})) \text{ for } i = 0, 2, 4, \ldots \ . \tag{6.1}$$

If this sequence is not eventually constant, it will be monotonic and will have as limit a pseudo-fixed point. Terminating the sequence, at will, at $x_N = u$, and reading the equation as a forwards iteration, establishes a unique sequence $\{v_r(u)\}$ defined by its initial value u, which is monotonic and converging as $r \to \infty$ towards a **nearest pseudo-fixed point** (to u) of $F(x, z_1, z_2)$. The function F is piecewise linear with slope either zero or δ^2 in each maximal interval Δ of linearity, thus the convergence is uniform. It is for this reason that we may consider the infinite negotiation game as terminating after some large, but reasonable, number of rounds depending on the tolerance level ε admitted when looking at ε-equilibria.

As regards the multiplicity of fixed-points, one can show that the largest equilibrium division of the negotiation subgame arises in the limit, as the horizon recedes to infinity, of a terminated subgame, à la Ståhl (1972), in which Player 1 as last bidder takes the whole asset. See Binmore (1981); the smallest equilibrium similarly corresponds to Player 2 taking all as last bidder. There may be an additional third stationary strategy equilibrium division, and this is characterized as the nearest fixed point from some fixed, alternative intermediate division of the asset in the final round (and horizon taken to infinity). We refer the reader to Binmore, Piccione and Samuelson (1998) for a discussion of the support a perspective of evolutionary stability offers to the significance of stationary strategies.

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