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Normed versus topological groups: Dichotomy and duality

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Abstract

The key vehicle of the recent development of a topological theory of regular variation based on topological dynamics [BOst-TRI], and embracing its classical univariate counterpart (cf. [BGT]) as well as fragmentary multivariate (mostly Euclidean) theories (eg [MeSh], [Res], [Ya]), are groups with a right-invariant metric carrying flows. Following the vector paradigm, they are best seen as *normed groups*. That concept only occasionally appears explicitly in the literature despite its frequent disguised presence, and despite a respectable lineage traceable back to the Pettis closed-graph theorem, to the Birkhoff-Kakutani metrization theorem and further back still to Banach's *Théorie des opérations linéaires*. We collect together known salient features and develop their theory including Steinhaus theory unified by the Category Embedding Theorem [**?**], the associated themes of subadditivity and convexity, and a topological duality inherent to topological dynamics. We study the latter both for its independent interest and as a foundation for topological regular variation.

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1. Introduction

Group-norms, which behave like the usual vector norms except that scaling is restricted to the basic scalars of group theory (the units ± 1 in an abelian context and the exponents ± 1 in the non-commutative context), have played a part in the early development of topological group theory. They appear naturally in the study of groups of homeomorphisms. Although ubiquitous, they lack a clear and unified exposition. This lack is our motivation here, since they offer the right context for the recent theory of *topological regular variation*. This extends the classical theory (for which see, e.g. [BGT]) from the real line to metrizable topological groups. Normed groups are just groups carrying a right-invariant metric. The basic metrization theorem for groups, the Birkhoff-Kakutani Theorem of 1936 ([Bir], [Kak], see [Kel, Ch.6 Problems N-R], [Klee], [Bour, Part 2, Section 4.1], and [ArMa], compare also [Eng, Exercise 8.1.G and Th. 8.1.21]), is usually stated as asserting that a first-countable Hausdorff group has a right-invariant metric. It is properly speaking a 'normability' theorem in the style of Kolmogorov's Theorem ([Kol], or [Ru, Th. 1.39]; in this connection see also [Jam], where strong forms of connectedness are used in an abelian setting to generate norms), as we shall see below. Indeed the metric construction in [Kak] is reminiscent of the more familiar construction of a Minkowski functional (for which see [Ru, Sect. 1.33]), but is implicitly a supremum norm – as defined below; in Rudin's derivation of the metric (for a topological vector space setting, [Ru, Th. 1.24]) this norm is explicit. Early use by A. D. Michal and his collaborators was in providing a canonical setting for differential calculus (see the review [Michal2] and as instance [JMW]) and included the noteworthy generalization of the implicit function theorem by Bartle [Bart] (see Th. 10.10). In name the group-norm makes an explicit appearance in 1950 in [Pet1] in the course of his classic closed-graph theorem (in connection with Banach's closed-graph theorem and the Banach-Kuratowski category dichotomy for groups). It reappears in the group context in 1963 under the name 'length function', motivated by word length, in the work of R. C. Lyndon [Lyn2] (cf. [LynSch]) on Nielsen's Subgroup Theorem, that a subgroup of a free group is a free group. (Earlier related usage for function spaces is in [EH].) The latter name is conventional in geometric group theory despite the parallel usage in algebra (cf. [Far]) and the recent work on norm extension (from a normal subgroup) of Bökamp [Bo].

When a group is topologically complete and also abelian, then it admits a metric which is *bi-invariant*, i.e. is both right- and left-invariant, as [Klee] first showed (in course of solving a problem of Banach). In Section 3 we characterize non-commutative groups that have a bi-invariant metric, a context of significance for the calculus of regular variation (in the study of products of regularly varying functions with range a normed group) – see [BOst-TRIII]. In a normed group topological completeness yields a powerful Shift Theorem, generalizing the following theorem on the reals about shift embedding of sequences (due in a weak form to Kestelman and in a Lebesgue-measure setting to Borwein and Ditor). We say *generically all* to mean 'off a meagre/null set', according to whether the context is (Baire) category, where we also say *quasi all,* or (Lebesgue, or more generally Haar) measure, where we say *almost all.*

THEOREM 1.1 (Kestelman-Borwein-Ditor Theorem, KBD). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null, or Baire and non-meagre, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

A stronger form still is derived in [BOst-Funct] (the Generic Reflection Theorem); see also [BOst-StOstr] Section 3.1 Note 3, [BOst-LBI] Section 3.1 Note 1. For proofs see the original papers [Kes] and [BoDi]; for a unified treatment in the real-variable case see [BOst-Funct]. Applications of shift embedding are implicit in Banach [Ban-Eq] and explicit though not by name in Banach [Ban-T] in the proofs that a measurable/Baire additive function is continuous (see the commentary by Henryk Fast loc. cit. p. 314 for various one-way implications among related results). The present paper is motivated precisely by normed groups being the natural setting for generalizations of the KBD Theorem and its numerous important applications (initially noticed in the Uniform Convergence Theorem of the theory of regular variation). Normed groups, as we will see, are subject to a dichotomy centered on automatic continuity (for background see Section 3.3 and Section 11), as to whether or not inner automorphisms $x \to gxg^{-1}$ are continuous: normed groups are thus either *topological groups* or *pathological groups.* That is, a smidgen of regularity tips the normed group over to a topological group. We are thus mostly concerned with the former; but even so in general, in the presence of completeness, they support a generalization of KBD from which one may derive a *Squared Pettis Theorem* (that $(AA^{-1})^2$, for A Baire non-meagre, has the identity as an interior point, Th.5.8); that in turn guarantees in the category of normed groups the Banach-Mehdi Continuity Theorem for Baire-continuous homomorphisms (Th 11.10), the Baire Homomorphism Theorem (Th.11.11) and the Souslin Graph Theorem (Th. 11.12). The origin of the squaring is the following first of several generalizations of KBD (cf. Th. 5.1).

THEOREM 1.2 (Kestelman-Borwein-Ditor Theorem – Normed Groups). *In a topologically complete normed group X, if $\{z_n\} \to e_X$ (a null sequence converging to the identity), T is Baire and non-meagre under the right norm topology, then there are $t, t_m \in T$ and an infinite set $\mathbb{M}_t$ such that*

$$\{tt_m^{-1}z_mt_m : m \in \mathbb{M}_t\} \subseteq T.$$

Topological completeness is a natural assumption here, but it is unnecessarily strong. Respectably defined subgroups of even a compact topological group need not be $\mathcal{G}_\delta$ (see [ChMa] and [FaSol] for such examples). In Section 5 we employ the weaker notion of *almost complete metrizability* which is applicable to non-meagre Souslin-$\mathcal{F}$ subspaces of a topologically complete subgroup, so embracing the non-complete examples just cited. Critical result like Th. 1.2 will be developed below using a;most completeness; elsewhere, or simplicity, we often work with topological completeness.

Fresh interest in metric groups dates back to the seminal work of Milnor [Mil] in 1968 on the metric properties of the fundamental group of a manifold and is key to the global study of manifolds initiated by Gromov [Gr1], [Gr2] in the 1980s (and we will see quasi-isometries in the duality theory of normed groups in Section 12), for which see [BH] and also [Far] for an early account; [PeSp] contains a variety of generalizations and their uses in interpolation theory (but the context is abelian groups).

The very recent [CSC] (see Sect. 2.1.1, Embedding quasi-normed groups into Banach spaces) employs norms in considering Ulam's problem (see [Ul]) on the *global* approximation of nearly additive functions by additive functions. This is a topic related to regular variation, where the weaker concept of *asymptotic* additivity is the key. Recall the classical definition of a regularly varying function, namely a function $h : \mathbb{R} \to \mathbb{R}$ for which the limit

$$\partial_{\mathbb{R}} h(t) := \lim_{x\to\infty} h(tx)h(x)^{-1} \qquad \text{(rv-limit)}$$

exists everywhere; for f Baire, the limit function is a continuous homomorphism (i.e. a multiplicative function). Following the pioneering study of [BajKar] launching a general (i.e., topological) theory of regular variation, [BOst-TRI] has re-interpreted (rv-limit), by replacing $|x| \to \infty$ with $\|x\| \to \infty$, for functions $h : X \to H$, with tx being the image of x under a T-flow on X (cf. Th. 2.7 and preceding definition), and with X, T, H all groups with right-invariant metric (right because of the division on the right) – i.e. normed groups (making ∂h_X a differential at infinity, in Michal's sense [Michal1]). In concrete applications the groups may be the familiar Banach groups of functional analyis, the associated flows either the ubiquitous domain translations of Fourier transform theory or convolutions from the related contexts of abstract harmonic analysis (e.g. Wiener's Tauberian theory so relevant to classical regular variation – see e.g. [BGT, Ch. 4]). In all of these one is guaranteed right-invariant metrics. Likewise in the foundations of regular variation the first tool is the group $\mathcal{H}(X)$ of bounded self-homeomorphisms of the group X under a supremum metric (and acting transitively on X); the metric is again right-invariant and hence a group-norm. It is thus natural, in view of the applications and the Birkhoff-Kakutani Theorem, to favour right-invariance.

We show in Section 4 and 10 that normed groups offer a natural setting for subadditivity and for (mid-point) convexity.

2. Metric versus normed groups

This section is devoted to group-norms and their associated metrics. We collect here some pertinent information (some of which is scattered in the literature). A central tool for applications is the introduction of the subgroup of bounded homeomorphisms of a given group $\mathcal{G}$ of self-homeomorphisms of a topological group X; the subgroup possesses a guaranteed right-invariant metric. This is the archetypal example of the symbiosis of norms and metrics, and it bears repetition that, in applications just as here, it is helpful to work simultaneously with a right-invariant metric and its associated group-norm.

We say that the group X is *normed* if it has a group-norm as defined below (cf. [DDD]).

Definition. We say that $\|\cdot\| : X \to \mathbb{R}_+$ is a *group-norm* if the following properties hold:
(i) Subadditivity (Triangle inequality): $\|xy\| \le \|x\| + \|y\|$;
(ii) Positivity: $\|x\| > 0$ for $x \ne e$ and $||e|| = 0$;
(iii) Inversion (Symmetry): $\|x^{-1}\| = \|x\|$.

If (i) holds we speak of a group *semi-norm*; if (i) and (iii) and $\|e\| = 0$ holds one speaks of a *pseudo-norm* (cf. [Pet1]); if (i) and (ii) hold we speak of a group *pre-norm* (see [Low] for a full vocabulary).

We say that a group pre-norm, and so also a group-norm, is *abelian*, or more precisely *cyclically permutable,* if

(iv) Abelian norm (cyclic permutation): $\|xy\| = \|yx\|$ for all x, y.

Other properties we wish to refer to are :

(i)$_K$ for all $x, y : \|xy\| \le K(\|x\| + \|y\|)$,

(i)$_{\mathrm{ult}}$ for all $x, y : \|xy\| \le \max\{\|x\|, \|y\|\}$.

Remarks 1. 1. Mutatis mutandis this is just the usual vector norm, but with scaling restricted to the units ± 1. The notation and language thus mimick the vector space counterparts, with subgroups playing the role of subspaces; for example, for a symmetric, subbadditive $p : X \to \mathbb{R}_+$, the set $\{x : p(x) = 0\}$ is a subgroup. Indeed the analysis of Baire subadditive functions (see Section 4) is naturally connected with norms, via regular variation. That is why normed groups occur naturally in regular variation theory.

2. When (i)$_K$, for some constant K, replaces (i), one speaks of quasi-norms (see [CSC], cf. 'distance spaces' [Rach] for a metric analogue). When (i)$_{\mathrm{ult}}$ replaces (i) one speaks of an ultra-norm, or non-Archimedean norm. For an example of the latter, in connection with the p-adic topology of a group, see [Fu, I.7.2].

3. Note that (i) implies joint continuity of multiplication at the identity e_X, while (iii) implies continuity of inversion at e_X, a matter we return to in Th. 2.19$'$ and in Section 3. (Montgomery [Mon1] shows that joint continuity is implied by separate continuity when the group is locally complete – cf. Th. 3.47; Ellis [Ell1] considers when one-sided continuity implies joint continuity in the case of locally compact abelian groups.) In a related theme Żelazko [Zel] considers a locally complete metric structure under which an

abelian group has separately continuous multiplication and shows this to be a topological group. See below for the stronger notion of uniform continuity invoked in the Uniformity Theorem of Conjugacy (Th. 12.4).

4. Abelian groups with ordered norms may also be considered, cf. [JMW].

Remarks 2. Subadditivity implies that $\|e\| \geq 0$ and this together with symmetry implies that $\|x\| \geq 0$, since $\|e\| = \|xx^{-1}\| \leq 2\|x\|$; thus a group-norm cannot take negative values. Subadditivity also implies that $\|x^n\| \leq n\|x\|$, for natural n. The norm is said to be 2-*homogeneous* if $\|x^2\| = 2\|x\|$; see [CSC] Prop. 4.12 (Ch. IV.3 p.38) for a proof that if a normed group is *amenable* or *weakly commutative* (defined in [CSC] to mean that, for given x, y, there is m of the form 2^n, for some natural number n, with $(xy)^m = x^m y^m$), then it is embeddable as a subgroup of a Banach space. In the case of an abelian group 2-homogeneity corresponds to sublinearity, and here Berz's Theorem characterizes the norm (see [Berz] and [BOst-GenSub]). The abelian property implies only that $\|xyz\| = \|zxy\| = \|yzx\|$, hence the alternative name of 'cyclically permutable'. Harding [H], in the context of quantum logics, uses this condition to guarantee that the group operations are jointly continuous (cf. Theorem 2 below) and calls this a *strong norm.* See [Kel, Ch. 6 Problem O] (which notes that a locally compact group with abelian norm has a bi-invariant Haar measure). We note Ellis' Theorem that, for X a locally compact group, continuity of the inverse follows from the separate continuity of multiplication (see [Ell2], or [HS, Section 2.5]). The more recent literature concerning when joint continuity of $(x, y) \to xy$ follows from separate continuity reaches back to Namioka [Nam] (see e.g. [Bou1], [Bou2], [HT], [CaMo]).

Convention. For a variety of purposes and for the sake of clarity, when we deal with a metrizable group X if we assume a metric d^X on X is right/left invariant we will write d^X_R or d^X_L, omitting the superscript and perhaps the subscript if context permits.

Remarks 3. For X a metrizable group with right-invariant metric d^X and identity e_X, the canonical example of a group-norm is identified in Proposition 2.3 below as

$$\|x\| := d^X(x, e_X).$$

It is convenient to use the above notation irrespective of whether the metric d^X is invariant.

Remarks 4. If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, subadditive with $f(0) = 0$, and $\|x\|_1$ is a group-norm, then

$$\|x\|_2 := f(\|x\|_1)$$

is also a group-norm. See [BOst-GenSub] for recent work on Baire (i.e., having the Baire property) subadditive functions. These will appear in Sections 3 and 4.

We begin with two key definitions.

Definition and Notation. For X a metric space with metric d^X and $\pi : X \to X$ a bijection the *π-permutation* metric is defined by

$$d^X_\pi(x,y) := d^X(\pi(x),\pi(y)).$$

When X is a group we will also say that d^X_π is the *π-conjugate* of d^X. We write

$$\|x\|_\pi := d^X(\pi(x),\pi(e)),$$

and for d any metric on X

$$B^d_r(x) := \{y : d(x,y) < r\},$$

suppressing the superscript for $d = d^X$; however, for $d = d^X_\pi$ we adopt the briefer notation

$$B^\pi_r(x) := \{y : d^X_\pi(x,y) < r\}.$$

Following [BePe] $Auth(X)$ denotes the algebraic group of self-homeomorphisms (or auto-homeomorphisms) of X under composition, i.e. without a topological structure. We denote by id_X the identity map $id_X(x) = x$ on X.

Examples A. Let X be a group with metric d^X. The following permutation metrics arise naturally in this study. (We use the notation $\|x\| := d^X(x, e_X)$, for an arbitrary metric.)

1. With $\pi(x) = x^{-1}$ we refer to the π-permutation metric as the involution-conjugate, or just the conjugate, metric and write

$$\tilde{d}^X(x,y) = d^X_\pi(x,y) = d^X(x^{-1},y^{-1}), \qquad \text{so that} \qquad \|x\|_\pi = \|x^{-1}\|.$$

2. With $\pi(x) = \gamma_g(x) := gxg^{-1}$, the inner automorphism, we have (dropping the additional subscript, when context permits):

$$d^X_\gamma(x,y) = d^X(gxg^{-1},gyg^{-1}), \qquad \text{so that} \qquad \|x\|_\gamma = \|gxg^{-1}\|.$$

3. With $\pi(x) = \lambda_g(x) := gx$, the left-shift by g, we refer to the π-permutation metric as the g-conjugate metric, and we write

$$d^X_g(x,y) = d^X(gx,gy).$$

If d^X is right-invariant, cancellation on the right gives

$$d^X(gxg^{-1},gyg^{-1}) = d^X(gx,gy), \text{ i.e. } d^X_\gamma(x,y) = d^X_g(x,y) \text{ and } \|x\|_g = \|gxg^{-1}\|.$$

For d^X right-invariant, $\pi(x) = \rho_g(x) := xg$, the right-shift by g, gives nothing new:

$$d^X_\pi(x,y) = d^X(xg,yg) = d^X(x,y).$$

But, for d^X left-invariant, we have

$$\|x\|_\pi = \|g^{-1}xg\|.$$

4 (Topological permutation). For $\pi \in Auth(X)$, i.e. a homeomorphism and x fixed, note that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$d_\pi(x, y) = d(\pi(x), \pi(y)) < \varepsilon,$$

provided $d(x, y) < \delta$, i.e.

$$B_\delta(x) \subset B^\pi_\varepsilon(x).$$

Take $\xi = \pi(x)$ and write $\eta = \pi(y)$; there is $\mu > 0$ such that

$$d(x, y) = d_{\pi^{-1}}(\xi, \eta) = d(\pi^{-1}(\xi), \pi^{-1}(\eta)) < \varepsilon,$$

provided $d_\pi(x, y) = d(\pi(x), \pi(y)) = d(\xi, \eta) < \mu$, i.e.

$$B^\pi_\mu(x) \subset B_\varepsilon(x).$$

Thus the topology generated by d_π is the same as that generated by d. This observation applies to all the previous examples provided the permutations are homeomorphisms (e.g. if X is a topological group under d^X). Note that for d^X right-invariant

$$\|x\|_\pi = \|\pi(x)\pi(e)^{-1}\|.$$

5. For $g \in Auth(X)$, $h \in X$, the bijection $\pi(x) = g(\rho_h(x)) = g(xh)$ is a homeomorphism provided right-shifts are continuous. We refer to this as the *shifted g-h-permutation metric*

$$d^X_{g\text{-}h}(x, y) = d^X(g(xh), g(yh)),$$

which has the associated g-h-shifted norm

$$\|x\|_{g\text{-}h} = d^X(g(xh), g(h)).$$

6 (Equivalent Bounded norm). Set $d_b(x, y) = \min\{d^X(x, y), 1\}$. Then d_b is an equivalent metric (cf. [Eng, Th. 4.1.3, p. 250]). We refer to

$$\|x\|_b := d_b(x, e) = \min\{d^X(x, e), 1\} = \min\{\|x\|, 1\}$$

as the *equivalent bounded norm.*

7. For $\mathcal{A} = Auth(X)$ the evaluation pseudo-metric at x on $\mathcal{A}$ is given by

$$d^{\mathcal{A}}_x(f, g) = d^X(f(x), g(x)),$$

and so

$$\|f\|_x = d^{\mathcal{A}}_x(f, id) = d^X(f(x), x)$$

is a pseudo-norm.

Definition (Refinements). 1 (cf. [GJ, Ch. 15.3] which works with pseudometrics). Let $\Delta = \{d^X_i : i \in I\}$ be a family of metrics on a group X. The *weak (Tychonov) Δ-refinement topology* on X is defined by reference to the local base at x obtained by finite intersections of ε-balls about x :

$$\bigcap\nolimits_{i\in F} B^i_\varepsilon(x), \text{ for } F \text{ finite, i.e. } B^{i_1}_\varepsilon(x) \cap ... \cap B^{i_n}_\varepsilon(x), \text{ if } F = \{i_1, ..., i_n\},$$

where

$$B^i_\varepsilon(x) := \{y \in X : d^X_i(x,y) < \varepsilon\}.$$

2. The *strong Δ-refinement topology* on X is defined by reference to the local base at x obtained by full intersections of ε-balls about x :

$$\bigcap\nolimits_{d\in\Delta} B^d_\varepsilon(x). \tag{Str}$$

Clearly

$$\bigcap\nolimits_{d\in\Delta} B^d_\varepsilon(x) \subset \bigcap\nolimits_{i\in F} B^i_\varepsilon(x), \text{ for } F \text{ finite},$$

hence the name. We will usually be concerned with a family Δ of conjugate metrics. We note the following, which is immediate from the definition. (For (ii) see the special case in [dGMc, Lemma 2.1], [Ru, Ch. I 1.38(c)], or [Eng, Th. 4.2.2 p. 259], which uses a sum in place of a supremum, and identify X with the diagonal of $\prod_{d\in\Delta}(X,d)$; see also [GJ, Ch. 15].)

PROPOSITION 2.1. *(i) The strong Δ-refinement topology is generated by the supremum metric*

$$d^X_\Delta(x,y) = \sup\{d^X_i(x,y) : i \in I\}.$$

(ii) For Δ a countable family of metrics indexed by $I = \mathbb{N}$, the weak Δ-refinement topology is generated by the weighted-supremum metric

$$d^X_\Delta(x,y) = \sup\nolimits_{i\in I} 2^{1-i}\frac{d^X_i(x,y)}{1+d^X_i(x,y)}.$$

This corresponds to the metric of first-difference in a product of discrete metric spaces, e.g. in the additive group $\mathbb{Z}^{\mathbb{Z}}$. (That is, $d^X_\Delta(\{x_i\},\{y_i\}) = 2^{-n(x,y)}$, where the two sequences first differ at index $i = n(x,y)$.)

EXAMPLES B. 1. For X a group we may take $\Delta = \{d^X_z : z \in X\}$ to obtain

$$d^X_\Delta(x,y) = \sup\{d^X(zx,zy) : z \in X\},$$

and if d^X is right-invariant

$$\|x\|_\Delta = \sup\nolimits_z \|zxz^{-1}\|.$$

2. For X a topological group we may take $\Delta = \{d^X_h : h \in Auth(X)\}$, to obtain

$$d^X_\Delta(x,y) = \sup\{d^X(h(x),h(y)) : h \in Auth(X)\}.$$

3. In the case $\mathcal{A} = Auth(X)$ we may take $\Delta = \{d^{\mathcal{A}}_x : x \in X\}$, the evaluation pseudo-metrics, to obtain

$$\begin{aligned} d^{\mathcal{A}}_\Delta(f,g) &= \sup\nolimits_x d^{\mathcal{A}}_x(f,g) = \sup\nolimits_x d^X(f(x),g(x)), \qquad \text{and} \\ \|f\|_\Delta &= \sup\nolimits_x d^{\mathcal{A}}_x(f,id_X) = \sup\nolimits_x d^X(f(x),x). \end{aligned}$$

In Proposition 2.12 we will show that the strong Δ-refinement topology restricted to the subgroup $\mathcal{H}(X) := \{f \in \mathcal{A} : \|f\|_\Delta < \infty\}$ is the topology of uniform convergence. The weak Δ-refinement topology here is just the topology of pointwise convergence.

The following result is simple; we make use of it in the Definition which follows Lemma 3.23.

Proposition 2.2 (Symmetrization refinement). *If* $\|x\|_0$ *is a group pre-norm, then the symmetrization refinement*

$$\|x\| := \max\{\|x\|_0, \|x^{-1}\|_0\}$$

is a group-norm.

Proof. Positivity is clear, likewise symmetry. Noting that, for any A, B,

$$a + b \leq \max\{a, A\} + \max\{b, B\},$$

and supposing without loss of generality that

$$\max\{\|x\|_0 + \|y\|_0, \|y^{-1}\|_0 + \|x^{-1}\|_0\} = \|x\|_0 + \|y\|_0,$$

we have

$$\begin{aligned} \|xy\| &= \max\{\|xy\|_0, \|y^{-1}x^{-1}\|_0\} \leq \max\{\|x\|_0 + \|y\|_0, \|y^{-1}\|_0 + \|x^{-1}\|_0\} \\ &= \|x\| + \|y\|_0 \leq \max\{\|x\|_0, \|x^{-1}\|_0\} + \max\{\|y\|_0, \|y^{-1}\|_0\} \\ &= \|x\| + \|y\|. \end{aligned}$$

■

Remark. One can use summation and take $\|x\| := \|x\|_0 + \|x^{-1}\|_0$, as

$$\|xy\| = \|xy\|_0 + \|y^{-1}x^{-1}\|_0 \leq \|x\|_0 + \|y\|_0 + \|y^{-1}\|_0 + \|x^{-1}\|_0 = \|x\| + \|y\|.$$

However, here and below, we prefer the more general use of a supremum or maximum, because it corresponds directly to the intersection formula (Str) which defines the refinement topology. We shall shortly see a further cogent reason (in terms of the refinement norm).

Proposition 2.3. *If* $\|\cdot\|$ *is a group-norm, then* $d(x,y) := \|xy^{-1}\|$ *is a right-invariant metric; equivalently,* $\tilde{d}(x,y) := d(x^{-1}, y^{-1}) = \|x^{-1}y\|$ *is the conjugate left-invariant metric on the group.*
Conversely, if d *is a right-invariant metric, then* $\|x\| := d(e,x) = \tilde{d}(e,x)$ *is a group-norm.*
Thus the metric d *is bi-invariant iff* $\|xy^{-1}\| = \|x^{-1}y\| = \|y^{-1}x\|$, *i.e. iff the group-norm is abelian.*
Furthermore, for $(X, \|\cdot\|)$ *a normed group, the inversion mapping* $x \to x^{-1}$ *from* (X,d) *to* $(X,\tilde{d})$ *is an isometry and hence a homeomorphism.*

Proof. Given a group-norm put $d(x,y) = \|xy^{-1}\|$. Then $\|xy^{-1}\| = 0$ iff $xy^{-1} = e$, i.e. iff $x = y$. Symmetry follows from inversion as $d(x,y) = \|(xy^{-1})^{-1}\| = \|yx^{-1}\| = d(y,x)$. Finally, d obeys the triangle inequality, since

$$\|xy^{-1}\| = \|xz^{-1}zy^{-1}\| \leq \|xz^{-1}\| + \|zy^{-1}\|.$$

As for the converse, given a right-invariant metric d, put $\|x\| := d(e,x)$. Now $\|x\| = d(e,x) = 0$ iff $x = e$. Next, $\|x^{-1}\| = d(e,x^{-1}) = d(x,e) = \|x\|$, and so

$$d(xy,e) = d(x,y^{-1}) \le d(x,e) + d(e,y^{-1}) = \|x\| + \|y\|.$$

Also $d(xa,ya) = \|xaa^{-1}y^{-1}\| = d(x,y)$.
If d is bi-invariant iff $d(e,yx^{-1}) = d(x,y) = d(e,x^{-1}y)$ iff $\|yx^{-1}\| = \|x^{-1}y\|$. Inverting the first term yields the abelian property of the group-norm.
Finally, for $(X,\|\cdot\|)$ a normed group and with the notation $d(x,y) = \|xy^{-1}\|$ etc., the mapping $x \to x^{-1}$ from $(X,d_R^X) \to (X,d_L^X)$ is an isometry and so a homeomorphism, as $d_L(x^{-1},y^{-1}) = d_R(x,y)$. ■

The two (inversion) conjugate metrics separately define a right and left uniformity; their common refinement is the *symmetrized metric*

$$d_S^X(x,y) := \max\{d_R^X(x,y), d_L^X(x,y)\},$$

defining what is known as the *ambidextrous uniformity*, the only one of the three capable in the case of topological groups of being complete – see [Br-1], [Hal-ET, p. 63] (the case of measure algebras), [Kel, Ch. 6 Problem Q] , and also [Br-2]. We return to these matters in Section 3. Note that

$$d_S^X(x,e_X) = d_R^X(x,e_X) = d_L^X(x,e_X),$$

i.e. the symmetrized metric defines the same norm.

Definitions. 1. For d_R^X a right-invariant metric on a group X, we are justified by Proposition 2.2 in defining the *g-conjugate norm* from the g-conjugate metric by

$$\|x\|_g := d_g^X(x,e_X) = d_R^X(gx,g) = d_R^X(gxg^{-1},e_X) = \|gxg^{-1}\|.$$

2. For Δ a family of right-invariant metrics on X we put $\Gamma = \{\|.\|_d : D \in \Delta\}$, the set of corresponding norms defined by

$$\|x\|_d := d(x,e_X), \text{ for } d \in \Delta.$$

The *refinement norm* is then, as in Proposition 2.1,

$$\|x\|_\Gamma := \sup_{d\in\Delta} d(x,e_X) = \sup_{d\in\Gamma} \|x\|_d.$$

We will be concerned with special cases of the following definition.

Definition ([Gr1], [Gr2], [BH, Ch. I.8]). For constants $\mu \ge 1, \gamma \ge 0$, the metric spaces X and Y are said to be *(μ-γ)-quasi-isometry* under the mapping $\pi : X \to Y$ if

$$\frac{1}{\mu}d^X(a,b) - \gamma \le d^Y(\pi a,\pi b) \le \mu d^X(a,b) + \gamma \qquad (a,b \in X),$$
$$d^Y(y,\pi[X]) \le \gamma \qquad (y \in Y).$$

Corollary 2.4. *For π a homomorphism, the normed groups X, Y are (μ-γ)-quasi-isometric under π for the corresponding metrics iff the associated norms are (μ-γ)-quasi-equivalent, i.e.*

$$\frac{1}{\mu}\|x\|_X - \gamma \leq \|\pi(x)\|_Y \leq \mu\|x\|_X + \gamma \qquad (a, b \in X),$$
$$d^Y(y, \pi[X]) \leq \gamma \qquad (y \in Y).$$

Proof. This follows from $\pi(e_X) = e_Y$ and $\pi(xy^{-1}) = \pi(x)\pi(y)^{-1}$. ■

Remark. Note that $p(x) = \|\pi(x)\|_Y$ is subadditive and bounded at $x = e$. It will follow that p is locally bounded at every point when we later prove Lemma 4.3.

The following result (which we use in [BOst-TRII]) clarifies the relationship between the conjugate metrics and the group structure. We define the ε-*swelling* of a set K in a metric space X for a given (e.g. right-invariant) metric d^X, to be

$$B_\varepsilon(K) := \{z : d^X(z, k) < \varepsilon \text{ for some } k \in K\} = \bigcup\nolimits_{k\in K} B_\varepsilon(k)$$

and for the conjugate (resp. left-invariant) case we can write similarly

$$\tilde{B}_\varepsilon(K) := \{z : \tilde{d}^X(z, k) < \varepsilon \text{ for some } k \in K\}.$$

We write $B_\varepsilon(x_0)$ for $B_\varepsilon(\{x_0\})$, so that

$$B_\varepsilon(x_0) := \{z : \|zx_0^{-1}\| < \varepsilon\} = \{wx_0 : w = zx_0^{-1}, \|w\| < \varepsilon\} = B_\varepsilon(e)x_0.$$

When $x_0 = e_X$, the ball $B_\varepsilon(e_X)$ is the same under either of the conjugate metrics, as

$$B_\varepsilon(e_X) := \{z : \|z\| < \varepsilon\}.$$

Proposition 2.5. *(i) In a locally compact group X, for K compact and for $\varepsilon > 0$ small enough so that the closed ε-ball $B_\varepsilon(e_X)$ is compact, the swelling $B_{\varepsilon/2}(K)$ is pre-compact.*
(ii) $B_\varepsilon(K) = \{wk : k \in K, \|w\|_X < \varepsilon\} = B_\varepsilon(e_X)K$, where the notation refers to swellings for d^X a right-invariant metric; similarly, for $\tilde{d}^X$, the conjugate metric, $\tilde{B}_\varepsilon(K)$ $= KB_\varepsilon(e_X)$.

Proof. (i) If $x_n \in B_{\varepsilon/2}(K)$, then we may choose $k_n \in K$ with $d(k_n, x_n) < \varepsilon/2$. Without loss of generality k_n converges to k. Thus there exists N such that, for $n > N$,$d(k_n, k) <$ $\varepsilon/2$. For such n, we have $d(x_n, k) < \varepsilon$. Thus the sequence x_n lies in the compact closed ε-ball centred at k and so has a convergent subsequence.
(ii) Let $d^X(x, y)$ be a right-invariant metric, so that $d^X(x, y) = \|xy^{-1}\|$. If $\|w\| < \varepsilon$, then $d^X(wk, k) = d^X(w, e) = \|w\| < \varepsilon$, so $wk \in B_\varepsilon(K)$. Conversely, if $\varepsilon > d^X(z, k) =$ $d^X(zk^{-1}, e)$, then, putting $w = zk^{-1}$, we have $z = wk \in B_\varepsilon(K)$. ■

For further information on norms with the *Heine-Borel* property (for which compact sets are precisely those sets which are closed in the right norm topology and norm-bounded) see [**?**]).

The significance of the following simple corollary is wide-ranging. It explicitly demonstrates that small either-sided translations λ_x, ρ_y do not much alter the norm. Its main effect is on the analysis of subadditive functions.

COROLLARY 2.6. *With* $\|x\| := d^X(x,e)$, *where* d^X *is a right-invariant metric on* X,

$$|(\|x\| - \|y\|)| \leq \|xy\| \leq \|x\| + \|y\|.$$

Proof. By Proposition 2.2, the triangle inequality and symmetry holds for norms, so $\|y\| = \|x^{-1}xy\| \leq \|x^{-1}\| + \|xy\| = \|x\| + \|xy\|$. ■

We now generalize (rv-limit), by letting T, X be subgroups of a normed group G with X invariant under T.

DEFINITION. We say that a function $h : X \to H$ is *slowly varying on* X *over* T if $\partial_X h(t) = e_H$, that is, for each t in T

$$h(tx)h(x)^{-1} \to e_H, \text{ as } \|x\| \to \infty \text{ for } x \in X.$$

We omit mention of X and T when context permits. In practice G will be an internal direct product of two normal subgroups $G = TX$. (For a topological view on the *internal direct product*, see [Na, Ch. 2.7] ; for an algebraic view see [vdW, Ch. 6, Sect. 47], [J] Ch. 9 and 10, or [Ga] Section 9.1.) We may verify the property of h just defined by comparison with a slowly varying function.

THEOREM 2.7 (Comparison criterion). $h : X \to H$ *is slowly varying iff for some slowly varying function* $g : X \to H$ *and some* $\mu \in H$,

$$\lim_{\|x\|\to\infty} h(x)g(x)^{-1} = \mu.$$

Proof. If this holds for some slowly varying g and some μ,

$$h(tx)h(x)^{-1} = h(tx)g(tx)^{-1}g(tx)g(x)^{-1}g(x)h(x)^{-1} \to \mu e_H \mu^{-1} = e_H,$$

so h is slowly varying; the converse is trivial. ■

THEOREM 2.8. *For* d^X *a right-invariant metric on a group* X, *the norm* $\|x\| := d^X(x,e)$, *as a function from* X *to the multiplicative positive reals* $\mathbb{R}_+^*$, *is slowly varying in the multiplicative sense, i.e., for any* $t \in X$,

$$\lim_{\|x\|\to\infty} \frac{\|tx\|}{\|x\|} = 1.$$

Hence also

$$\lim_{\|x\|\to\infty} \frac{\|gxg^{-1}\|}{\|x\|} = 1.$$

More generally, for T a one-parameter subgroup of X, any sub-additive Baire function $p : X \to \mathbb{R}_+^*$ with

$$\|p\|_T := \lim_{x\in T,\ \|x\|\to\infty} \frac{p(x)}{\|x\|} > 0$$

is multiplicatively slowly varying. (The limit exists by the First Limit Theorem for Baire subadditive functions, see [BOst-GenSub].)

Proof. By Corollary 2.6, for $x \neq e$,

$$1 - \frac{\|t\|}{\|x\|} \leq \frac{\|tx\|}{\|x\|} \leq 1 + \frac{\|t\|}{\|x\|},$$

which implies slow variation. We regard p as mapping to $\mathbb{R}_+^*$, the strictly positive reals (since $p(x) = 0$ iff $x = e_X$). Taking $h = p$ and $\mu = \|p\|_T > 0$, the assertion follows from the Comparison Criterion (Th. 2.7) above (with $g(x) = \|x\|$). Explicitly, for $x \neq e$,

$$\frac{p(xy)}{p(x)} = \frac{p(xy)}{\|xy\|} \cdot \frac{\|xy\|}{\|x\|} \cdot \frac{\|x\|}{p(x)} \to \|p\|_T \cdot 1 \cdot \frac{1}{\|p\|_T} = 1.$$

■

Corollary 2.9. *If $\pi : X \to Y$ is a group homomorphism and $\|\cdot\|_Y$ is (1-γ)-quasi-isometric to $\|\cdot\|_X$ under the mapping π, then the subadditive function $p(x) = \|\pi(x)\|_Y$ is slowly varying. For general (μ-γ)-quasi-isometry the function p satisfies*

$$\mu^{-2} \leq p_*(z) \leq p^*(z) \leq \mu^2,$$

where

$$p^*(z) = \limsup_{\|x\|\to\infty} p(zx)p(x)^{-1} \qquad p_*(z) = \liminf_{\|x\|\to\infty} p(zx)p(x)^{-1}.$$

Proof. Subadditivity of p follows from π being a homomorphism, since $p(xy) = \|\pi(xy)\|_Y$ $= \|\pi(x)\pi(y)\|_Y \leq \|\pi(x)\|_Y + \|\pi(y)\|_Y$. Assuming that, for $\mu = 1$ and $\gamma > 0$, the norm $\|\cdot\|_Y$ is (μ-γ)-quasi-isometric to $\|\cdot\|_X$, we have, for $x \neq e$,

$$1 - \frac{\gamma}{\|x\|_X} \leq \frac{p(x)}{\|x\|_X} \leq 1 - \frac{\gamma}{\|x\|_X}.$$

So

$$\lim_{\|x\|\to\infty} \frac{p(x)}{\|x\|} = 1 \neq 0,$$

and the result follows from the Comparison Criterion (Th. 2.7) and Theorem 2.5.
If, for general $\mu \geq 1$ and $\gamma > 0$, the norm $\|\cdot\|_Y$ is (μ-γ)-quasi-isometric to $\|\cdot\|_X$, we have, for $x \neq e$,

$$\mu^{-1} - \frac{\gamma}{\|x\|_X} \leq \frac{p(x)}{\|x\|_X} \leq \mu - \frac{\gamma}{\|x\|_X}.$$

So for y fixed

$$\frac{p(xy)}{p(x)} = \frac{p(xy)}{\|xy\|} \cdot \frac{\|xy\|}{\|x\|} \cdot \frac{\|x\|}{p(x)} \leq \left(\mu - \frac{\gamma}{\|xy\|_X}\right) \cdot \frac{\|xy\|}{\|x\|} \cdot \left(\mu^{-1} - \frac{\gamma}{\|x\|_X}\right)^{-1},$$

giving, by Theorem 2.8 and because $\|xy\| \geq \|x\| - \|y\|$,

$$p^*(y) := \limsup_{x\to\infty} \frac{p(xy)}{p(x)} \leq \mu^2.$$

The left-sided inequality is proved dually (interchanging the roles of the upper and lower bounds on $\|\pi(x)\|_Y$). ∎

Remarks. 1. In the case of the *general* $(\mu$-$\gamma)$-*quasi-isometry,* p exhibits the normed-groups O-analogue of slow-variation; compare [BGT, Cor. 2.0.5 p. 65].

2. When $X = \mathbb{R}$ the weaker boundedness property: "$p^*(y) < \infty$ on a large enough set of ys" implies that p satisfies

$$z^d \leq p_*(z) \leq p^*(z) \leq z^c, \qquad (z \geq Z)$$

for some constants c, d, Z (so is *extended regularly varying* in the sense of [BGT, Ch. 2, 2.2 p. 65]). Some generalizations are given in Theorems 7.10 and 7.11.

3. We pause to consider briefly some classical examples. If $X = H = \mathbb{R}$ is construed additively, so that $e_H = e_X = 0$ and $\|x\| := |x - 0| = |x|$ in both cases, and with the action tx denoting $t + x$, the function $f(x) := |x|$ is not slowly varying, because $(x + t) - x = t \nrightarrow 0 = e_H$. On the other hand a multiplicative construction on $H = \mathbb{R}^*_+$, for which $e_H = 1$ and $\|h\|_H := |\log h|$, but with $X = \mathbb{R}$ still additive and tx still meaning $t + x$, yields f as having slow variation (as in the Theorem 2.8), as

$$f(tx)f(x)^{-1} = (x + t)/x \to 1 = e_H \text{ as } x \to \infty.$$

We note that in this context the regularly varying functions h on X have $h(tx)h(x)^{-1} = h(t + x) - h(x) \to at$, for some constant a.
Note that, for $X = H = \mathbb{R}^*_+$, and with tx meaning $t\cdot x$, since $\|x\| = |\log x|$ (as just noted) is the group-norm, we have here

$$f(tx)f(x)^{-1} = \|tx\|/\|x\| = \frac{|\log tx|}{|\log x|} = \frac{|\log t + \log x|}{|\log x|} \to 1 = e_H, \text{ as } x \to \infty,$$

which again illustrates the content of Theorem 2.7. Here the regularly varying functions $h(tx)h(x)^{-1} \to e^{at}$, for some constant a. See [BGT, Ch. 1] for background on additive and multiplicative formulations of regular variation in the classical setting of functions $f : G \to H$ with $G, H = \mathbb{R}$ or $\mathbb{R}_+$.

Definitions. 1. Say that $\xi \in X$ is *infinitely divisible* if, for each positive integer n, there is x with $x^n = \xi$. (Compare Section 3.)
2. Say that the infinitely divisible element ξ is *embeddable* if, for some one-parameter subgroup T in X, we have $\xi \in T$. When such a T exists it is unique (the elements $\xi^{m/n}$, for m, n integers, are dense in T); we write $T(\xi)$ for it.

Clearly any element of a one-parameter subgroup is both infinitely divisible and embeddable. For results on this see Davies [D], Heyer [Hey], McCrudden [McC]. With these definitions, our previous analysis allows the First Limit Theorem for subadditive functions (cf. Th. 2.8 and [BOst-GenSub]) to be restated in the context of normed groups.

Proposition 2.10. *Let ξ be infinitely divisible and embeddable in the one-parameter subgroup $T(\xi)$ of X. Suppose that $\lim_{n\to\infty}\|x^n\|=\infty$ for $x\neq e_X$. Then for any Baire subadditive $p:X\to\mathbb{R}_+$ and $t\in T(\xi)$,*

$$\partial_{T(\xi)}p(t):=\lim_{s\in T,\ \|s\|\to\infty}\frac{p(ts)}{\|s\|}=\|p\|_T,$$

i.e., treating the subgroup $T(\xi)$ as a direction, the limit function is determined by the direction.

Proof. By subadditivity, $p(s)=p(t^{-1}ts)\leq p(t^{-1})+p(ts)$, so

$$p(s)-p(t^{-1})\leq p(ts)\leq p(t)+p(s).$$

For $s\in T$ with $s\neq e$, divide through by $\|s\|$ and let $\|s\|\to\infty$ (as in Th. 2.8):

$$\|p\|_T\leq\partial_T p(t)\leq\|p\|_T.$$

■

(We consider this in detail in Section 4.)

Definition (Supremum metric, supremum norm). Let X have a metric d^X. As before $\mathcal{G}$ is a fixed subgroup of $Auth(X)$, for example $Tr_L(X)$ the group of left-translations λ_x (cf. Th. 3.12), defined by

$$\lambda_x(z)=xz.$$

For $g,h\in\mathcal{G}$, define the possibly infinite number

$$\hat{d}^{\mathcal{G}}(g,h),\ \text{or}\ \hat{d}^X(g,h):=\sup_{x\in X}d^X(g(x),h(x)),$$

where the notation identifies either the domain of the metric or the source metric d^X. Put

$$\mathcal{H}(X)=\mathcal{H}(X,\mathcal{G}):=\{g\in\mathcal{G}:\hat{d}^{\mathcal{G}}(g,id_X)<\infty\},$$

and call these the *bounded elements* of $\mathcal{G}$. We write $\hat{d}^{\mathcal{H}}$ for the metric $\hat{d}^{\mathcal{G}}$ restricted to $\mathcal{H}=\mathcal{H}(X)$ and call $\hat{d}^{\mathcal{H}}(g,h)$ the *supremum metric* on $\mathcal{H}$; the associated norm

$$\|h\|_{\mathcal{H}}=\|h\|_{\mathcal{H}(X)}:=\hat{d}^{\mathcal{H}}(h,id_X)=\sup_{x\in X}d^X(h(x),x)$$

is the *supremum norm*. This metric notion may also be handled in the setting of uniformities (cf. the notion of functions limited by a cover $\mathcal{U}$ arising in [AnB, Section 2]; see also [BePe, Ch. IV Th. 1.2]); in such a context excursions into invariant measures rather than use of Haar measure (as in Section 6) would refer to corresponding results established by Itzkowitz [Itz] (cf. [SeKu, §7.4]).

Our next result justifies the terminology of the definition above.

Proposition 2.11 (Group-norm properties in $\mathcal{H}(X)$). *If* $\|h\| = \|h\|_{\mathcal{H}}$, *then* $\|\cdot\|$ *is a group-norm: that is, for* $h, h' \in \mathcal{H}(X)$,

$$\|h\| = 0 \text{ iff } h = e, \ \ \|h \circ h'\| \leq \|h\| + \|h'\| \text{ and } \|h\| = \|h^{-1}\|.$$

Proof. Evidently $\hat{d}(h, id_X) = \sup_{x \in X} d(h(x), x) = 0$ iff $h(x) = id_X$. We have

$$\|h\| = \hat{d}(h, id_X) = \sup_{x \in X} d(h(x), x) = \sup_{y \in X} d(y, h^{-1}(y)) = \|h^{-1}\|.$$

Next note that

$$\hat{d}(id_X, h \circ h') = \sup_{x \in X} d(hh'(x), x) = \sup_{y \in X} d(h(y), h'^{-1}(y)) = \hat{d}(h, h'^{-1}). \quad \text{(right-inv)}$$

But

$$\begin{aligned}\hat{d}(h, h') &= \sup_{x \in X} d(h(x), h'(x)) \leq \sup_{x \in X} [d(h(x), x) + d(x, h'(x))] \\ &\leq \hat{d}(h, id) + \hat{d}(h', id) < \infty.\end{aligned}$$

■

Theorem 2.12. *The set* $\mathcal{H}(X)$ *of bounded self-homeomorphisms of a metric group* X *is a group under composition, metrized by the right-invariant supremum metric* $\hat{d}^X$.

Proof. The identity, id_X, is bounded. For right-invariance (cf. (right-inv)),

$$\hat{d}(g \circ h, g' \circ h) = \sup_{x \in X} d(g(h(x)), g'(h(x)) = \sup_{y \in X} d(g(y), g'(y)) = \hat{d}(g, g').$$

■

Theorem 2.13 ([BePe, Ch. IV Th 1.1]). *Let* d *be a bounded metric on* X. *As a group under composition,* $\mathcal{A} = Auth(X)$ *is a topological group under the weak* Δ*-refinement topology for* $\Delta := \{\hat{d}_\pi : \pi \in \mathcal{A}\}$.

Proof. To prove continuity of inversion at F, write $H = F^{-1}$ and for any x put $y = f^{-1}(x)$. Then

$$d_\pi(f^{-1}(x), F^{-1}(x)) = d_\pi(H(F(y)), H(f(y))) = d_{\pi H}(F(y), f(y)),$$

and so

$$\hat{d}_\pi(f^{-1}, F^{-1}) = \sup_x d_\pi(f^{-1}(x), F^{-1}(x)) = \sup_y d_{\pi H}(F(y), f(y)) = \hat{d}_{\pi H}(f, F).$$

Thus f^{-1} is in any $\hat{d}_\pi$ neighbourhood of F^{-1} provided f is in any $\hat{d}_{\pi H}$ neighbourhood of F.

As for continuity of composition at F, G, we have for fixed x that

$$\begin{aligned}d_\pi(f(g(x)), F(G(x))) &\leq d_\pi(f(g(x)), F(g(x))) + d_\pi(F(g(x)), F(G(x))) \\ &= d_\pi(f(g(x)), F(g(x)) + d_{\pi F}(g(x), G(x)) \\ &\leq \hat{d}_\pi(f, F) + \hat{d}_{\pi F}(g, G).\end{aligned}$$

Hence

$$\hat{d}_\pi(fg, FG) \leq \hat{d}_\pi(f, F) + \hat{d}_{\pi F}(g, G),$$

so that fg is in the $\hat{d}_\pi$-ball of radius ε of FG provided f is in the $\hat{d}_\pi$-ball of radius $\varepsilon/2$ of F and g is in the $\hat{d}_{\pi H}$-ball of radius $\varepsilon/2$ of G. ■

Remark (The compact-open topology). In similar circumstances, we show in Theorem 3.17 below that under the *strong* Δ-refinement topology, so a finer topology, $Auth(X)$ is a normed group and a topological group. Rather than use weak or strong refinement of metrics in $Auth(X)$, one may consider the compact-open topology (the topology of uniform convergence on compacts, introduced by Fox and studied by Arens in [Ar1], [Ar2]). However, in order to ensure the kind of properties we need (especially in flows), the metric space X would then need to be restricted to a special case. Recall some salient features of the compact-open topology. For composition to be continuous local compactness is essential ([Dug, Ch. XII.2], [McCN], [BePe, Section 8.2], or [vM2, Ch.1]). When T is compact the topology is admissible (i.e. $Auth(X)$ is a topological group under it), but the issue of admissibility in the non-compact situation is not currently fully understood (even in the locally compact case for which counter-examples with non-continuous inversion exist, and so additional properties such as local connectedness are usually invoked – see [Dij] for the strongest results). In applications the focus of interest may fall on separable spaces (e.g. function spaces), but, by a theorem of Arens, if X is separable metric and further the compact-open topology on $\mathcal{C}\ (X,\mathbb{R})$ is metrizable, then X is necessarily locally compact and σ-compact, and conversely (see e.g [Eng, p.165 and 266]).

We will now apply the supremum-norm construction to deduce that right-invariance may be arranged if for every $x \in X$ the left translation λ_x has finite sup-norm:

$$\|\lambda_x\|_{\mathcal{H}} = \sup\nolimits_{z\in X} d^X(xz,z) < \infty.$$

We will need to note the connection with conjugate norms.

Definition. Recall the g-conjugate norm is defined by

$$\|x\|_g := \|gxg^{-1}\|.$$

The *conjugacy refinement norm* corresponding to the family of all the g-conjugate norms $\Gamma = \{\|.\|_g : g \in G\}$ will be denoted by

$$\|x\|_\infty := \sup\nolimits_g \|x\|_g,$$

in contexts where this is finite.

Clearly, for any g,

$$\|x\|_\infty = \|gxg^{-1}\|_\infty,$$

and so $\|x\|_\infty$ is an abelian norm (substitute xg for x). Evidently, if the metric d_L^X is left-invariant we have

$$\|x\|_\infty = \sup\nolimits_g \|x\|_g = \sup\nolimits_{z\in X} d_L^X(z^{-1}xz, e) = \sup\nolimits_{z\in X} d_L^X(xz,z). \qquad \text{(shift)}$$

One may finesse the left-invariance assumption, using (shift), as we will see in Proposition 2.14.

Example C. As $\mathcal{H}(X)$ is a group and $\hat{d}^{\mathcal{H}}$ is right-invariant, the norm $\|f\|_{\mathcal{H}}$ gives rise to a conjugacy refinement norm. Working in $\mathcal{H}(X)$, suppose that $f_n \to f$ under the supremum norm $\hat{d}^X = \hat{d}^{\mathcal{H}}$. Let $g \in \mathcal{H}(X)$. Then pointwise

$$\lim_n f_n(g(x)) = f(g(x)).$$

Hence, as f^{-1} is continuous, we have for any $x \in X$,

$$f^{-1}(\lim_n f_n(g(x))) = \lim_n f^{-1} f_n(g(x)) = g(x).$$

Likewise, as g^{-1} is continuous, we have for any $x \in X$,

$$g^{-1}(\lim_n f^{-1} f_n(g(x))) = \lim_n g^{-1} f^{-1} f_n(g(x)) = x.$$

Thus

$$g^{-1} f^{-1} f_n g \to id_X \text{ pointwise.}$$

This result is generally weaker than the assertion $\|f^{-1} f_n\|_g \to 0$, which requires uniform rather than pointwise convergence.

We need the following notion of admissibility (with the norm $\|.\|_\infty$ in mind; compare also Section 3).

Definitions. 1. Say that the metric d^X satisfies the *metric admissibility condition* on $H \subset X$ if, for any $z_n \to e$ in H under d^X and arbitrary y_n,

$$d^X(z_n y_n, y_n) \to 0.$$

2. If d^X is left-invariant, the condition may be reformulated as a *norm admissibility condition* on $H \subset X$, since

$$\|y_n^{-1} z_n y_n\| = d_L^X(y_n^{-1} z_n y_n, e) = d_L^X(z_n y_n, y_n) \to 0. \qquad \text{(H-adm)}$$

3. We will say that the group X satisfies the *topological admissibility condition* on $H \subset X$ if, for any $z_n \to e$ in H and arbitrary y_n

$$y_n^{-1} z_n y_n \to e.$$

The next result extends the usage of $\|\cdot\|_{\mathcal{H}}$ beyond $\mathcal{H}$ to X itself (via the left shifts).

Proposition 2.14 (Right-invariant sup-norm). *For any metric d^X on a group X, put*

$$H_X := H = \{x \in X : \sup_{z \in X} d^X(xz, z) < \infty\},$$
$$\|x\|_{\mathcal{H}} := \sup d^X(xz, z), \text{ for } x \in H.$$

For $x, y \in H$, let $\bar{d}_{\mathcal{H}}(x, y) := \hat{d}^{\mathcal{H}}(\lambda_x, \lambda_y) = \sup_z d^X(xz, yz)$. Then:
(i) $\bar{d}_{\mathcal{H}}$ is a right-invariant metric on H, and $\bar{d}_{\mathcal{H}}(x, y) = \|xy^{-1}\|_{\mathcal{H}} = \|\lambda_x \lambda_y^{-1}\|_{\mathcal{H}}$.
(ii) If d^X is left-invariant, then $\bar{d}_{\mathcal{H}}$ is bi-invariant on H, and so $\|x\|_\infty = \|x\|_{\mathcal{H}}$ and the norm is abelian on H.

(iii) The $\bar{d}_{\mathcal{H}}$-topology on H is equivalent to the d^X-topology on H iff d^X satisfies (H-adm), the metric admissibility condition on H.
(iv) In particular, if d^X is right-invariant, then $H = X$ and $\bar{d}_{\mathcal{H}} = d^X$.
(v) If X is a compact topological group under d^X, then $\bar{d}_{\mathcal{H}}$ is equivalent to d^X.

Proof. (i) The argument relies implicitly on the natural embedding of X in $Auth(X)$ as $Tr_L(X)$ (made explicit in the next section). For $x \in X$ we write

$$\|\lambda_x\|_{\mathcal{H}} := \sup\nolimits_z d^X(xz, z).$$

For $x \neq e$, we have $0 < \|\lambda_x\|_{\mathcal{H}} \leq \infty$. By Proposition 2.12, $\mathcal{H}(X) = \mathcal{H}(X, Tr_L(X)) = \{\lambda_x : \|\lambda_x\|_{\mathcal{H}} < \infty\}$ is a subgroup of $\mathcal{H}(X, Auth(X))$ on which $\|\cdot\|_{\mathcal{H}}$ is thus a norm. Identifiying $\mathcal{H}(X)$ with the subset $H = \{x \in X : \|\lambda_x\| < \infty\}$ of X, we see that on H

$$\bar{d}_{\mathcal{H}}(x, y) := \sup\nolimits_z d^X(xz, yz) = \hat{d}_{\mathcal{H}}(\lambda_x, \lambda_y)$$

defines a right-invariant metric, as

$$\bar{d}_{\mathcal{H}}(xv, yv) = \sup\nolimits_z d^X(xvz, yvz) = \sup\nolimits_z d^X(xz, yz) = \bar{d}_{\mathcal{H}}(x, y).$$

Hence with

$$\|x\|_{\mathcal{H}} = \bar{d}_{\mathcal{H}}(x, e) = \|\lambda_x\|_{\mathcal{H}},$$

by Proposition 2.11

$$\|\lambda_x \lambda_y^{-1}\|_{\mathcal{H}} = \bar{d}_{\mathcal{H}}(x, y) = \|xy^{-1}\|_{\mathcal{H}},$$

as asserted.
If d^X is left-invariant, then

$$\bar{d}_{\mathcal{H}}(vx, vy) = \sup\nolimits_z d_L^X(vxz, vyz) = \sup\nolimits_z d_L^X(xz, yz) = \bar{d}_{\mathcal{H}}(x, y),$$

and so $\bar{d}_{\mathcal{H}}$ is both left-invariant and right-invariant.
Note that

$$\|x\|_{\mathcal{H}} = \bar{d}_{\mathcal{H}}(x, e) = \sup\nolimits_z d_L^X(xz, z) = \sup\nolimits_z d_L^X(z^{-1}xz, e) = \sup\nolimits_z \|x\|_z = \|x\|_\infty.$$

(ii) We note that

$$d^X(z_n, e) \leq \sup\nolimits_y d^X(z_n y, y).$$

Thus if $z_n \to e$ in the sense of $d_{\mathcal{H}}$, then also $z_n \to e$ in the sense of d^X. Suppose that the metric admissibility condition holds but the metric $d_{\mathcal{H}}$ is not equivalent to d^X. Thus for some $z_n \to e$ (in H and under d^X) and $\varepsilon > 0$,

$$\sup\nolimits_y d^X(z_n y, y) \geq \varepsilon.$$

Thus there are y_n with

$$d^X(z_n y_n, y_n) \geq \varepsilon/2,$$

which contradicts the admissibility condition.
For the converse, if the metric $d_{\mathcal{H}}$ is equivalent to d^X, and $z_n \to e$ in H and under d^X, then $z_n \to e$ also in the sense of $d_{\mathcal{H}}$; hence for y_n given and any $\varepsilon > 0$, there is N such that for $n \geq N$,

$$\varepsilon > \bar{d}_{\mathcal{H}}(z_n, e) = \sup\nolimits_y d^X(z_n y, y) \geq d^X(z_n y_n, y_n).$$

Thus $d^X(z_n y_n, y_n) \to 0$, as required.
(iii) If d^X is right-invariant, then $d^X(z_n y_n, y_n) = d^X(z_n, e) \to 0$ and the admissibility condition holds on H. Of course $\|\lambda_x\|_{\mathcal{H}} = \sup_z d^X(xz, z) = d^X(x, e) = \|x\|_X$ and so $H = X$.
(iv) If d^X is right-invariant, then $\bar{d}_{\mathcal{H}}(x, y) := \sup_z d^X(xz, yz) = d^X(x, y)$.
(v) If X is compact, then $H = H_X$ as $z \to d^X(xz, z)$ is continuous. If $z_n \to e$ and y_n are arbitrary, suppose that the admissibility condition fails. Then for some $\varepsilon > 0$ we have without loss of generality

$$d^X(z_n y_n, y_n) \geq \varepsilon.$$

Passing down a subsequence $y_m \to y$ and assuming that X is a topological group we obtain

$$0 = d^X(ey, y) \geq \varepsilon,$$

a contradiction. ∎

As a corollary we obtain the following known result ([HR, 8.18]; cf. Theorem 3.3.4 in [vM2] p. 101, for a different proof).

Proposition 2.15. *In a first-countable topological group X the (topological admissibility) condition $y_n^{-1} z_n y_n \to e$ on X as $z_n \to e$ is equivalent to the existence of an abelian norm (equivalently, a bi-invariant metric).*

Proof. We shall see below in the Birkhoff-Kakutani Theorem (Th.2. 19) that the topology of X may be induced by a left-invariant metric, d_L^X say; we may assume without loss of generality that it is bounded (take $d = \max\{d_L^X, 1\}$, which is also left-invariant, cf. Example A6 towards the start of this Section). Then $H_X = X$, and the assumed topological admissibility condition $y_n^{-1} z_n y_n \to e$ on X implies (H-adm), the metric admissibility condition on H for d_L^X. The metric d_L^X thus induces the norm $\|x\|_{\mathcal{H}}$, which is abelian, and in turn, by Proposition 2.3, defines an equivalent bi-invariant metric on X. Conversely, if the norm $\|.\|_X$ is abelian, then the topological admissibility condition follows from the observation that

$$\|y_n^{-1} z_n y_n\| = \|y_n y_n^{-1} z_n\| = \|z_n\| \to 0.$$

∎

Application. Let S, T be normed groups. For $\alpha : S \to T$ an arbitrary function we define the possibly infinite number

$$\|\alpha\| := \sup\{\|\alpha(s)\|_T / \|s\|_S : s \in S\} = \inf\{M : \|\alpha(s)\| \leq M\|s\| \ (\forall s \in S)\}.$$

α is called *bounded* if $\|\alpha\|$ is finite. The bounded functions form a group G under the pointwise multiplication $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Clearly $\|\alpha\| = 0$ implies that $\alpha(t) = e$, for all t. Symmetry is clear. Also

$$\|\alpha(t)\beta(t)\| \leq \|\alpha(t)\| + \|\beta(t)\| \leq [\|\alpha\| + \|\beta\|]\|t\|,$$

so

$$\|\alpha\beta\| \leq \|\alpha\| + \|\beta\|.$$

We say that a function $\alpha : S \to T$ is *multiplicative* if α is bounded and

$$\alpha(ss') = \alpha(s)\alpha(s').$$

A function $\gamma : S \to T$ is *asymptotically multiplicative* if $\gamma = \alpha\beta$, where α is multiplicative and bounded and β is bounded. In the commutative situation with S, T normed vector spaces, the norm here reduces to the operator norm. This group-norm is studied extensively in [CSC] in relation to Ulam's problem. We consider in Section 3.2 the case $S = T$ and functions α which are inner automorphisms. In Proposition 3.42 we shall see that the oscillation of a group X is a bounded function from X to $\mathbb{R}$ in the sense above.

Proposition 2.16 (Magnification metric). *Let $T = \mathcal{H}(X)$ with group-norm $\|t\| = d^T(t, e_T) = d^{\mathcal{H}}(t, e_T)$ and $\mathcal{A}$ a subgroup (under composition) of $Auth(T)$ (so, for $t \in T$ and $\alpha \in \mathcal{A}$, $\alpha(t) \in \mathcal{H}(X)$ is a homeomorphism of X). For any $\varepsilon \geq 0$, put*

$$d^{\varepsilon}_{\mathcal{A}}(\alpha, \beta) := \sup\nolimits_{\|t\| \leq \varepsilon} \hat{d}^T(\alpha(t), \beta(t)).$$

Suppose further that X distinguishes the maps $\{\alpha(e_{\mathcal{H}(X)}) : \alpha \in \mathcal{A}\}$, i.e., for $\alpha, \beta \in \mathcal{A}$, there is $z = z_{\alpha,\beta} \in X$ with $\alpha(e_{\mathcal{H}(X)})(z) \neq \beta(e_{\mathcal{H}(X)})(z)$.
Then $d^{\varepsilon}_{\mathcal{A}}(\alpha, \beta)$ is a metric; furthermore, $d^{\varepsilon}_{\mathcal{A}}$ is right-invariant for translations by $\gamma \in \mathcal{A}$ such that γ^{-1} maps the ε-ball of X to the ε-ball.

Proof. To see that this is a metric, note that for $t = e_{\mathcal{H}(X)} = id_T$ we have $\|t\| = 0$ and

$$\begin{aligned}\hat{d}^T(\alpha(e_{\mathcal{H}(X)}), \beta(e_{\mathcal{H}(X)})) &= \sup\nolimits_z d^X(\alpha(e_{\mathcal{H}(X)})(z), \beta(e_{\mathcal{H}(X)})(z)) \\ &\geq d^X(\alpha(e_{\mathcal{H}(X)})(z_{\alpha,\beta}), \beta(e_{\mathcal{H}(X)})(z_{\alpha,\beta})) > 0.\end{aligned}$$

Symmetry is clear. Finally the triangle inequality follows as usual:

$$\begin{aligned}d^{\varepsilon}_{\mathcal{A}}(\alpha, \beta) &= \sup\nolimits_{\|t\| \leq 1} \hat{d}^T(\alpha(t), \beta(t)) \leq \sup\nolimits_{\|t\| \leq 1} [\hat{d}^T(\alpha(t), \gamma(t)) + \hat{d}^T(\gamma(t), \beta(t))] \\ &\leq \sup\nolimits_{\|t\| \leq 1} \hat{d}^T(\alpha(t), \gamma(t)) + \sup\nolimits_{\|t\| \leq 1} \hat{d}^T(\gamma(t), \beta(t)) \\ &= d^{\varepsilon}_{\mathcal{A}}(\alpha, \gamma) + d^{\varepsilon}_{\mathcal{A}}(\gamma, \beta).\end{aligned}$$

One cannot hope for the metric to be right-invariant in general, but if γ^{-1} maps the ε-ball to the ε-ball, one has

$$\begin{aligned}d^{\varepsilon}_{\mathcal{A}}(\alpha\gamma, \beta\gamma) &= \sup\nolimits_{\|t\| \leq \varepsilon} \hat{d}^T(\alpha(\gamma(t)), \beta(\gamma(t)) \\ &= \sup\nolimits_{\|\gamma^{-1}(s)\| \leq \varepsilon} \hat{d}^T(\alpha(s), \beta(s)).\end{aligned}$$

■

In this connection we note the following.

Proposition 2.17. *In the setting of Proposition 2.16, denote by $\|.\|_{\varepsilon}$ the norm induced by $d^{\varepsilon}_{\mathcal{A}}$; then*

$$\sup\nolimits_{\|t\| \leq \varepsilon} \|\gamma(t)\|_T - \varepsilon \leq \|\gamma\|_{\varepsilon} \leq \sup\nolimits_{\|t\| \leq \varepsilon} \|\gamma(t)\|_T + \varepsilon.$$

Proof. By definition, for t with $\|t\| \leq \varepsilon$, we have

$$\|\gamma\|_\varepsilon = \sup_{\|t\|\leq\varepsilon} \hat{d}^T(\gamma(t), t) \leq \sup_{\|t\|\leq\varepsilon}[\hat{d}^T(\gamma(t), e) + \hat{d}^T(e, t)] \leq \sup_{\|t\|\leq\varepsilon} \|\gamma(t)\|_T + \varepsilon,$$

$$\begin{aligned}\|\gamma(t)\|_T &= \hat{d}^T(\gamma(t), e) \leq \hat{d}^T(\gamma(t), t) + \hat{d}^T(t, e) \\ &\leq \|t\| + \|\gamma\|_\varepsilon \leq \varepsilon + \|\gamma\|_\varepsilon.\end{aligned}$$

■

Theorem 2.18 (Invariance of Norm Theorem – for (b) cf. [Klee]). *(a) The group-norm is abelian (and the metric is bi-invariant) iff*

$$\|xy(ab)^{-1}\| \leq \|xa^{-1}\| + \|yb^{-1}\|,$$

for all x, y, a, b, or equivalently,

$$\|uabv\| \leq \|uv\| + \|ab\|,$$

for all x, y, u, v.
(b) Hence a metric d on the group X is bi-invariant iff the Klee property holds:

$$d(ab, xy) \leq d(a, x) + d(b, y). \tag{Klee}$$

In particular, this holds if the group X is itself abelian.
(c) The group-norm is abelian iff the norm is preserved under conjugacy (inner automorphisms).

Proof. (a) If the group-norm is abelian, then by the triangle inequality

$$\begin{aligned}\|xyb^{-1} \cdot a^{-1}\| &= \|a^{-1}xyb^{-1}\| \\ &\leq \|a^{-1}x\| + \|yb^{-1}\|.\end{aligned}$$

For the converse we demonstrate bi-invariance in the form $\|ba^{-1}\| = \|a^{-1}b\|$. In fact it suffices to show that $\|yx^{-1}\| \leq \|x^{-1}y\|$; for then bi-invariance follows, since taking $x = a, y = b$ we get $\|ba^{-1}\| \leq \|a^{-1}b\|$, whereas taking $x = b^{-1}, y = a^{-1}$ we get the reverse $\|a^{-1}b\| \leq \|ba^{-1}\|$. As for the claim, we note that

$$\|yx^{-1}\| \leq \|yx^{-1}yy^{-1}\| \leq \|yy^{-1}\| + \|x^{-1}y\| = \|x^{-1}y\|.$$

(b) Klee's result is deduced as follows. If d is a bi-invariant metric, then $\|\cdot\|$ is abelian. Conversely, for d a metric, let $\|x\| := d(e, x)$. Then $\|.\|$ is a group-norm, as

$$d(ee, xy) \leq d(e, x) + d(e, y).$$

Hence d is right-invariant and $d(u, v) = \|uv^{-1}\|$. Now we conclude that the group-norm is abelian since

$$\|xy(ab)^{-1}\| = d(xy, ab) \leq d(x, a) + d(y, b) = \|xa^{-1}\| + \|yb^{-1}\|.$$

Hence d is also left-invariant.
(c) Suppose the norm is abelian. Then for any g, by the cyclic property $\|g^{-1}bg\| = \|gg^{-1}b\| = \|b\|$. Conversely, if the norm is preserved under automorphism, then we have bi-invariance, since $\|ba^{-1}\| = \|a^{-1}(ba^{-1})a\| = \|a^{-1}b\|$. ■

Remark. Note that, taking $b = v = e$, we have the triangle inequality. Thus the result (a) characterizes maps $\|\cdot\|$ with the positivity property as group pre-norms which are abelian. In regard to conjugacy, see also the Uniformity Theorem for Conjugation (Th. 12.4). We now state the following classical result.

Theorem 2.19 (Normability Theorem for Groups – Birkhoff-Kakutani Theorem). *Let X be a first-countable topological group and let V_n be a symmetric local base at e_X with*

$$V_{n+1}^4 \subseteq V_n.$$

Let $r = \sum_{n=1}^{\infty} c_n(r)2^{-n}$ be a terminating representation of the dyadic number r, and put

$$A(r) := \sum\nolimits_{n=1}^{\infty} c_n(r)V_n.$$

Then

$$p(x) := \inf\{r : x \in A(r)\}$$

is a group-norm. If further X is locally compact and non-compact, then p may be arranged such that p is unbounded on X, but bounded on compact sets.

For a proof see that offered in [Ru] for Th. 1.24 (p. 18-19), which derives a metrization of a topological vector space in the form $d(x, y) = p(x - y)$ and makes no use of the scalar field (so note how symmetric neighbourhoods here replace the 'balanced' ones in a topological vector space). That proof may be rewritten verbatim with xy^{-1} substituting for the additive notation $x - y$ (cf. Proposition 2.2).

Remark. In fact, a close inspection of Kakutani's metrizability proof in [Kak] (cf. [SeKu] §7.4) for topological groups yields the following characterization of normed groups – for details see [Ost-LB3].

Theorem 2.19′ (Normability Theorem for right topological groups – Birkhoff-Kakutani Theorem). *A first-countable right topological group X is a normed group iff inversion and multiplication are continuous at the identity.*

We close with some information concerning commutators, which arise in Theorems 3.7, 6.3, 10.7 and 10.9.

Definition. The right-sided and left-sided *commutators* are defined by

$$[x, y]_L := xyx^{-1}y^{-1}.$$
$$[x, y]_R := x^{-1}y^{-1}xy = [x^{-1}, y^{-1}]_L.$$

As

$$xy = [x, y]_L yx \text{ and } xy = yx[x, y]_R,$$

these express in terms of shifts the distortion arising from commuting factors, and so their continuity here is significant. Let $[x, y]$ denote either a right or left commutator;

we call the maps $x \to [x, y]$ and $y \to [x, y]$ *commutator maps* and in the context of a specified norm topology (either!), we say that the commutator $[\cdot, \cdot]$ is:

(i) *left continuous* if *for all* y the map $x \to [x, y]$ is continuous at each x;

(ii) *right continuous* if *for all* x the map $y \to [x, y]$ is continuous at each y,

(iii) *separately continuous* if it is left and right continuous.

We show that the commutators are like homomorphisms, in that their continuity may be implied by continuity at the identity e_X, but this does require that *all* the commutator maps be continuous at the identity.

THEOREM 2.20. *In a normed group an either-sided commutator is left continuous iff it is right continuous and so iff it is separately continuous.*

We deduce the above theorem from the following two more detailed results; see also Theorem 3.4 for further insights on this result.

PROPOSITION 2.21. *In a normed group under either norm topology the following are equivalent for $y \in X$:*
(i) the commutator map $x \to [x, y]_L$ is (left) continuous at $x = y$,
(ii) the commutator map $x \to [x, y]_L$ is (left) continuous at e, i.e. $[z_n, y]_L \to e$, as $z_n \to e$,
(iii) the commutator map $x \to [y, x]_L$ is (right) continuous at e, i.e. $[y, z_n]_L \to e$, as $z_n \to e$,
(iv) the commutator map $z \to [y, z]_L$ is (right) continuous at $z = y$,
(v) the commutator map $z \to [y^{-1}, z]_R$ is (right) continuous at $z = y^{-1}$,
(vi) the commutator map $x \to [x, y^{-1}]_R$ is (left) continuous at $x = y^{-1}$.

Proof. As the conclusions are symmetric without loss of generality we work in the right norm topology generated by the right-invariant metric d_R and write $\to_R$ to show that the convergence is in d_R. Note that $y_n \to_R x$ iff $y_n x^{-1} \to e$; there is no need for a subscript for convergence to e, as the ball $B_\varepsilon(e_X)$ is the same under either of the conjugate metrics (cf. Prop. 2.15). Indeed, writing $y_n = z_n y$, we have $d_R(z_n y, y) = d_R(z_n, e) \to 0$.
We first prove the chain of equivalences: (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv). The remaining equivalences follow from the observation that

$$[z_n, y]_L = [z_n^{-1}, y^{-1}]_R$$

and z_n^{-1} is a null sequence iff z_n is.
In regard to the first equivalence, employing the notation $y_n = z_n y$, the identity

$$[z_n, y]_L = z_n y z_n^{-1} y^{-1} = (z_n y) y (y^{-1} z_n^{-1}) y^{-1} = y_n y y_n^{-1} y^{-1} = [y_n, y]_L,$$

i.e.

$$[y_n, y]_L = [y_n y^{-1}, y]_L,$$

shows that $[z_n, y]_L \to e$ iff $[y_n, y]_L \to e$, i.e. (i)$\Leftrightarrow$(ii).
Turning to the second equivalence in the chain, we see from continuity of inversion at e (or inversion invariance) that for any y

$$[z_n, y]_L = z_n y z_n^{-1} y^{-1} \to e \text{ iff } [y, z_n]_L = y z_n y^{-1} z_n^{-1} \to e,$$

giving (ii)$\Leftrightarrow$(iii). Finally, with the notation $y_n = z_n y$, the identity

$$[y, y_n]_L = y y_n y^{-1} y_n^{-1} = y(z_n y) y^{-1}(y^{-1} z_n^{-1}) = y z_n y^{-1} z_n^{-1} = [y, z_n]_L$$

shows that $[y, z_n]_L \to e$ iff $[y, y_n]_L \to e$, i.e. (iii)$\Leftrightarrow$(iv). ■

PROPOSITION 2.22. *For a normed group X, with the right norm topology, and for $g, h \in X$, the commutator map $x \to [x, h]_L$ is continuous at $x = g$ provided the map $x \to [x, hgh^{-1}]_L$ is continuous at $x = e$.*
Hence if all the commutator maps $x \to [x, y]_L$ for $y \in X$ are continuous at $x = e$, then they are all continuous everywhere.

Proof. For fixed g, h and with $h_n = z_n h$ we have the identity

$$\begin{aligned}[h_n, g]_L &= [z_n, hgh^{-1}]_L [h, g]_L \\ &= (z_n hgh^{-1} z_n^{-1} hg^{-1} h^{-1})(hgh^{-1} g^{-1}).\end{aligned}$$

Suppose $x \to [x, hgh^{-1}]_L$ is continuous at $x = e$. The identity above now yields $[h_n, g]_L \to [h, g]_L$ as $h_n \to_R h$; indeed $z_n = h_n h^{-1} \to e_X$ so $w_n := [z_n, hgh^{-1}]_L \to e_X$, and thus with $a := [h, g]_L$ we have $\rho_a(w_n) = w_n a \to a$. ■

REMARKS. 1. If the group-norm is abelian, then we have the *left-right commutator inequality*

$$\|[x, y]_L\| \leq 2\|xy^{-1}\| = 2d_R(x, y),$$

because

$$\|[x, y]_L\| = \|xyx^{-1}y^{-1}\| \leq \|xy^{-1}\| + \|yx^{-1}\| = 2\|xy^{-1}\|.$$

The commutator inequality thus implies separate continuity of the commutator by Lemma 2.21.
2. If the group-norm is arbitrary, this inequality may be stated via the symmetrized metric:

$$\begin{aligned}\|[x, y^{-1}]_R\| &\leq \|xy^{-1}\| + \|x^{-1}y\| = d_R(x, y) + d_L(x, y) \\ &\leq 2\max\{d_R(x, y), d_L(x, y)\} := 2d_S(x, y).\end{aligned}$$

3. Take $u = f(tx), v = f(x)^{-1}$ etc.; then, assuming the Klee Property, we have

$$\begin{aligned}\|f(tx)g(tx)[f(x)g(x)]^{-1}\| &= \|f(tx)g(tx)g(x)^{-1}f(x)^{-1}\| \\ &\leq \|f(tx)f(x)^{-1}\| + \|g(tx)g(x)^{-1}\|,\end{aligned}$$

showing that the product of two slowly varying functions is slowly varying, since

$$f(tx)f(t)^{-1} \to e \text{ iff } \|f(tx)f(t)^{-1}\| \to 0.$$

3. Normed versus topological groups

By the Birkhoff-Kakutani Theorem above (Th. 2.19) any *metrizable* topological group has a right-invariant equivalent metric, and hence is a normed group. Theorem 3.4 below establishes a converse: a normed group is a topological group provided all its shifts (both right and left-sided) are continuous, i.e. provided the normed group is *semitopological* (see [ArRez]). This is not altogether surprising, in the light of known results on semitopological groups: assuming that a group T is metrizable, non-meagre and analytic in the metric, and that both left and right-shifts are continuous, then T is a topological group (see e.g. [THJ] for several results of this kind in [Rog2, p. 352]; compare also [Ell2] and the literature cited under Remarks 2 in Section 2). The results here are cognate, and new because a normed group has a one-sided rather than a two-sided topology. We will also establish the equivalent condition that all conjugacies $\gamma_g(x) := gxg^{-1}$ are continuous; this has the advantage of being stated in terms of the norm, rather than in terms of one of the associated metrics. As inner automorphisms are homomorphisms, this condition ties the structure of normed groups to issues of automatic continuity of homomorphisms: automatic continuity forces a normed group to be a topological group (and the homomorphisms to be homeomorphisms). Normed groups are thus *either topological or pathological*, as noted in the Introduction.

The current section falls into three parts. In the first we characterize topological groups in the category of normed groups and so in particular, using norms, characterize also the Klee groups (topological groups which have an equivalent bi-invariant metric). Then we study continuous automorphisms in relation to Lipschitz norms. In the third subsection we demonstrate that a small amount of regularity forces a normed group to be a topological group.

3.1. Left versus right-shifts: Equivalence Theorem. As we have seen in Th. 2.3, a group-norm defines two metrics: the right-invariant metric which we denote as usual by $d_R(x,y) := \|xy^{-1}\|$ and the conjugate left-invariant metric, here to be denoted $d_L(x,y) := d_R(x^{-1},y^{-1}) = \|x^{-1}y\|$. There is correspondingly a right and left metric topology which we term the *right* or *left norm topology*. We favour this over 'right' or 'left' normed groups rather than follow the [HS] paradigm of 'right' and 'left' topological semigroups. We write $\to_R$ for convergence under d_R etc. Recall that both metrics give rise to the same norm, since $d_L(x,e) = d_R(x^{-1},e) = d_R(e,x) = \|x\|$, and hence define the same balls centered at the origin e:

$$B_R^d(e,r) := \{x : d(e,x) < r\} = B_L^d(e,r).$$

Denoting this commonly determined set by $B(r)$, we have seen in Proposition 2.5 that

$$B_R(a,r) = \{x : x = ya \text{ and } d_R(a,x) = d_R(e,y) < r\} = B(r)a,$$
$$B_L(a,r) = \{x : x = ay \text{ and } d_L(a,x) = d_L(e,y) < r\} = aB(r).$$

Thus the open balls are right- or left-shifts of the norm balls at the origin. This is best viewed in the current context as saying that under d_R the right-shift $\rho_a : x \to xa$ is right uniformly continuous, since

$$d_R(xa, ya) = d_R(x,y),$$

and likewise that under d_L the left-shift $\lambda_a : x \to ax$ is left uniformly continuous, since

$$d_L(ax, ay) = d_L(x,y).$$

In particular, under d_R we have $y \to_R b$ iff $yb^{-1} \to_R e$, as $d_R(e, yb^{-1}) = d_R(y,b)$. Likewise, under d_L we have $x \to_L a$ iff $a^{-1}x \to_L e$, as $d_L(e, a^{-1}x) = d_L(x,a)$.

Thus either topology is determined by the neighbourhoods of the identity (origin) and according to choice makes the appropriately sided shift continuous; said another way, the topology is determined by the neighbourhoods of the identity and the chosen shifts. We noted earlier that the triangle inequality implies that multiplication is jointly continuous at the identity e, as a mapping from (X, d_R) to (X, d_R). Likewise inversion is also continuous at the identity by the symmetry axiom. (See Theorem 2.19′.) To obtain similar results elsewhere one needs to have continuous conjugation, and this is linked to the equivalence of the two norm topologies (see Th. 3.4). The conjugacy map under $g \in G$ (inner automorphism) is defined by

$$\gamma_g(x) := gxg^{-1}.$$

Recall that the inverse of γ_g is given by conjugation under g^{-1} and that γ_g is a *homomorphism*. Its continuity, as a mapping from (X, d_R) to (X, d_R), is thus determined by behaviour at the identity, as we verify below. We work with the right topology (under d_R), and sometimes leave unsaid equivalent assertions about the isometric case of (X, d_L) replacing (X, d_R).

Lemma 3.1. *The homomorphism γ_g is right-to-right continuous at any point iff it is right-to-right continuous at e.*

Proof. This is immediate since $x \to_R a$ if and only if $xa^{-1} \to_R e$ and $\gamma_g(x) \to_R \gamma_g(a)$ iff $\gamma_g(xa^{-1}) \to_R \gamma_g(e)$, since

$$\|gxg^{-1}(gag^{-1})^{-1}\| = \|gxa^{-1}g^{-1}\|.$$

■

We note that, by the Generalized Darboux Theorem (Th. 11.22), if γ_g is locally norm-bounded and the norm is $\mathbb{N}$-subhomogeneous (i.e. a Darboux norm – there are constants $\kappa_n \to \infty$ with $\kappa_n\|z\| \le \|z^n\|$), then γ_g is continuous. Working under d_R, we will relate inversion to left-shifts. We begin with the following, a formalization of an earlier observation.

Lemma 3.2. *If inversion is right-to-right continuous, then*

$x \to_R a$ *iff* $a^{-1}x \to_R e$.

Proof. For $x \to_R a$, we have $d_R(e, a^{-1}x) = d_R(x^{-1}, a^{-1}) \to 0$, assuming continuity. Conversely, for $a^{-1}x \to_R e$ we have $d_R(a^{-1}x, e) \to 0$, i.e. $d_R(x^{-1}, a^{-1}) \to 0$. So since inversion is assumed to be right-continuous and $(x^{-1})^{-1} = x$, etc, we have $d_R(x, a) \to 0$. ■

We now expand this.

Theorem 3.3. *The following are equivalent:*
(i) inversion is right-to-right continuous,
(ii) left-open sets are right-open,
(iii) for each g the conjugacy γ_g is right-to-right continuous at e, i.e. for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$gB(\delta)g^{-1} \subset B(\varepsilon),$$

(iv) left-shifts are right-continuous.

Proof. We show that (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv).
Assume (i). For any a and any $\varepsilon > 0$, by continuity of inversion at a, there is $\delta > 0$ such that, for x with $d_R(x, a) < \delta$, we have $d_R(x^{-1}, a^{-1}) < \varepsilon$, i.e. $d_L(x, a) < \varepsilon$. Thus

$$B(\delta)a = B_R(a, \delta) \subset B_L(a, \varepsilon) = aB(\varepsilon), \tag{incl}$$

i.e. left-open sets are right-open, giving (ii). For the converse, we just reverse the last argument. Let $\varepsilon > 0$. As $a \in B_L(a, \varepsilon)$ and $B_L(a, \varepsilon)$ is left open, it is right open and so there is $\delta > 0$ such that

$$B_R(a, \delta) \subset B_L(a, \varepsilon).$$

Thus for x with $d_R(x, a) < \delta$, we have $d_L(x, a) < \varepsilon$, i.e. $d_R(x^{-1}, a^{-1}) < \varepsilon$, i.e. inversion is right-to-right continuous, giving (i).
To show that (ii)$\Longleftrightarrow$(iii) note that the inclusion (incl) is equivalent to

$$a^{-1}B(\delta)a \subset B(\varepsilon),$$

i.e. to

$$\gamma_a^{-1}[B(\delta)] \subset B(\varepsilon),$$

that is, to the assertion that $\gamma_a(x)$ is continuous at $x = e$ (and so continuous, by Lemma 3.1). The property (iv) is equivalent to (iii) since the right-shift is right-continuous and $\gamma_a(x)a = \lambda_a(x)$ is equivalent to $\gamma_a(x) = \lambda_a(x)a^{-1}$. ■

We saw in the Birkhoff-Kakutani Theorem (Th. 2.19) that metrizable topological groups are normable (equivalently, have a right-invariant metric); we now formulate a converse, showing when the right-invariant metric derived from a group-norm equips its group with a topological group structure. As this is a characterization of metric topological groups, we will henceforth refer to them synonymously as *normed topological groups*.

THEOREM 3.4 (Equivalence Theorem). *A normed group is a topological group under the right (resp. left) norm topology iff each conjugacy*

$$\gamma_g(x) := gxg^{-1}$$

is right-to-right (resp. left-to-left) continuous at $x = e$ (and so everywhere), i.e. for $z_n \to_R e$ and any g

$$gz_ng^{-1} \to_R e. \tag{adm}$$

Equivalently, it is a topological group iff left/right-shifts are continuous for the right/left norm topology, or iff the two norm topologies are themselves equivalent.
In particular, if also the group structure is abelian, then the normed group is a topological group.

Proof. Only one direction needs proving. We work with the d_R topology, the right topology. By Theorem 3.3 we need only show that under it multiplication is jointly right-continuous. First we note that multiplication is right-continuous iff

$$d_R(xy, ab) = \|xyb^{-1}a^{-1}\|, \text{ as } (x, y) \to_R (a, b).$$

Here, we may write $Y = yb^{-1}$ so that $Y \to_R e$ iff $y \to_R b$, and we obtain the equivalent condition

$$d_R(xYb, ab) = d_R(xY, a) = \|xYa^{-1}\|, \text{ as } (x, Y) \to_R (a, e).$$

By Theorem 3.3, as inversion is right-to-right continuous, Lemma 3.2 justifies re-writing the second convergence condition with $X = a^{-1}x$ and $X \to_R e$, yielding the equivalent condition

$$d_R(aXYb, ab) = d_R(aXY, a) = \|aXYa^{-1}\|, \text{ as } (X, Y) \to_R (e, e).$$

But, by Lemma 3.1, this is equivalent to continuity of conjugacy. ∎

The final is related to a result of Żelazko [Zel] (cf. [Com, §11.6]). We will later apply the Equivalence Theorem several times in conjunction with the following result (see also Lemma 3.34 for a strengthening).

LEMMA 3.5 (Weak continuity criterion). *For fixed x, if for all null sequences w_n, we have $\gamma_x(w_{n(k)}) \to e_X$ down some subsequence $w_{n(k)}$, then γ_x is continuous.*

Proof. We are to show that for every $\varepsilon > 0$ there is $\delta > 0$ and N such that for all $n > N$

$$xB(\delta)x^{-1} \subset B(\varepsilon).$$

Suppose not. Then there is $\varepsilon > 0$ such that for each $k = 1, 2, ..$ and each $\delta = 1/k$ there is $n = n(k) > k$ and w_k with $\|w_k\| < 1/k$ and $\|xw_kx^{-1}\| > \varepsilon$. So $w_k \to 0$. By assumption, down some subsequence $n(k)$ we have $\|xw_{n(k)}x^{-1}\| \to 0$, but this contradicts $\|xw_{n(k)}x^{-1}\| > \varepsilon$. ∎

Corollary 3.6. *For X a topological group under its norm, the left-shifts $\lambda_a(x) := ax$ are bounded and uniformly continuous in norm.*

Proof. We have $\|\lambda_a\| = \|a\|$ as

$$\sup_x d_R(x, ax) = d_R(e, a) = \|a\|.$$

We also have

$$d_R(ax, ay) = d_R(axy^{-1}a^{-1}, e) = \|\gamma_a(xy^{-1})\|.$$

Hence, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for $\|z\| < \delta$

$$\|\gamma_a(z)\| \le \varepsilon.$$

Thus provided $d_R(x, y) = \|xy^{-1}\| < \gamma$, we have $d_R(ax, ay) < \varepsilon$. ∎

Remarks. 1 **(Klee property).** If the group has an abelian norm (in particular if the group is abelian), then the norm has the Klee property (see [Klee] for the original metric formulation, or Th. 2.18), and then it is a *topological group* under the norm-topology. Indeed the Klee property is that

$$\|xyb^{-1}a^{-1}\| \le \|xa^{-1}\| + \|yb^{-1}\|,$$

and so if $x \to_R a$ and $y \to_R b$, then $xy \to_R ab$. This may also be deduced from the observation that γ_g is continuous, since here

$$\|gxg^{-1}\| = \|gxeg^{-1}\| \le \|gg^{-1}\| + \|xe\| = \|x\|.$$

Compare [vM2] Section 3.3, especially Example 3.3.6 of a topological group of real matrices which fails to have an abelian norm (see also [HJ, p.354] p.354).
2. Theorem 3.4 may be restated in the language of commutators, introduced at the end of Section 2 (see Th. 2.20). These are of interest in Theorems 6.3, 10.7 and 10.9.

Corollary 3.7. *If the L-commutator is right continuous as a map from (X, d_R) to (X, d_R), then (X, d_R) is a topological group. The same conclusion holds for left continuity and for the R-commutator.*

Proof. Fix g. We will show that γ_g is continuous at e; so let $z_n \to e$.
First we work with the L-commutator and assume it to be, say right continuous, at e (which is equivalent to being left continuous at e, by Lemma 2.21). From the identity

$$\gamma_g(z_n) := gz_ng^{-1}(z_n^{-1}z_n) = [g, z_n]_L z_n,$$

the assumed right continuity implies that $w_n := [g, z_n]_L \to e$; but then $w_n z_n \to e$, by the triangle inequality. Thus γ_g is continuous. By Theorem 3.4 (X, d_R) is a topological group.
Next we work with the R-commutator and again assume that to be right continuous at e. Noting that $[g, z_n]_L = [g^{-1}, z_n^{-1}]_L$ and $z_n^{-1} \to e$ we may now interpet the previous

argument as again proving that γ_g is continuous; indeed we may now read the earlier identity as asserting that

$$\gamma_g(z_n) := gz_n g^{-1}(z_n^{-1}z_n) = [g^{-1}, z_n^{-1}]_R z_n,$$

for which the earlier argument continues to hold. ■

3. For T a normed group with right-invariant metric d_R one is led to study the associated supremum metric on the group of bounded homeomorphisms h from T to T (i.e. having $\sup_T d(h(t),t) < \infty$) with composition $\circ$ as group operation:

$$d_{\mathcal{A}}(h,h') = \sup\nolimits_T d(h(t),h'(t)).$$

This is a right-invariant metric which generates the norm

$$\|h\|_{\mathcal{A}} := d_{\mathcal{A}}(h,e_{\mathcal{A}}) = \sup\nolimits_T d(h(t),t).$$

It is of interest from the perspective of topological flows, in view of the following observation.

Lemma 3.8 ([Dug, XII.8.3, p. 271]). *Under $d_{\mathcal{A}}$ on $\mathcal{A} = Auth(T)$ and d^T on T, the evaluation map $(h,t) \to h(t)$ from $\mathcal{A}\times T$ to T is continuous.*

Proof. Fix h_0 and t_0. The result follows from continuity of h_0 at t_0 via

$$\begin{aligned} d^T(h_0(t_0),h(t)) &\le d^T(h_0(t_0),h_0(t)) + d^T(h_0(t),h(t)) \\ &\le d^T(h_0(t_0),h_0(t)) + d_{\mathcal{A}}(h,h_0). \end{aligned}$$

■

4. Since the conjugate metric of a right-invariant metric need not be continuous, one is led to consider the earlier defined *symmetrization refinement of a metric d*, which we recall is given by

$$d_S(g,h) = \max\{d(g,h), d(g^{-1},h^{-1})\}. \tag{sym}$$

This metric need not be translation invariant on either side (cf. [vM2, Example 1.4.8]); however, it is inversion-invariant:

$$d_S(g,h) = d^S(g^{-1},h^{-1}),$$

so one expects to induce topological group structure with it, as we do in Th. 3.13 below. When $d = d_R^X$ is right-invariant and so induces the group-norm $\|x\| := d(x,e)$ and $d(x^{-1},y^{-1}) = d_L^X(x,y)$, we may use (sym) to define

$$\|x\|_S := d_S^X(x,e).$$

Then

$$\|x\|_S = \max\{d_R^X(x,e), d_R^X(x^{-1},e)\} = \|x\|,$$

which is a group-norm, even though d_S^X need not be either left- or right-invariant. This motivates the following result, which follows from the Equivalence Theorem (Th. 3.4) and Example A4 (Topological permutations), given towards the start of Section 2.

Theorem 3.9 (Ambidextrous Refinement). *For X a normed group with norm $\|.\|$, put*
$$d_S^X(x,y) := \max\{\|xy^{-1}\|, \|x^{-1}y\|\} = \max\{d_R^X(x,y), d_L^X(x,y)\}.$$
Then X is a topological group under the right (or left) norm topology iff X is a topological group under the symmetrization refinement metric d_S^X iff the topologies of d_S^X and of d_R^X are identical.

Proof. Suppose that under the right-norm topology X is a topological group. Then d_L^X is d_R^X-continuous, by Th. 3.4 (continuity of inversion), and hence d_S^X is also d_R^X-continuous. Thus if $x_n \to x$ under d_R^X, then also, by continuity of d_L^X, one has $x_n \to x$ under d_S^X. Now if $x_n \to x$ under d_S^X, then also $x_n \to x$ under d_R^X, as $d_R^X \leq d_S^X$. Thus d_S^X generates the topology and so X is a topological group under d_S^X.
Conversely, suppose that X is a topological group under d_S^X. As X is a topological group, its topology is generated by the neighbourhoods of the identity. But as already noted,
$$d_S^X(x,e) := \|x\|,$$
so the d_S^X-neighbourhoods of the identity are also generated by the norm; in particular any left-open set $aB(\varepsilon)$ is d_S^X-open (as left shifts are homeomorphisms) and so right-open (being a union of right shifts of neighbourhoods of the identity). Hence by Th. 3.4 (or Th. 3.3) X is a topological group under either norm topology.

As for the final assertion, if the d_S^X topology is identical with the d_R^X topology then inversion is d_R^X-continuous and so X is a topological group by Th. 3.4. The argument of the first paragraph shows that if d_R^X makes X into a topological group then d_R^X and d_S^X generate the same topology. ∎

Thus, according to the Ambidextrous Refinement Theorem, a symmetrization that creates a topological group structure from a norm structure is in fact *redundant*. We are about to see such an example in the next theorem.

Given a metric space (X,d), we let $\mathcal{H}_{unif}(X)$ denote the subgroup of uniformly continuous homeomorphisms (relative to d), i.e. homeomorphisms α satisfying the condition that, for each $\varepsilon > 0$, there is $\delta > 0$ such that
$$d(\alpha(x), \alpha(x')) < \varepsilon, \text{ for } d(x,x') < \delta. \tag{u-cont}$$

Lemma 3.10 (Compare [dGMc, Cor. 2.13]). *(i) For fixed $\xi \in \mathcal{H}(X)$, the mapping $\rho_\xi : \alpha \to \alpha\xi$ is continuous.*
(ii) For fixed $\alpha \in \mathcal{H}_{unif}(X)$, the mapping $\lambda_\alpha : \beta \to \alpha\beta$ is in $\mathcal{H}_{unif}(X)$ – i.e. is uniformly continuous.
(iii) The mapping $(\alpha,\beta) \to \alpha\beta$ is continuous from $\mathcal{H}_{unif}(X) \times \mathcal{H}_{unif}(X)$ to $\mathcal{H}(X)$ under the supremum norm.

Proof. (i) We have
$$\hat{d}(\alpha\xi, \beta\xi) = \sup d(\alpha(\xi(t)), \beta(\xi(t))) = \sup d(\alpha(s), \beta(s)) = \hat{d}(\alpha, \beta).$$

(ii) For $\alpha \in \mathcal{H}_{unif}(X)$ and given $\varepsilon > 0$, choose $\delta > 0$, so that (u-cont) holds. Then, for β, γ with $\hat{d}(\beta, \gamma) < \delta$, we have $d(\beta(t), \gamma(t)) < \delta$ for each t, and hence

$$\hat{d}(\alpha\beta, \alpha\gamma) = \sup d(\alpha(\beta(t)), \alpha(\gamma(t))) \leq \varepsilon.$$

(iii) Again, for $\alpha \in \mathcal{H}_{unif}(X)$ and given $\varepsilon > 0$, choose $\delta > 0$, so that (u-cont) holds. Thus, for β, η with $\hat{d}(\beta, \eta) < \delta$, we have $d(\beta(t), \eta(t)) < \delta$ for each t. Hence for ξ with $\hat{d}(\alpha, \xi) < \varepsilon$ we obtain

$$\begin{aligned} d(\alpha(\beta(t)), \xi(\eta(t))) &\leq d(\alpha(\beta(t)), \alpha(\eta(t))) + d(\alpha(\eta(t)), \xi(\eta(t))) \\ &\leq \varepsilon + \hat{d}(\alpha, \xi) \leq \varepsilon + \varepsilon. \end{aligned}$$

Consequently, we have

$$\hat{d}(\alpha\beta, \xi\eta) = \sup d(\alpha(\beta(t)), \xi(\eta(t))) \leq 2\varepsilon.$$

■

Comment. See also [AdC] for a discussion of the connection between choice of metric and uniform continuity. The following result is of interest.

Proposition 3.11 (deGroot-McDowell Lemma, [dGMc, Lemma 2.2]). *Given Φ, a countable family of self-homeomorphism of X closed under composition (i.e. a semigroup in $Auth(X)$), the metric on X may be replaced by a topologically equivalent one such that each $\alpha \in \Phi$ is uniformly continuous.*

Definition. Say that a homeomorphism h is *bi-uniformly continuous* if both h and h^{-1} are uniformly continuous. Write

$$\mathcal{H}_u = \{h \in \mathcal{H}_{unif} : h^{-1} \in \mathcal{H}_{unif}\}.$$

Proposition 3.12 (Group of left-shifts). *For a normed topological group X with right-invariant metric d^X, the group $Tr_L(X)$ of left-shifts is (under composition) a subgroup of $\mathcal{H}_u(X)$ that is isometric to X.*

Proof. As X is a topological group, we have $Tr_L(X) \subseteq \mathcal{H}_u(X)$ by Cor. 3.6; $Tr_L(X)$ is a subgroup and $\lambda : X \to Tr_L(X)$ is an isomorphism, because

$$\lambda_x \circ \lambda_y(z) = \lambda_x(\lambda_y(z)) = x(\lambda_y(z) = xyz = \lambda_{xy}(z).$$

Moreover, λ is an isometry, as d^X is right-invariant; indeed, we have

$$d^T(\lambda_x, \lambda_y) = \sup_z d^X(xz, yz) = d^X(x, y).$$

■

We now offer a generalization which motivates the duality considerations of Section 12.

Theorem 3.13. *The family $\mathcal{H}_u(T)$ of bi-uniformly continuous bounded homeomorphisms of a complete metric space T is a complete topological group under the symmetrized supremum metric. Consequently, under the supremum metric it is a topological group and is topologically complete.*

Proof. Suppose that T is metrized by a complete metric d. The bounded homeomorphisms of T, i.e. those homeomorphisms h for which $\sup d(h(t),t) < \infty$, form a group $\mathcal{H} = \mathcal{H}(T)$ under composition. The subgroup

$$\mathcal{H}_u = \{h \in \mathcal{H} : h \text{ and } h^{-1} \text{ is uniformly continuous}\}$$

is complete under the supremum metric $\hat{d}(h,h') = \sup d(h(t),h'(t))$, by the standard 3ε argument. It is a topological semigroup since the composition map $(h,h') \to h \circ h'$ is continuous. Indeed, as in the proof of Proposition 2.13, in view of the inequality

$$\begin{aligned} d(h \circ h'(t), H \circ H'(t)) &\leq d(h \circ h'(t), H \circ h'(t)) + d(H \circ h'(t), H \circ H'(t)) \\ &\leq \hat{d}(h,H) + d(H \circ h'(t), H \circ H'(t)), \end{aligned}$$

for each $\varepsilon > 0$ there is $\delta = \delta(H,\varepsilon) < \varepsilon$ such that for $\hat{d}(h',H') < \delta$ and $\hat{d}(h,H) < \varepsilon$,

$$\hat{d}(h \circ h', H \circ H') \leq 2\varepsilon.$$

Likewise, mutatis mutandis, for their inverses; to be explicit, writing $g = h'^{-1}, G = H'^{-1}$ etc, for each $\varepsilon > 0$ there is $\delta' = \delta(G,\varepsilon) = \delta(H'^{-1},\varepsilon)$ such that for $\hat{d}(g',G') < \delta'$ and $\hat{d}(g,G) < \varepsilon$,

$$\hat{d}(g \circ g', G \circ G') \leq 2\varepsilon.$$

Set $\eta = \min\{\delta,\delta'\} < \varepsilon$. So for $\max\{\hat{d}(h',H'),\hat{d}(g,G)\} < \eta$ and $\max\{\hat{d}(h,H),\hat{d}(g',G')\} < \eta$, we have $\hat{d}(h',H') < \delta, \hat{d}(h,H) < \delta < \varepsilon$, and $\hat{d}(g',G') < \delta$ and $\hat{d}(g,G) < \varepsilon$. Since $(h \circ h')^{-1} = g \circ g'$ etc, we have

$$\max\{\hat{d}(h \circ h', H \circ H'), \hat{d}(g \circ g', G \circ G')\} \leq 2\varepsilon.$$

So composition is continuous under the symmetrized metric

$$d_S(g,h) = \max\{\hat{d}(g,h,),\hat{d}(g^{-1},h^{-1})\}.$$

But as this metric is inversion-invariant, i.e.

$$d_S(g,h) = d_S(g^{-1},h^{-1}),$$

this gives continuity of inversion. This means that $\mathcal{H}_u$ is a complete metric topological group under the symmetrized supremum metric.
The final assertion follows from the Ambidextrous Refinement Theorem, Th. 3.9. (The symmetrized metric topology and the supremum metric coincide.) ∎

We now deduce a corollary with important consequences for the Uniform Convergence Theorem of topological regular variation (for which see [BOst-TRI]). We need the following definitions and a result due to Effros (for a proof and related literature see [vM2]).

DEFINITION. A group $G \subset \mathcal{H}(X)$ acts *transitively* on a space X if for each x, y in X there is g in X such that $g(x) = y$.

The group acts *micro-transitively* on X if for U a neighbourhood of e in G and $x \in X$ the set $\{h(x) : h \in U\}$ is a neighbourhood of x.

THEOREM 3.14 (Effros' Open Mapping Principle, [Eff]). *Let G be a Polish topological group acting transitively on a separable metrizable space X. The following are equivalent.*
(i) G acts micro-transitively on X,
(ii) X is Polish,
(iii) X is of second category.

REMARK. van Mill [vM1] gives the stronger result for G an *analytic* group (see Section 11 for definition) that (iii) implies (i). See also Section 10 for definitions, references and the related classical Open Mapping Theorem (which follows from Th. 3.14: see [vM1]). Indeed, van Mill ([vM1]) notes that he uses (i) separately continuous action (see the final page of his proof), (ii) the existence of a sequence of symmetric neighbourhoods U_n of the identity with $U_{n+1} \subseteq U_{n+1}^2 \subseteq U_n$, and (iii) $U_1 = G$ (see the first page of his proof). By Th. 2.19' (Birkhoff-Kakutani Normability Theorem) van Mill's conditions under (ii) specify a normed group, whereas condition (iii) may be arranged by switching to the equivalent norm $||x||_1 := \max\{||x||, 1\}$ and then taking $U_n := \{x : ||x||_1 < 2^{-n}\}$. Thus in fact one has

THEOREM 3.14′ (Analytic Effros Open Mapping Principle). *For T an analytic* ***normed group*** *acting transitively and separately continuously on a separable metrizable space X: if X is non-meagre, then T acts micro-transitively on X.*

The normed-group result is of interest, as some naturally occurring normed groups are not complete (see Charatonik et Maćkowiak [ChMa] for Borel normed groups that are not complete, and [FaSol] for a study of Borel subgroups of Polish groups).

THEOREM 3.15 (Crimping Theorem). *Let T be a Polish space with a complete metric d. Suppose that a closed subgroup $\mathcal{G}$ of $\mathcal{H}_u(T)$ acts on T transitively, i.e. for any s, t in T there is h in $\mathcal{G}$ such that $h(t) = s$. Then for each $\varepsilon > 0$ and $t \in T$, there is $\delta > 0$ such that for any s with $d^T(s, t) < \delta$, there exists h in $\mathcal{G}$ with $\|h\|_{\mathcal{H}} < \varepsilon$ such that $h(t) = s$. Consequently:*
(i) if y, z are in $B_\delta(t)$, then there exists h in $\mathcal{G}$ with $\|h\|_{\mathcal{H}} < 2\varepsilon$ such that $h(y) = z$;
(ii) Moreover, for each $z_n \to t$ there are h_n in $\mathcal{G}$ converging to the identity such that $h_n(t) = z_n$.

Proof. As T is Polish, $\mathcal{G}$ is Polish, and so by Effros' Theorem, $\mathcal{G}$ acts micro-transitively on T; that is, for each t in T and each $\varepsilon > 0$ the set $\{h(t) : h \in \mathcal{H}_u(T)$ and $\|h\|_{\mathcal{H}} < \varepsilon\}$

is a neighbourhood of t, i.e. for some $\delta = \delta(\varepsilon) > 0$, $B_\delta(t) \subset \{h(t) : \|h\| < \varepsilon\}$. Hence if $d^T(s,t) < \delta$ we have for some h in $\mathcal{G}$ with $\|h\|_{\mathcal{H}} < \varepsilon$ that $h(t) = s$.
If $y, z \in B_\delta(t)$, there is h, k in $\mathcal{G}$ with $\|h\| < \varepsilon$ and $\|k\| < \varepsilon$ such that $h(t) = y$ and $k(t) = z$. Thus $kh^{-1}$is in $\mathcal{G}$, $kh^{-1}(y) = z$ and

$$\|kh^{-1}\| \leq \|k\| + \|h^{-1}\| = \|k\| + \|h\| \leq 2\varepsilon,$$

as the norm is inversion symmetric.
For the final conclusion, taking for ε successively the values $\varepsilon_n = 1/n$, we define $\delta_n = \delta(\varepsilon_n)$. Let $z_n \to t$. By passing to a subsequence we may assume that $d^T(z_n, t) < \delta_n$. Now there exists h_n in $\mathcal{G}$ such that $\|h_n\| < 2\varepsilon_n$ and $h_n(t) = z_n$. As $h_n \to id$, we have constructed the 'crimping sequence' of homeomorphisms asserted. ■

Remark. By Proposition 3.12, this result applies also to the closed subgroup of left translations on T for T a Polish topological group.

The Crimping Theorem implies the following classical result.

Theorem 3.16 (Ungar's Theorem, [Ung], [vM2, Th. 2.4.1, p. 78]). *Let $\mathcal{G}$ be a subgroup of $\mathcal{H}(X)$. Let X be a compact metric space on which $\mathcal{G}$ acts transitively. For each $\varepsilon > 0$, there is $\delta > 0$ such that for x, y with $d(x,y) < \delta$ there is $h \in \mathcal{G}$ such that $h(x) = y$ and $\|h\| < \varepsilon$.*

Proof. X is a Polish space, and $\mathcal{H}(X) = \mathcal{H}_u(X)$, as X is compact. Let $\varepsilon > 0$. By the Crimping Theorem, for each $x \in X$ there is $\delta = \delta(x, \varepsilon) > 0$ such that for $y, z \in B_\delta(x)$ there is $h \in \mathcal{G}$ with $h(y) = z$ and $\|h\| < \varepsilon$. Thus $\{B_{\delta(x,\varepsilon)}(x) : x \in X\}$ covers X. By compactness, for some finite set $F = \{x_1, ..., x_N\}$, the space X is covered by $\{B_{\delta(x,\varepsilon)}(x) : x \in F\}$. The conclusion of the theorem follows on taking $\delta = \min\{\delta(x, e) : x \in F\}$. ■

Definition. Let G be a normed group with group-norm $\|.\|$. For $g \in G$, recall that the *g-conjugate norm* is defined by

$$\|x\|_g := \|\gamma_g(x)\| = \|gxg^{-1}\|.$$

If left and right-shifts are continuous in G (in particular if G is a semitopological group), then $\|z_n\| \to 0$ iff $\|z_n\|_g \to 0$.

Example. For X a normed group with metric d^X, take $G = \mathcal{H}_u(X)$ normed by $\|h\| := \|h\|_{\mathcal{H}}$. Then

$$\|h\|_g = \sup\nolimits_x d^X(ghg^{-1}(x), x) = \sup\nolimits_z d^X(g(h(z)), z).$$

We now give an explicit construction of a equivalent bi-invariant metric on G when one exists (compare [HR, Section 8.6]), namely

$$\|x\|_\infty := \sup\{\|x\|_g : g \in G\}.$$

We recall from Section 2 that the group-norm satisfies the *norm admissibility condition* (on X) if, for $z_n \to e$ and g_n arbitrary,

$$\|g_n z_n g_n^{-1}\|_G \to 0. \tag{n-adm}$$

Evidently in view of the sequence $\{g_n\}$, this is a sharper version of (adm).

THEOREM 3.17. *For G with group-norm $\|.\|_G$, suppose that $\|.\|_\infty$ is finite on G. Then $\|x\|_\infty$ is an equivalent norm iff the $\|.\|_G$ meets the norm admissibility condition (n-adm). In particular, for $|x| := \min\{\|x\|, 1\}$ the corresponding norm $|x|_\infty := \sup\{|x|_g : g \in G\}$ is an equivalent abelian norm iff the admissibility condition (n-adm) holds.*

Proof. First assume (n-adm) holds. As $\|x\| = \|x\|_e \le \|x\|_\infty$ we need to show that if $z_n \to e$, then $\|z_n\|_\infty \to 0$. Suppose otherwise; then for some $\varepsilon > 0$, without loss of generality $\|z_n\|_\infty \ge \varepsilon$, and so there is for each n an element g_n such that

$$\|g_n z_n g_n^{-1}\| \ge \varepsilon/2.$$

But this contradicts the admissibility condition (n-adm).
As to the abelian property of the norm, we have

$$\|yzy^{-1}\|_\infty = \sup\{\|gyzy^{-1}g^{-1}\| : g \in G\} = \sup\{\|gyz(gy)^{-1}\| : g \in G\} = \|z\|_\infty,$$

and so taking $z = xy$ we have $\|yx\| = \|xy\|$.
For the converse, assume $\|x\|_\infty$ is an equivalent norm. For g_n arbitrary, suppose that $\|z_n\| \to 0$ and $\varepsilon > 0$. For some N and all $n \ge N$ we thus have $\|z_n\|_\infty < \varepsilon$. Hence for $n \ge N$,

$$\|g_n z_n g_n^{-1}\| \le \|z_n\|_\infty < \varepsilon,$$

verifying the condition (n-adm). ■

THEOREM 3.18. *Let G be a normed topological group which is compact under its norm $\|.\|_G$. Then*

$$\|x\|_\infty := \sup\{\|x\|_g : g \in G\}$$

is an abelian (hence bi-invariant) norm topologically equivalent to $\|x\|$.

Proof. We write $\|.\|$ for $\|.\|_G$. Suppose, for some x, that $\{\|x\|_g : g \in G\}$ is unbounded. We may select g_n with

$$\|g_n x g_n^{-1}\| \to \infty.$$

Passing to a convergent subsequence we obtain a contradiction. Thus $\|x\|_\infty$ is finite and hence a norm. We verify the admissibility condition. Suppose to the contrary that for some $z_n \to e$, arbitrary g_n, and some $\varepsilon > 0$ we have

$$\|g_n z_n g_n^{-1}\| > \varepsilon.$$

Using compactness, we may pass to a convergent subsequence, $g_m \to g$ (in the norm $\|.\|_G$). Since multiplication is jointly continuous in G we obtain the contradiction that $\|geg^{-1}\| = \|e\| = 0 > \varepsilon$. ■

Remarks. 1. Suppose as usual that d_R is a right-invariant metric on a group G. The right-shift $\rho_g(x) = xg$ is uniformly continuous, as

$$d_R(xg, yg) = d_R(x, y).$$

However, it is not necessarily bounded, as

$$\|\rho_g\|_{\mathcal{H}} = \sup_x d_R(xg, x) = \sup_x \|g\|_x = \|g\|_\infty.$$

But on the subgroup $\{\rho_g : \|g\|_\infty < \infty\}$, the norm $\|\rho_g\|$ is bi-invariant, since $\|g\|_\infty$ is bi-invariant.

2. The condition (n-adm) used in Theorem 3.17 to check *admissibility* of the supremum norm may be reformulated, without reference to the group-norm, topologically thus:

$$g_n z_n g_n^{-1} \to e \text{ for } z_n \to e,$$

with g_n arbitrary. In a first-countable topological group this condition is equivalent to the *existence* of a bi-invariant metric (see Proposition 2.15; cf. Theorem 3.3.4 in [vM2, p. 101]). We will see several related conditions later: (ne) in Th.3.30, and (W-adm) and (C-adm) ahead of Lemma 3.33 below; we recall here the condition (H-adm) of Prop. 2.14.

3. Note that $SL(2,\mathbb{R})$, the set of 2×2 real matrices with determinant 1, under matrix multiplication and with the subspace topology of $\mathbb{R}^4$ forms a (locally compact) topological group with no equivalent bi-invariant metric; for details see e.g. [HR, 4.24], or [vM2] Example 3.3.6 (p.103), where matrices a_n, g_n are exhibited with $z_n := a_n g_n \to e$ and $g_n a_n \nrightarrow e$, so that $g_n(a_n g_n)g_n^{-1} \nrightarrow e$. (See also [HJ, p.354] for a further example.)

We now apply the last theorem and earlier results to an example of our greatest interest.

Example. Let X be a normed group with right-invariant metric d^X. Give the group $G = \mathcal{H}(X)$ the usual group-norm

$$\|f\|_{\mathcal{H}} := \sup_x d^X(f(x), x).$$

Finally, for $f, g \in G$ recall that the g-conjugate norm and the conjugacy refinement norm are

$$\|f\|_g := \|gfg^{-1}\|_{\mathcal{H}}, \text{ and } \|f\|_\infty := \sup\{\|f\|_g : g \in G\}.$$

Thus

$$\|f\|_\infty = \sup_x \sup_g d_g^X(f(x), x).$$

Theorem 3.19 (Abelian normability of $\mathcal{H}(X)$ – cf. [BePe, Ch. IV Th 1.1]). *For X a normed group, assume that $\|f\|_\infty$ is finite for each f in $\mathcal{H}(X)$– for instance if d^X is bounded, and in particular if X is compact.*
Then:
(i) $\mathcal{H}(X)$ under the abelian norm $\|f\|_\infty$ is a topological group.
(ii) The norm $\|f\|_\infty$ is equivalent to $\|f\|_\mathcal{H}$ iff the admissibility condition (n-adm) holds, which here reads: for $\|f_n\|_\mathcal{H} \to 0$ and any g_n in $\mathcal{H}(X)$,

$$\|g_n f_n g_n^{-1}\|_\mathcal{H} \to 0.$$

Equivalently, for $\|z_n\|_\mathcal{H} \to 0$ (i.e. z_n converging to the identity), any g_n in $\mathcal{H}(X)$, and any $y_n \in X$,

$$\|g_n(z_n(y_n))g_n(y_n)^{-1}\|_X \to 0.$$

(iii) In particular, if X is compact, $\mathcal{H}(X) = \mathcal{H}_u(X)$ is under $\|f\|_\mathcal{H}$ a topological group.

Proof. (i) and the first part of (ii) follow from Th. 3.17 (cf. Remarks 1 on the Klee property, after Cor. 3.6); as to (iii), this follows from Th. 3.14 and 3.9. Turning to the second part of (ii), suppose first that

$$\|g_n z_n g_n^{-1}\|_\mathcal{H} \to 0,$$

and let y_n be given. For any $\varepsilon > 0$ there is N such that, for $n \geq N$,

$$\varepsilon > \|g_n z_n g_n^{-1}\|_\mathcal{H} = \sup_x d(g_n z_n g_n^{-1}(x), x).$$

Taking x here as $x_n = g_n(y_n)$, we obtain

$$\varepsilon > d(g_n(z_n(y_n)), g_n(y_n)) = d(g_n z_n(y_n) g_n(y_n)^{-1}, e_X), \text{ for } n \geq N.$$

Hence $\|g_n(z_n(y_n))g_n(y_n)^{-1}\|_X \to 0$, as asserted.
For the converse direction, suppose next that

$$\|g_n z_n g_n^{-1}\|_\mathcal{H} \nrightarrow 0.$$

Then without loss of generality there is $\varepsilon > 0$ such that for all n

$$\|g_n z_n g_n^{-1}\|_\mathcal{H} = \sup_x d(g_n z_n g_n^{-1}(x), x) > \varepsilon.$$

Hence, for each n, there exists x_n such that

$$d(g_n z_n g_n^{-1}(x_n), x_n) > \varepsilon.$$

Equivalently, setting $y_n = g_n^{-1}(x_n)$ we obtain

$$d(g_n(z_n(y_n))g_n(y_n)^{-1}, e_X) = d(g_n(z_n(y_n)), g_n(y_n)) > \varepsilon.$$

Thus for this sequence y_n we have

$$\|g_n(z_n(y_n))g_n(y_n)^{-1}\|_X \nrightarrow 0.$$

■

Remark. To see the need for the refinement norm in verifying continuity of composition in $\mathcal{H}(X)$, we work with metrics and recall the permutation metric $d_g^X(x,y) := d^X(g(x),g(y))$. Recall also that the metric defined by the norm $\|f\|_g$ is the supremum metric $\hat{d}_g$ on $\mathcal{H}(X)$ arising from d_g on X. Indeed

$$\begin{aligned} d_g(h',h) = \|h'h^{-1}\|_g &= \sup\nolimits_z d^X(gh'h^{-1}g^{-1}(z),z) = \sup\nolimits_x d^X(g(h'(x)),g(h(x))) \\ &= \sup\nolimits_x d_g^X(h'(x)),h(x)). \end{aligned}$$

Since, as in Proposition 2.13,

$$\hat{d}_g(F_1G_1,FG) \le \hat{d}_g(F_1,F) + \hat{d}_{gF}(G_1,G) \le \hat{d}_\infty(F_1,F) + \hat{d}_\infty(G_1,G),$$

we may conclude that

$$\hat{d}_\infty(F_1G_1,FG) \le \hat{d}_\infty(F_1,F) + \hat{d}_\infty(G_1,G).$$

This reconfirms that composition is continuous. When $g = e$, the term $\hat{d}_F$ arises above and places conditions on how 'uniformly' close G_1 needs to be to G (as in Th. 3.13). For these reasons we find ourselves mostly concerned with $\mathcal{H}_u(X)$.

3.2. Lipschitz-normed groups. Below we weaken the Klee property, characterized by the condition $\|gxg^{-1}\| \le \|x\|$, by considering instead the existence of a real-valued function $g \to M_g$ such that

$$\|gxg^{-1}\| \le M_g\|x\|, \text{ for all } x.$$

This will be of use in the development of duality in Section 12 and partly in the consideration of the oscillation of a normed group in Section 3.3.

Remark. Under these circumstances, on writing xy^{-1} for x and with d^X the right-invariant metric defined by the norm, one has

$$d^X(gxg^{-1},gyg^{-1}) = d^X(gx,gy) \le M_g d^X(x,y),$$

so that the inner-automorphism γ_g is uniformly continuous (and a homeomorphism). Moreover, M_g is related to the *Lipschitz-1 norms* $\|g\|_1$ and $\|\gamma_g\|_1$, where

$$\|g\|_1 := \sup\nolimits_{x\ne y} \frac{d^X(gx,gy)}{d^X(x,y)}, \text{ and } \|\gamma_g\|_1 := \sup\nolimits_{x\ne y} \frac{d^X(gxg^{-1},gyg^{-1})}{d^X(x,y)},$$

cf. [Ru, Ch. I, Exercise 22]. This motivates the following terminology.

Definitions. 1. Say that an automorphism $f : G \to G$ of a normed group has the *Lipschitz property* if there is $M > 0$ such that

$$\|f(x)\| \le M\|x\|, \text{ for all } x \in G. \tag{Lip}$$

2. Say that a group-norm has the *Lipschitz property* , or that the group is *Lipschitz-normed*, if each *continuous* automorphism has the Lipschitz property under the group-norm.

Definitions. 1. Recall from the definitions of Section 2 that a group G is *infinitely divisible* if for each $x \in G$ and $n \in \mathbb{N}$ there is some $\xi \in G$ with $x = \xi^n$. We may write $\xi = x^{1/n}$ (without implying uniqueness).

2. Further recall that a group-norm is $\mathbb{N}$-*homogeneous* if it is n-*homogeneous* for each $n \in \mathbb{N}$, i.e. for each $n \in \mathbb{N}$, $\|x^n\| = n\|x\|$for each x. Thus if $\xi^n = x$, then $\|\xi\| = \frac{1}{n}\|x\|$ and, as $\xi^m = x^{m/n}$, we have $\frac{m}{n}\|x\| = \|x^{m/n}\|$, i.e. for rational $q > 0$ we have $q\|x\| = \|x^q\|$.

Theorem 3.20 below relates the Lischitz property of a norm to local behaviour. One should expect local behaviour to be critical, as asymptotic properties are trivial, since by the triangle inequality

$$\lim_{\|x\| \to \infty} \frac{\|x\|_g}{\|x\|} = 1.$$

As this asserts that $\|x\|_g$ is slowly varying (see Section 2) and $\|x\|_g$ is continuous, the Uniform Convergence Theorem (UCT) applies (see [BOst-TRI]; for the case $G = \mathbb{R}$ see [BGT]), and so this limit is uniform on compact subsets of G. Theorem 3.21 identifies circumstances when a group-norm on G has the Lipschitz property and Theorem 3.22 considers the Lipschitz property of the supremum norm in $\mathcal{H}_u(X)$.

On a number of occasions, the study of group-norm behaviour is aided by the presence of the following property. Its definition is motivated by the notion of an 'invariant connected metric' as defined in [Var, Ch. III.4] (see also [NSW]). The property expresses scale-comparability between word-length and distance, in keeping with the key notion of quasi-isometry.

Definition (Word-net). Say that a normed group G has a group-norm $\|.\|$ with a *vanishingly small word-net* (which may be also *compactly generated,* as appropriate) if, for any $\varepsilon > 0$, there is $\eta > 0$ such that, for all δ with $0 < \delta < \eta$ there is a set (a compact set) of generators Z_δ in $B_\delta(e)$ and a constant M_δ such that, for all x with $\|x\| > M_\delta$, there is some word $w(x) = z_1...z_{n(x)}$ using generators in Z_δ with $\|z_i\| = \delta(1 + \varepsilon_i)$, with $|\varepsilon_i| < \varepsilon$, where

$$d(x, w(x)) < \delta$$

and

$$1 - \varepsilon \leq \frac{n(x)\delta}{\|x\|} \leq 1 + \varepsilon.$$

Say that the word-net is *global* if $M_\delta = 0$.

Remarks. 1. $\mathbb{R}^d$ has a vanishingly small compactly generated global word-net and hence so does the sequence space l_2.
2. An infinitely divisible group X with an $\mathbb{N}$-homogenous norm has a vanishingly small global word-net. Indeed, given $\delta > 0$ and $x \in X$ take $n(x) = \|x\|/\delta$, then if $\xi^n = x$ we have $\|x\| = n\|\xi\|$, and so $\|\xi\| = \delta$ and $n(x)\delta/\|x\| = 1$.

Theorem 3.20. *Let G be a locally compact topological group with a norm having a compactly generated, vanishingly small global word-net. For f a continuous automorphism (e.g. $f(x) = gxg^{-1}$), suppose*

$$\beta := \limsup_{\|x\|\to 0+} \frac{\|f(x)\|}{\|x\|} < \infty.$$

Then

$$M = \sup_x \frac{\|f(x)\|}{\|x\|} < \infty.$$

We defer the proof to Section 4 as it relies on the development there of the theory of subadditive functions.

Theorem 3.21. *If G is an infinitely divisible group with an $\mathbb{N}$-homogeneous norm, then its norm has the Lipschitz property, i.e. if $f : G \to G$ is a continuous automorphism, then for some $M > 0$*

$$\|f(x)\| \le M\|x\|.$$

Proof. Suppose that $\delta > 0$. Fix $x \neq e$. Define

$$p_\delta(x) := \sup\{q \in \mathbb{Q}^+ : \|x^q\| < \delta\} = \delta/\|x\|.$$

Let f be a continuous automorphism. As $f(e) = e$, there is $\delta > 0$ such that, for $\|z\| \le \delta$,

$$\|f(z)\| < 1.$$

If $\|x^q\| < \delta$, then

$$\|f(x^q)\| < 1.$$

Thus for each $q < p_\delta(x)$ we have

$$\|f(x)\| < 1/q.$$

Taking limits, we obtain, with $M = 1/\delta$,

$$\|f(x)\| \le 1/p_\delta(x) = M\|x\|.$$

■

Definitions. 1. Let G be a Lipschitz-normed topological group. We may now take $f(x) = \gamma_g(x) := gxg^{-1}$, since this homomorphism is continuous. The *Lipschitz norm* is defined by

$$M_g := \sup_{x\neq e} \|\gamma_g(x)\|/\|x\| = \sup_{x\neq e} \|x\|_g/\|x\|.$$

(As noted before the introduction of the Lipschitz property this is the Lipschitz-1 norm.) Thus

$$\|x\|_g := \|gxg^{-1}\| \le M_g\|x\|.$$

2. For X a normed group with right-invariant metric d^X and $g \in \mathcal{H}_u(X)$ denote the following (inverse) *modulus of continuity* by

$$\delta(g) = \delta_1(g) := \sup\{\delta > 0 : d^X(g(z), g(z')) \le 1, \text{ for all } d^X(z, z') \le \delta\}.$$

THEOREM 3.22 (Lipschitz property in $\mathcal{H}_u$). *Let X be a normed group with a right-invariant metric d^X having a vanishingly small global word-net. Then*

$$\|h\|_g \leq \frac{2}{\delta(g)}\|h\|, \text{ for } g,h \in \mathcal{H}_u(X),$$

and so $\mathcal{H}_u(X)$ has the Lipschitz property.

Proof. We have for $d(z,z') < \delta(g)$ that

$$d(g(z),g(z')) < 1.$$

For given x put $y = h(x)x^{-1}$. In the definition of the word-net take $\varepsilon < 1$. Now suppose that $w(y) = w_1...w_{n(y)}$ with $\|z_i\| = \frac{1}{2}\delta(1+\varepsilon_i)$ and $|\varepsilon_i| < \varepsilon$, where $n(y) = n(y,\delta)$ satisfies

$$1-\varepsilon \leq \frac{n(y)\delta(g)}{\|y\|} \leq 1+\varepsilon.$$

Put $y_0 = e$,

$$y_{i+1} = w_i y_i$$

for $0 < i < n(y)$, and $y_{n(x)+1} = y$; the latter is within δ of y. Now

$$d(y_i, y_{i+1}) = d(e, w_i) = \|w_i\| < \delta.$$

Finally put $z_i = y_i x$, so that $z_0 = x$ and $z_{n(y)+1} = h(x)$. As

$$d(z_i, z_{i+1}) = d(y_i x, y_{i+1}x) = d(y_i, y_{i+1}) < \delta,$$

we have

$$d(g(z_i), g(z_{i+1})) \leq 1.$$

Hence

$$\begin{aligned} d(g(x), g(h(x))) &\leq n(y) + 1 < 2\|y\|/\delta(g) \\ &= \frac{2}{\delta(g)} d(h(x), x). \end{aligned}$$

Thus

$$\|h\|_g = \sup_x d(g(x), g(h(x))) \leq \frac{2}{\delta(g)} \sup_x d(h(x), x) = \frac{2}{\delta(g)}\|h\|.$$

■

LEMMA 3.23 (Bi-Lipschitz property). *In a Lipschitz-normed group $M_e = 1$ and $M_g \geq 1$, for each g; moreover $M_{gh} \leq M_g M_h$ and for any g and all x in G,*

$$\frac{1}{M_{g^{-1}}}\|x\| \leq \|x\|_g \leq M_g\|x\|.$$

Thus in particular $\|x\|_g$ is an equivalent norm.

Proof. Evidently $M_e = 1$. For $g \neq e$, as $\gamma_g(g) = g$, we see that

$$\|g\| = \|g\|_g \leq M_g\|g\|,$$

and so $M_g \geq 1$, as $\|g\| > 0$. Now for any g and all x,

$$\|g^{-1}xg\| \leq M_{g^{-1}}\|x\|.$$

So with gxg^{-1} in place of x, we obtain

$$\|x\| \leq M_{g^{-1}}\|gxg^{-1}\|, \text{ or } \frac{1}{M_{g^{-1}}}\|x\| \leq \|x\|_g.$$

■

Definition. In a Lipschitz-normed group, put $|\gamma_g| := \log M_g$ and define the *symmetrization pseudo-norm* $\|\gamma_g\| := \max\{|\gamma_g|, |\gamma_g^{-1}|\}$ (cf. Prop. 2.2). Furthermore, put

$$Z_\gamma(G) := \{g \in G : \|\gamma_g\| = 0\}.$$

Since $M_g \geq 1$ and $M_{gh} \leq M_g M_h$ the symmetrization in general yields, as we now show, a pseudo-norm (unless $Z_\gamma = \{e\}$) on the inner-automorphism subgroup

$$\mathcal{I}nn := \{\gamma_g : g \in G\} \subset Auth(G).$$

Evidently, one may adjust this deficiency, e.g. by considering $\max\{\|\gamma_g\|, \|g\|\}$, as $\gamma_g(g) = g$(cf. [Ru, Ch. I Ex. 22]).

Theorem 3.24. *Let G be a Lipschitz-normed topological group. The set Z_γ is the subgroup of elements g characterized by*

$$M_g = M_{g^{-1}} = 1,$$

equivalently by the 'norm-central' property

$$\|gx\| = \|xg\| \textit{ for all } x \in G,$$

and so $Z_\gamma(G) \subseteq Z(G)$, the centre of G.

Proof. The condition $\max\{|\gamma_g|, |\gamma_g^{-1}|\} = 0$ is equivalent to $M_g = M_{g^{-1}} = 1$. Thus Z_γ is closed under inversion; the inequality $1 \leq M_{gh} \leq M_g M_h = 1$ shows that Z_γ is closed under multiplication. For $g \in Z_\gamma$, as $M_g = 1$, we have $\|gxg^{-1}\| \leq \|x\|$ for all x, which on substitution of xg for x is equivalent to

$$\|gx\| \leq \|xg\|.$$

Likewise $M_{g^{-1}} = 1$ yields the reverse inequality:

$$\|xg\| \leq \|g^{-1}x^{-1}\| \leq \|x^{-1}g^{-1}\| = \|gx\|.$$

Conversely, if $\|gx\| = \|xg\|$ for all x, then replacing x either by xg^{-1} or $g^{-1}x$ yields both $\|gxg^{-1}\| = \|x\|$ and $\|g^{-1}xg\| = \|x\|$ for all x, so that $M_g = M_{g^{-1}} = 1$. ■

Corollary 3.25. *$M_g = 1$ for all $g \in G$ iff the group-norm is abelian iff $\|ab\| \leq \|ba\|$ for all $a, b \in G$.*

Proof. $Z_\gamma = G$ (cf. Th. 2.18). ∎

The condition $M_g \equiv 1$ is not necessary for the existence of an equivalent bi-invariant norm, as we see below. The next result is similar to Th. 3.17 (where the Lipschitz property is absent).

THEOREM 3.26. *Let G be a Lipschitz-normed topological group. If $\{M_g : g \in G\}$ is bounded, then $\|x\|_\infty$ is an equivalent abelian (hence bi-invariant) norm.*

Proof. Let M be a bound for the set $\{M_g : g \in G\}$. Thus we have

$$\|x\|_\infty \le M\|x\|,$$

and so $\|x\|_\infty$ is again a norm. As we have

$$\|x\| = \|x\|_e \le \|x\|_\infty \le M\|x\|,$$

we see that $\|z_n\| \to 0$ iff $\|z_n\|_\infty \to 0$. ∎

THEOREM 3.27. *Let G be a compact, Lipschitz-normed, topological group. Then $\{M_g : g \in G\}$ is bounded, hence $\|x\|_\infty$ is an equivalent abelian (hence bi-invariant) norm.*

Proof. The mapping $|\gamma_\cdot| := g \to \log M_g$ is subadditive. For G a compact metric group, $|\gamma_\cdot|$ is Baire, since, by continuity of conjugacy,

$$\{g : a < M_g < b\} = proj_1\{(g,x) \in G^2 : \|gxg^{-1}\| > a\|x\|\} \cap \{g : \|gxg^{-1}\| < b\|x\|\},$$

and so is analytic, hence by Nikodym's Theorem (see [Jay-Rog, p. 42]) has the Baire property. As G is Baire, the subadditive mapping $|\gamma_\cdot|$ is locally bounded (the proof of Prop. 1 in [BOst-GenSub] is applicable here; cf. Th. 4.4), and so by the compactness of G, is bounded; hence Theorem 3.20 applies. ∎

DEFINITION. Let G be a Lipschitz-normed topological group. Put

$$\begin{aligned}
\mathcal{M}(g) &:= \{m : \|x\|_g \le m\|x\| \text{ for all } x \in G\}, \text{ and then}\\
M_g &:= \inf\{m : m \in \mathcal{M}(g)\},\\
\mu(g) &:= \{m > 0 : m\|x\| \le \|x\|_g \text{ for all } x \in G\}, \text{ and then}\\
m_g &:= \sup\{m : m \in \mu(g)\}.
\end{aligned}$$

PROPOSITION 3.28. *Let G be a Lipschitz-normed topological group. Then*

$$m_g^{-1} = M_{g^{-1}}.$$

Proof. For $0 < m < m_g$ we have for all x that

$$\|x\| \le \frac{1}{m}\|gxg^{-1}\|.$$

Setting $x = g^{-1}zg$ we obtain, as in Lemma 3.23,

$$\|g^{-1}zg\| \leq \frac{1}{m}\|z\|,$$

so $M_{g^{-1}} \leq 1/m$. ■

Definitions. (Cf. [Kur-1, Ch. I §18] and [Kur-2, Ch. IV§43] ; [Berg] Ch. 6 – where compact values are assumed – [Bor, Ch. 11] , [Ful]; the first unification of these ideas is attributed to Fort [For].)
1. The correspondence $g \to \mathcal{M}(g)$ *has closed graph* means that if $g_n \to g$ and $m_n \to m$ with $m_n \in \mathcal{M}(g_n)$, then $m \in \mathcal{M}(g)$.
2. The correspondence is *upper semicontinuous* means that for any open U with $\mathcal{M}(g) \subset U$ there is a neighbourhood V of g such that $\mathcal{M}(g') \subset U$ for $g' \in V$.
3. The correspondence is *lower semicontinuous* means that for any open U with $\mathcal{M}(g) \cap U \neq \emptyset$ there is a neighbourhood V of g such that $\mathcal{M}(g') \cap U \neq \emptyset$ for $g' \in V$.

Theorem 3.29. *Let G be a Lipschitz-normed topological group. The mapping $g \to \mathcal{M}(g)$ has closed graph and is upper semicontinuous.*

Proof. For the closed graph property: suppose $g_n \to g$ and $m_n \to m$ with $m_n \in \mathcal{M}(g_n)$. Fix $x \in G$. We have

$$\|g_n x g_n^{-1}\| \leq m_n\|x\|,$$

so passing to the limit

$$\|gxg^{-1}\| \leq m\|x\|.$$

As x was arbitrary, this shows that $m \in \mathcal{M}(g)$.
For the upper semicontinuity property: suppose otherwise. Then for some g and some open U with $\mathcal{M}(g) \subset U$ the property fails. We may thus suppose that $\mathcal{M}(g) \subset (m', \infty) \subset U$ for some $m' < M_g$ and that there are $g_n \to g$ and $m_n < m'$ with $m_n \in \mathcal{M}(g_n)$. Thus, for any n and all x,

$$\|g_n x g_n^{-1}\| \leq m_n\|x\|.$$

As $1 \leq m_n \leq m'$, we may pass to a convergent subsequence $m_n \to m$, so that we have in the limit that

$$\|gxg^{-1}\| \leq m\|x\|.$$

for arbitrary fixed x. Thus $m \in \mathcal{M}(g)$ and yet $m \leq m' < M_g$, a contradiction. ■

Definition. Say that the group-norm is *nearly abelian* if for arbitrary $g_n \to e$ and $z_n \to e$

$$\lim\nolimits_n \|g_n z_n g_n^{-1}\|/\|z_n\| = 1,$$

or equivalently

$$\lim\nolimits_n \|g_n z_n\|/\|z_n g_n\| = 1. \tag{ne}$$

THEOREM 3.30. *Let G be a Lipschitz-normed topological group. The following are equivalent:*
(i) the mapping $g \to M_g$ is continuous,
(ii) the mapping $g \to M_g$ is continuous at e,
(iii) the norm is nearly abelian, i.e. (ne) holds.
In particular, if in addition G is compact and condition (ne) holds, then $\{M_g : g \in G\}$ is bounded, and so again Theorem 3.24 applies, confirming that $\|x\|_\infty$ is an equivalent abelian (hence bi-invariant) norm.

Proof. Clearly (i)$\Longrightarrow$ (ii). To prove (ii)$\Longrightarrow$ (i), given continuity at e, we prove continuity at h as follows. Write $g = hk$; then $h = gk^{-1}$ and $g \to h$ iff $k \to e$ iff $k^{-1} \to e$. Now by Lemma 3.23,

$$M_h = M_{gk^{-1}} \le M_g M_{k^{-1}},$$

so since $M_{k^{-1}} \to M_e = 1$, we have

$$M_h \le \lim\nolimits_{g\to h} M_g.$$

Since $M_k \to M_e = 1$ and

$$M_g = M_{hk} \le M_h M_k,$$

we also have

$$\lim\nolimits_{g\to h} M_g \le M_h.$$

Next we show that (ii)$\Longrightarrow$(iii). By Lemma 3.23, we have

$$1/M_{g_n^{-1}} \le \|g_n z_n g_n^{-1}\|/\|z_n\| \le M_{g_n}.$$

By assumption, $M_{g_n} \to M_e = 1$ and $M_{g_n^{-1}} \to M_e = 1$, so

$$\lim\nolimits_n \|g_n^{-1} z_n g_n\|/\|z_n\| = 1.$$

Finally we show that (iii)$\Longrightarrow$(ii). Suppose that the mapping is not continuous at e. As $M_e = 1$ and $M_g \ge 1$, for some $\varepsilon > 0$ there is $g_n \to e$ such that $M_{g_n} > 1+\varepsilon$. Hence there are $x_n \ne e$ with

$$(1+\varepsilon)\|x_n\| \le \|g_n x_n g_n^{-1}\|.$$

Suppose that $\|x_n\|$ is unbounded. We may suppose that $\|x_n\| \to \infty$. Hence

$$(1+\varepsilon) \le \frac{\|g_n x_n g_n^{-1}\|}{\|x_n\|} \le \frac{\|g_n\| + \|x_n\| + \|g_n^{-1}\|}{\|x_n\|},$$

and so as $\|g_n\| \to 0$ and $\|x_n\| \to \infty$ we have

$$(1+\varepsilon) \le \lim\nolimits_{n\to\infty} \left(\frac{\|g_n\| + \|x_n\| + \|g_n\|}{\|x_n\|}\right) = \lim\nolimits_{n\to\infty} \left(1 + \frac{2}{\|x_n\|} \cdot \|g_n\|\right) = 1,$$

again a contradiction. We may thus now suppose that $\|x_n\|$ is bounded and so without loss of generality convergent, to $\xi \ge 0$ say. If $\xi > 0$, we again deduce the contradition that

$$(1+\varepsilon) \le \lim\nolimits_{n\to\infty} \frac{\|g_n\| + \|x_n\| + \|g_n^{-1}\|}{\|x_n\|} = \frac{0+\xi+0}{\xi} = 1.$$

Thus $\xi = 0$, and hence $x_n \to e$. So our assumption of (iii) yields

$$(1+\varepsilon) \leq \lim_{n\to\infty} \frac{\|g_n x_n g_n^{-1}\|}{\|x_n\|} = 1,$$

a final contradiction. ∎

We note the following variant on Theorem 3.30.

Theorem 3.31. *Let G be a Lipschitz-normed topological group. The following are equivalent:*

(i) *the mapping $g \to \mathcal{M}(g)$ is continuous,*
(ii) *the mapping $g \to \mathcal{M}(g)$ is continuous at e,*
(iii) *the norm is nearly abelian, i.e. for arbitrary $g_n \to e$ and $z_n \to e$*

$$\lim_n \|g_n z_n g_n^{-1}\|/\|z_n\| = 1.$$

Proof. Clearly (i)$\Longrightarrow$ (ii). To prove (ii)$\Longrightarrow$ (iii), suppose the mapping is continuous at e, then by the continuity of the maximization operation (cf. [Bor, Ch.12]) $g \to M_g$ is continuous at e, and Theorem 3.30 applies.
To prove (iii)$\Longrightarrow$ (ii), assume the condition; it now suffices by Theorem 3.30 to prove lower semicontinuity (lsc) at $g = e$. So suppose that, for some open U, $U \cap \mathcal{M}(e) \neq \varnothing$. Thus $U \cap (1,\infty) \neq \varnothing$. Choose $m' < m''$ with $1 < m$ such that $(m', m'') \subset U \cap \mathcal{M}(e)$. If $\mathcal{M}$ is not lsc at e, then there is $g_n \to e$ such

$$(m', m'') \cap \mathcal{M}(g_n) = \varnothing.$$

Take, e.g., $m := \frac{1}{2}(m' + m'')$. As $m' < m < m''$, there is $x_n \neq e$ such that

$$m\|x_n\| < \|g_n x_n g_n^{-1}\|.$$

As before, if $\|x_n\|$ is unbounded we may assume $\|x_n\| \to \infty$, and so obtain the contradiction

$$1 < m \leq \lim_{n\to\infty} \frac{\|g_n\| + \|x_n\| + \|g_n^{-1}\|}{\|x_n\|} = 1.$$

Now assume $\|x_n\| \to \xi \geq 0$. If $\xi > 0$ we have the contradiction

$$m \leq \lim_{n\to\infty} \frac{\|g_n\| + \|x_n\| + \|g_n^{-1}\|}{\|x_n\|} = \frac{0 + \xi + 0}{\xi} = 1.$$

Thus $\xi = 0$. So we obtain $x_n \to 0$, and now deduce that

$$1 < m \leq \lim_{n\to\infty} \frac{\|g_n x_n g_n^{-1}\|}{\|x_n\|} = 1,$$

again a contradiction. ∎

Remark. On the matter of continuity a theorem of Mueller ([Mue, Th. 3] , see Th. 4.6 below) asserts that in a locally compact group a subadditive p satisfying

$$\liminf_{x\to e}(\limsup_{y\to x} p(y)) \leq 0$$

is continuous almost everywhere.

3.3. Cauchy Dichotomy. In this section we demonstrate the impact on the structure of normed groups of the classic *Cauchy dichotomy* of homomorphisms; conjugacy is a homomorphism so, in an appropriate setting, it is either continuous or highly discontinuous and so pathological, as mentioned in the Introduction. Thus, since conjugacy is at the heart of normed groups, normed groups are in turn either topological or pathological (see e.g. Theorems 3.39 - 3.41 below, inspired by automatic continuity). The key here is Darboux's classical result on automatic continuity [Dar], that an additive function on the reals is continuous if it is locally bounded, and its later weakening via 'regularity' to the Baire property (for which see below) or Haar-measurability in a locally compact context. Our aim here is to develop connections between continuity of automorphisms and three areas: completeness, the Baire property and 'boundedness' of automorphisms. In respect of completeness, the findings (see e.g. Th. 3.37) are in keeping with tradition as exemplified by [Kel, Problem 6.Q]; the Baire property yields normed groups as 'automatically topological' (see the earlier cited theorems); boundedness is far more illuminating – in particular we see that if the KBD holds in the right norm topology with a left-shift in the standard form (i.e., as in Th.1.1, with tz_m in lieu of $t+z_m$), as opposed to the special form of Th.1.2 (yet to be established in Section 5), then the normed group is topological.

In view of the importance of the Baire property to subsequent arguments, we recall that a set is *meagre* if it is a countable union of nowhere dense sets, a set is *Baire* if it is open modulo a meagre set, or equivalently if it is closed modulo a meagre set (cf. Engelking [Eng] especially p.198 Section 4.9 and Exercises 3.9.J, although we prefer 'meagre' to 'of first category'). For examples, see Section 11.

Definition. Noting that $d_L(x_n, x_m) = d_R(x_n^{-1}, x_m^{-1})$, call a sequence $\{x_n\}$ *bi-Cauchy* (or *two-sided Cauchy*) if $\{x_n\}$ is both d_R- and d_L-Cauchy, i.e. both $\{x_n\}$ and $\{x_n^{-1}\}$ are d_R-Cauchy sequences. Thus a sequence is bi-Cauchy iff it is d_S-Cauchy, where $d_S = \max\{d_R, d_L\}$ is the *symmetrization* metric. Recall that d_S induces a norm, the originating norm, as $\|x\|_S := d_S(x, e) = \|x\|$, but does not in general induce the same topology as d_R.

Discontinuity of automorphisms may be approached through *bi-Cauchy* sequences. Indeed, if a normed group is not topological, then by Th. 3.4 there are a null sequence z_n, a point t, and $\varepsilon > 0$ such that

$$\varepsilon \leq \lim_n \|tz_nt^{-1}\|;$$

then $d_R(tz_n, t) \geq \varepsilon$, and tz_n is prevented from converging to t. Thus one asks whether $\{tz_n\}$ has a convergent subsequence $\{tz_m\}_{m\in\mathbb{M}}$ (such a sequence would be bi-Cauchy, as $\{z_n^{-1}t^{-1}\}$ is Cauchy, since for w_n null $w_nx \to_R x$, for any x).

This approach suggests another: if $y = \lim_{\mathbb{M}} tz_m$, the distance $d_R(y,t)$ measures the discontinuity of λ_t in the corresponding direction $\{z_m\}_{\mathbb{M}}$, since

$$\|yt^{-1}\| = d_R(y,t) = \lim_{\mathbb{M}} d_R(tz_m, t) = \lim_{\mathbb{M}} \|\gamma_t(z_m)\|,$$

leading to a study of the properties of the *oscillation* (at e_X) of γ_t as t varies over the group, which we address later in this subsection. (Note that here $\|y\| = \|t\|$, so the maximum dispersion by a left-shift of the null sequence, away from where the right-shift takes it, is $\|yt^{-1}\| \leq 2\|t\|$; it is this that the oscillation measures.) We shall see later that if a normed group is not topological than the oscillation is bounded away from zero on a non-empty open set, suggesting a considerable amount (in topological terms) of pathology.

Returning to the bi-Cauchy approach, in order to draw our work closer to the separate continuity literature (esp. Bouziad [Bou2]), we restate it in terms of the following notion of continuity due to Fuller [Ful] (in his study of the preservation of compactness), adapted here from nets to sequences, because of our metric context. (Here one is reminded of compact operators – cf. [Ru, 4.16])

Definition. A function f between metric spaces is said to be *subcontinuous* at x if for each sequence x_n with limit x, the sequence $f(x_n)$ has a convergent subsequence.

Thus for $f(x) = \lambda_t(x) = tx$, with t fixed, λ_t is subcontinuous under d_R at e iff for each null z_n there exists a convergent subsequence $\{tz_n\}_{n\in\mathbb{M}}$. We note that λ_t is subcontinuous under d_R at e iff it is subcontinuous at some/all points x (since $tz_m x \to_R yx$ iff $tz_m \to_R y$ down the same subsequence $\mathbb{M}$, and $x_n \to_R x$ iff $z_n := x_n x^{-1} \to e$ so that $z_n x \to_R x$.)

One criterion for subcontinuity is provided by a form of the Heine-Borel Theorem, which motivates a later definition.

Proposition 3.32 (cf. [Ost-Mn, Prop. 2.8]). *Suppose that $Y = \{y_n : n = 1, 2, ..\}$ is an infinite subset of a normed group X. Then Y contains a subsequence $\{y_{n(k)}\}$ which is either d_R-Cauchy or is uniformly separated (i.e. for some m satisfies $d_R(y_{n(k)}, y_{n(h)}) \geq 1/m$, for all h, k). In particular, if X is locally compact, z_n is null, and the ball $B_{\|x\|}(e)$ is precompact, then $y_n = xz_n$ contains a d_R-Cauchy sequence.*

Proof. We may assume without loss of generality that y_n is injective and so identify Y with $\mathbb{N}$. Define a colouring M on $\mathbb{N}$ by setting $M(h,k) = m$ iff m is the smallest integer such that $d_R(y_h, y_k) \geq 1/m$. If an infinite subset I of $\mathbb{N}$ is monochromatic with colour n, then $\{y_i : i \in I\}$ is a discrete subset in X. Now partition $\mathbb{N}^3$ by putting $\{u,v,w\}$ in the cells $C_<, C_=, C_>$ according as $M(u,v) < M(v,w)$, $M(u,v) = M(v,w)$, or $M(u,v) > M(v,w)$. By Ramsey's Theorem (see e.g. [GRS, Ch.1]), one cell contains an infinite set I^3. As $C_>$ cannot contain an infinite (descending) sequence, the infinite subset is either in $C_=$, when $\{y_i : i \in I\}$ is uniformly separated, or in $C_<$, when $\{y_i : i \in I\}$ is a d_R-Cauchy sequence.

As for the conclusion, the set $B_{\|x\|+\varepsilon}(e)$ has compact closure for some $\varepsilon > 0$. But $\|xz_n\| \leq \|x\| + \|z_n\|$, so for large enough n the points xz_n lie in the compact set $B_{\|x\|+\varepsilon}(e)$, hence contain a convergent subsequence. ■

Our focus on metric completeness is needed in part to supply background to assumptions in Section 5 (e.g. Th. 5.1). We employ definitions inspired by weakening the admissibility condition (adm). We recall from Th. 2.15 that a normed group with (adm) is a topological group, more in fact: a Klee group, as it has an equivalent abelian norm. We will see that the property of (only) being a topological group is equivalent to a weakened admissibility property; a second (less weakened) notion of admissibility – the Cauchy-admissibility property – ensures that (X, d_R) has a group completion. This motivates the use in Section 5 of the weaker property still that (X, d_R) is topologically complete, i.e. there is a complete metric d on X equivalent to d_R.

DEFINITIONS. 1. Say that the normed group satisfies the *weak-admissibility condition,* or (W-adm) for short, if for every *convergent* $\{x_n\}$ and null $\{w_n\}$

$$x_n w_n x_n^{-1} \to e, \text{ as } n \to \infty. \tag{W-adm}$$

Note that the (W-adm) condition has a reformulation as the *joint continuity* of the left commutator $[x, y]_L$, at $(x, y) = (w, e)$, when the convergent sequence $\{x_n\}$ has limit x; indeed

$$x_n w_n x_n^{-1} = x_n w_n x_n^{-1} w_n^{-1} w_n = [x_n, w_n]_L w_n.$$

Likewise, if the sequence $\{x_n^{-1}\}$ has limit x^{-1}, then one can write

$$x_n w_n x_n^{-1} = x_n w_n x_n^{-1} w_n w_n^{-1} = [x_n^{-1}, w_n^{-1}]_R w_n^{-1}.$$

2. Say that the normed group satisfies the *Cauchy-admissibility condition,* or (C-adm) for short, if for every *Cauchy* $\{x_n\}$ and null $\{w_n\}$

$$x_n w_n x_n^{-1} \to e, \text{ as } n \to \infty. \tag{C-adm}$$

In what follows, we have some flexibility as to when x_n is a Cauchy sequence. One interpretation is that x_n is d_R-Cauchy, i.e. $\|x_n x_m^{-1}\| = d_R(x_n, x_m) \to 0$. The other is that x_n is d_L-Cauchy; but then $y_m = x_n^{-1}$ is d_R-Cauchy and we have

$$x_n w_n x_n^{-1} = y_n^{-1} w_n y_n \to e.$$

The distinction is only in the positioning of the inverse; hence in arguments, as below, which do not appeal to continuity of inversion, either format will do.

LEMMA 3.33. *In a normed group the condition (C-adm) is equivalent to the following uniformity condition holding for all $\{x_n\}$ Cauchy:*
for each $\varepsilon > 0$ there is $\delta > 0$ and N such that for all $n > N$ and all $\|w\| < \delta$

$$\|x_n w x_n^{-1}\| < \varepsilon.$$

Proof. For the direct implication, suppose otherwise. Then for some Cauchy $\{x_n\}$ and some $\varepsilon > 0$ and each $\delta = 1/k$ ($k = 1, 2, ...$) there is $n = n(k) > k$ and w_k with $\|w_k\| < 1/k$ such that

$$\|x_{n(k)} w_k x_{n(k)}^{-1}\| > \varepsilon.$$

But w_k is null and $x_{n(k)}$ is Cauchy, so from (C-adm) it follows that $x_{n(k)} w_k x_{n(k)}^{-1} \to e$, a contradiction.
The converse is immediate. ∎

Definition. For an arbitrary sequence x_n, putting

$$\gamma_{x(n)}(w) := x_n w x_n^{-1},$$

we say that $\{\gamma_{x(n)}\}$ is *uniformly continuous* at e if the uniformity condition of Lemma 3.33 holds. Thus that Lemma may be interpreted as asserting that $\{\gamma_{x(n)}\}$ is *uniformly continuous* at e for all Cauchy $\{x_n\}$ iff (C-adm) holds.

Our next result strengthens Lemma 3.5 in showing that for inner automorphisms a weak form of continuity implies continuity.

Lemma 3.34 (Weak Continuity Criterion). *For any fixed sequence x_n, if for all null sequences w_n we have $\gamma_{x(n(k))}(w_{n(k)}) \to e_X$ down some subsequence $w_{n(k)}$, then $\{\gamma_{x(n)}\}$ is uniformly continuous at e.*
In particular, for a fixed x and all null sequences w_n, if $\gamma_x(w_{n(k)}) \to e_X$ down some subsequence $w_{n(k)}$, then γ_x is continuous.

Proof. We are to show that for every $\varepsilon > 0$ there is $\delta > 0$ and N such that for all $n > N$

$$x_n B(\delta) x_n^{-1} \subset B(\varepsilon).$$

Suppose not. Then there is $\varepsilon > 0$ such that for each $k = 1, 2, ..$ and each $\delta = 1/k$ there is $n = n(k) > k$ and w_k with $\|w_k\| < 1/k$ and $\|x_{n(k)} w_k x_{n(k)}^{-1}\| > \varepsilon$. So $w_k \to 0$. By assumption, down some subsequence $k(h)$ we have $\|x_{n(k(h))} w_{k(h)} x_{n(k(h))}^{-1}\| \to 0$. But this contradicts $\|x_{n(k(h))} w_{k(h)} x_{n(k(h))}^{-1}\| > \varepsilon$.
The last assertion is immediate from taking $x_n \equiv x$, as the uniform continuity condition at e reduces to continuity at e. ∎

Theorem 3.35. *In a normed group, the condition (C-adm) holds iff the product of Cauchy sequences is Cauchy.*

Proof. We work in the right norm topology and refer to d_R-Cauchy sequences.
First we assume (C-adm). Let x_n and y_n be Cauchy. For m, n large we are to show that $d_R(x_n y_n, x_m y_m) = \|x_n y_n y_m^{-1} x_m^{-1}\|$ is small. We note that

$$\begin{aligned}\|x_n w x_m^{-1}\| &= \|x_n w x_n^{-1} x_n x_m^{-1}\| \le \|x_n w x_n^{-1}\| + \|x_n x_m^{-1}\| \\ &\le \|x_n w x_n^{-1}\| + d_R(x_n, x_m).\end{aligned}$$

By (C-adm), we may apply Lemma 3.33 to deduce for $w = y_n y_m^{-1}$ and m, n large that $\|x_n w x_m^{-1}\|$ is small. Hence so also is $d_R(x_n y_n, x_m y_m)$. That is, the product of Cauchy sequences is Cauchy.

Before considering the converse, observe that if $w_n = y_n y_{n+1}^{-1}$ is given with y_n Cauchy, then w_n is null and with $m = n+1$ we have as $n \to \infty$ that

$$\|x_n w_n x_n^{-1}\| = \|x_n y_n y_m^{-1} x_m^{-1} x_m x_n^{-1}\| \leq \|(x_n y_n)(x_m y_m)^{-1}\| + \|x_m x_n^{-1}\| \to 0,$$

provided $x_n y_n$ is a Cauchy sequence. We refine this observation below.

Returning to the converse: assume that in X the product of Cauchy sequences is Cauchy. Let x_n and y_n be Cauchy.

Let w_n be an arbitrary null sequence. By Lemma 3.34 it is enough to show that down a subsequence $\gamma_{x(n(k))}(w_{n(k)}) \to e_X$. Since we seek an appropriate subsequence, we may assume (by passing to a subsequence) that without loss of generality $\|w_n\| \leq 2^{-n}$. We may now solve the equation $w_n = z_{n-1} z_n^{-1}$ for $n = 1, 2, \ldots$ with z_n null, by taking $z_0 = e$ and inductively

$$z_n = w_n^{-1} z_{n-1} = w_n^{-1} w_{n-1}^{-1} \ldots w_1^{-1}.$$

Indeed z_n is null, since

$$\|z_n\| \leq 2^{-(1+2+\ldots+n)} \to 0.$$

Now, as $n \to \infty$ we have

$$\begin{aligned} \|x_{n+1} w_{n+1} x_{n+1}^{-1}\| &= \|x_{n+1} x_n^{-1} (x_n z_n z_{n-1}^{-1} x_{n-1}^{-1}) x_{n-1} x_n^{-1}\| \\ &\leq d(x_n z_n, x_{n-1} z_{n-1}) + d(x_{n+1}, x_n) + d(x_n, x_{n-1}) \to 0, \end{aligned}$$

since x_n and $x_n z_n$ are Cauchy. By Lemma 3.34 $\{\gamma_{x(n)}\}$ is uniformly continuous at e, and so by Lemma 3.33, (C-adm) holds. ■

Remark. The proof in fact shows that it is enough to consider products with x_n Cauchy and y_n null; since the general case gives

$$d_R(x_n y_n, x_m y_m) = x_n y_n y_m^{-1} x_n^{-1} (x_n x_m^{-1})$$

and, for $y_n y_m^{-1}$ small, this is small by an appeal to (C-adm).

Lemma 3.36. *(W-adm) is satisfied iff γ_x is continuous for all x.*

Proof. We work in the right norm topology. Asume (W-adm) holds. As the constant sequence $x_n \equiv x$ is convergent, it is immediate that γ_x is continuous. For the converse, suppose $x_n \to_R x$ and put $z_n = x_n x^{-1}$ (which is null); then $x_n = z_n x \to_R x$ and

$$x_n w_n x_n^{-1} = z_n (x w_n x^{-1}) z_n^{-1} \to e,$$

by the triangle inequality (since $\|z_n (x w_n x^{-1}) z_n^{-1}\| \leq \|x w_n x^{-1}\| + 2\|z_n\|$) and so (W-adm) holds. ■

THEOREM 3.37. *For X a normed group, if products of Cauchy sequences are Cauchy then X is a topological group.*

Proof. By Th.3.35 (C-adm) holds. The latter implies the weak admissibility condition which, by Lemma 3.33 and Th. 3.4, implies that X is a topological group. ∎

DEFINITIONS. 1. Say that a normed group is *bi-Cauchy complete* if each bi-Cauchy sequence has a limit.
2. For any metric d on X, the relation $\{x_n\} \sim_d \{y_n\}$ between d-Cauchy sequences, defined by requiring $d(x_n, y_n) \to 0$ (as $n \to \infty$), is an equivalence (by the triangle inequality). In particular for $d_S = \max\{d_R, d_L\}$, it defines an equivalence $\{x_n\} \sim \{y_n\}$ between bi-Cauchy sequences. It is the intersection of the right and left equivalence relations, demanding both $d_R(x_n, y_n) \to 0$ and $d_L(x_n, y_n) \to 0$.
3. For X a normed group, working modulo $\sim$ put

$$\widetilde{X} := \{\{x_n\} : \{x_n\} \text{ is a bi-Cauchy sequence}\}.$$

Working in $\widetilde{X}$ define

$$\|\{x_n\}\| := \lim_n \|x_n\|, \text{ and } \{x_n\} \cdot \{y_n\} = \lim_n x_n y_n,$$
$$\widetilde{d}_R(\{x_n\}, \{y_n\}) := \lim_n \|x_n y_n^{-1}\| \text{ and } \widetilde{d}_L(\{x_n\}, \{y_n\}) := \lim_n \|x_n^{-1} y_n\|.$$

(Compare also the *sequence space* $\mathcal{C}(G)$ considered in Section 11.)

The following result is in a thin disguise the standard result on the completion of a topological group under its ambidextrous uniformity (as under our assumptions X is in fact a topological space), see e.g. [Kel, Problem 6Q]; here we are merely asserting additionally that the completion uniformity extends the originating norm and is normable, provided the uniformity on X is.

Of course $\widetilde{X}$ need not be $\widetilde{d}_R$-complete; indeed it will not be if there are Cauchy sequences that are not bi-Cauchy. However, $\widetilde{X}$ under $\widetilde{d}_R$ is topologically complete. Indeed we have $\widetilde{d}_S = \max\{\widetilde{d}_R, \widetilde{d}_L\}$, and by construction $(\widetilde{X}, \widetilde{d}_S)$ is complete, and being a topological group is homeomorphic to $(\widetilde{X}, \tilde{d}_R)$, by the Ambidextrous Refinement Principle (Th. 3.9). Note that $(\widetilde{X}, \widetilde{d}_R)$ as a metric space has a completion $(\hat{X}, \hat{d})$, not necessarily a group, in which of course $\widetilde{(X, d_R)}$ is embedded as a $\mathcal{G}_\delta$ set.

THEOREM 3.38 (Bi-Cauchy completion). *If the group-norm of X satisfies (C-adm), then $\widetilde{X}$ is a normed group extending X (isometrically), satisfying (C-adm) (so also a topological group), in which bi-Cauchy sequences are convergent.*

Proof. We work under the right norm topology. By Th.3.35, products of Cauchy sequences in X are Cauchy. Note that $\{x_n\} \cdot \{y_n\} \sim \{e\}$ implies that $\{y_n\} \sim \{x_n^{-1}\}$. So $X^{\sim}$ is equipped with an inversion.

We now verify that $\|\cdot\|$ is indeed a norm on $X^{\sim}$. We have

$$\|\{x_n\}\cdot\{y_n\}\| = \lim_n \|x_n y_n\| \leq \lim_n \|x_n\| + \lim_n \|y_n\|,$$

and

$$\|\{x_n\}\| = 0 \text{ iff } \lim_n \|x_n\| = 0 \text{ iff } x_n \to e,$$

so $\{x_n\} \sim \{e\}$; also

$$\|\{x_n^{-1}\}\| = \lim_n \|x_n^{-1}\| = \lim_n \|x_n\| = \|\{x_n\}\|.$$

We note that if $x_n \to_R x$, then

$$\|\{x_n\}\cdot\{x^{-1}\}\| = \lim_n \|x_n x^{-1}\| = 0,$$

so that $\{x_n\} \to_R \{x\}$ and hence the map $x \to \{x_n\}$ where $x_n \equiv x$ isometrically embeds X into $X^{\sim}$. This far $X^{\sim}$ is a normed group. Say that $\{x_n\}$ is d-regular if $d(x_n, x_m) \leq 2^{-n}$ for $m \geq n$. If $\{\{x_n^m\}_n\}_m$ is $\tilde{d}_R$-regular with each $\{x_n^m\}$ also d_R-regular, put $y_n = x_n^n$. Then $\{y_n\}$ is the limit of $\{\{x_n^m\}_n\}_m$.

Notice that if $\{w_n^m\} \to e$, then without loss of generality w_n^n is null, so we have

$$x_n^n w_n^n (x_n^n)^{-1} \to e_X,$$

and so $X^{\sim}$ also satisfies (C-adm). ■

Remarks. 1. The definition of $\widetilde{X}$ requires sequences to be *bi-Cauchy* to achieve bi-Cauchy completeness. Compare this two-sided condition to that of Prop. 3.13 which uses bi-uniformly continuous functions, and also [BePe] Prop 1.1, where in the context of $Auth(X)$ with the weak refinement topology (that defined in Th. 2.12, as opposed to that of Th. 3.19, where there is an abelian norm), the two-sided assumptions $\lim_n f_n = f \in X^X$ and $\lim_n f^{-1} = g \in X^X$ (limits in the supremum metric) yield $g = f^{-1} \in Auth(X)$. (On this last point see also Lemma 1 of [Ost-Joint].)

2. If X is complete under d_R there is no guarantee that X is closed under products of Cauchy sequences, so Th. 3.35 does not characterize (C-adm).

We now consider the impact of automatic continuity. Our first result captures the effect on automorphisms of the result, due to Darboux [Dar], that an additive function which is locally bounded is continuous.

Definition. Say that a group is *Darboux-normed* if there are constants κ_n with $\kappa_n \to \infty$ associated with the group-norm such that for all elements z of the group

$$\kappa_n \|z\| \leq \|z^n\|,$$

or equivalently

$$\|z^{1/n}\| \leq \frac{1}{\kappa_n}\|z\|.$$

Thus $z^{1/n} \to e$; a related condition was considered by McShane in [McSh] (cf. the Eberlein-McShane Theorem, Th. 10.1).

THEOREM 3.39. *A Darboux-normed group is a topological group.*

Proof. Fix x. By Theorem 3.4 we must show that γ_x is continuous at e. By Darboux's theorem (Th. 11.22), it suffices to show that γ_x is bounded in some ball $B_{1/n}(e)$. Suppose not: then there is $w_n \in B_{\varepsilon(n)}(e)$ with $\varepsilon(n) = 2^{-n}$ and

$$\|\gamma_x(w_n)\| \geq n.$$

Thus w_n is null. We may solve the equation $w_n = z_{n-1}z_n^{-1}$ for $n = 1, 2, ...$ with z_n null, by taking $z_0 = e$ and inductively

$$z_n = w_n^{-1}z_{n-1} = w_n^{-1}w_{n-1}^{-1}...w_1^{-1}.$$

Indeed z_n is null, since

$$\|z_n\| \leq 2^{-(1+2+...+n)} \to 0.$$

Applying the triangle inequality twice,

$$\begin{aligned} d(xz_n, xz_m) &\leq d(xz_n, e) + d(e, xz_m) \\ &= \|xz_n\| + \|xz_m\| \leq 2(\|x\| + 1), \end{aligned}$$

as $\|z_n\| \leq 1$. So for all n, we have

$$\|xw_{n+1}x^{-1}\| = \|x_n z_n z_{n-1}^{-1}x^{-1}\| \leq d(x_n z_n, x_{n-1}z_{n-1}) \leq 2(\|x\| + 1).$$

This contradicts the unboundedness of $\|\gamma_x(w_n)\|$. ■

Two more results on the effects of automatic continuity both come from the Banach-Mehdi Theorem on Homomorphism Continuity (Th. 11.11) or its generalization, the Souslin Graph Theorem (Th. 11.12), both of which belong properly to a later circle of ideas considered in Section 11 and employ the Baire property.

THEOREM 3.40. *For X a topologically complete, separable, normed group, if each automorphism $\gamma_g(x) = gxg^{-1}$ is Baire, then X is a topological group.*

Proof. We work under d_R. Fix g. As X is separable and γ_g Baire, γ_g is Baire-continuous (Th. 11.8) and so by the Banach-Mehdi Theorem (Th. 11.11) is continuous. As g is arbitrary, we deduce from Th. 3.4 that X is topological. ■

REMARK. Here by assumption (X, d_R) is a Polish space. In such a context, abandoning the Axiom of Choice, one may consistently assume that all functions are Baire and so that all topologically complete separable normed groups are topological. (See the models of set theory due to Solovay [So] and to Shelah [She].)

THEOREM 3.41 (On Borel/analytic inversion). *For X a topologically complete, separable normed group, if the inversion $x \to x^{-1}$ regarded as a map from (X, d_R) to (X, d_R) is a Borel function, or more generally has an analytic graph, then X is a topological group.*

Proof. To apply Th. 11.11 or Th. 11.12 we need to interpret inversion as a homomorphism between normed groups. To this end, define $X^* = (X, *, d^*)$ to be the metric group with underlying set X with multiplication $x * y := yx$ and metric $d^*(x, y) = d_R(x, y)$. Then X^* is isometric with $(X, \cdot, d_R)$ under the identity and d^* is left-invariant, since $d^*(x*y, x*z) = d_R(yx, zx) = d_R(y, z) = d^*(y, z)$. Thus X^* is separable and topologically complete. Now $f : X \to X^*$ defined by $f(x) = x^{-1}$ is a homomorphism, which is Borel/analytic (by the isometry). Hence f is continuous and so is right-to-right continuous. Now by the Equivalence Theorem (Th. 3.4), the normed group X is topological. ■

For the connection between continuity, openness and the closed graph property of homomorphisms, which we just exploited, see [Pet3] and the discussion in [Pet4, esp. VIII]. For related work see Solecki and Srivastava [SolSri], where the group is Baire, separable, metrizable with continuous right-shifts $\rho_t(s) = st$ and has Baire-measurable left-shifts $\lambda_s(t) = st$. Before leaving the issue of automatic continuity, we note that Th. 3.40 and 3.41 have analogues in locally compact, normed group having the Heine-Borel property (i.e. a set is compact iff it is closed and norm-bounded) – see [Ost-LB3]. A further automatic result that (X, d_R) is a topological group is derived in [Ost-Joint] from the hypotheses that (X, d_S) is non-meagre and (X, d_S) is Polish. (See also [Ost-AB] for the non-separable case which requires further conditions involving the notion of σ-discreteness.)

We now study the oscillation function in a normed group setting.

Definition. We put

$$\omega(t) = \lim\nolimits_{\delta \searrow 0} \omega_\delta(t), \text{ where } \omega_\delta(t) := \sup\nolimits_{\|z\| \le \delta} \|\gamma_t(z)\|,$$

and call $\omega(\cdot)$ the *oscillation function* of the group-norm. (We will see in Prop. 3.42 that these are finite quantities.) If $\omega(t) < \varepsilon$, then $\omega_\delta(t) < \varepsilon$, for some $\delta > 0$. In the light of this, we will need to refer to the related sets

$$\begin{aligned} \Omega(\varepsilon) &: = \{t : \omega(t) < \varepsilon\}, \quad \Omega_\delta(\varepsilon) := \{t : \omega_\delta(t) < \varepsilon\}, \\ \Lambda_\delta(\varepsilon) &: = \{t : d(t, tz) \le \varepsilon \text{ for all } \|z\| \le \delta\}, \end{aligned}$$

so that for $d = d_R$ we have $\Omega_\delta(\varepsilon) \subseteq \Lambda_\delta(\varepsilon)$ and

$$\Omega(\varepsilon) \subset \bigcup\nolimits_{\delta \in \mathbb{R}_+} \Lambda_\delta(\varepsilon) \subset \Omega(2\varepsilon). \tag{cover}$$

It is convenient on occasion to allow the d in $\Lambda_\delta(\varepsilon)$ to be a general metric compatible with the topology of X (not necessarily right-invariant).

Remarks. 1. Of course if $\omega(t) = 0$, then γ_t is continuous.
2. For fixed z and $\varepsilon > 0$, the sets

$$F_\varepsilon(z) = \{t : d(t, tz) \le \varepsilon\}, \text{ and } G_\varepsilon(z) = \{t : d(t, tz) < \varepsilon\},$$

are closed, respectively open, if $\rho_z(x) = xz$ is continuous under d, and so

$$\Lambda_\delta(\varepsilon) := \{t : d_R(t, tz) \le \varepsilon \text{ for all } \|z\| \le \delta\} = \bigcap\nolimits_{\|z\|\le\delta} F_\varepsilon(z)$$

is closed. Evidently $e_X \in G_\varepsilon(z)$ for $\|z\| < \varepsilon$.

Proposition 3.42 (Uniform continuity of oscillation). *For X a normed group*

$$\omega(t) - 2\|s\| \le \omega(st) \le \omega(t) + 2\|s\|, \text{ for all } s, t \in X.$$

Hence

$$0 \le \omega(s) \le 2\|s\|, \text{ for all } s \in X,$$

and the oscillation function is uniformly continuous and norm-bounded.

Proof. We prove the right-hand side of the first inequality. Fix s, t. By the triangle inequality, for all $0 < \delta < 1$ and $\|z\| \le \delta$ we have that

$$\|stzt^{-1}s^{-1}\| \le 2\|s\| + 2\|t\| + \delta \le 2\|s\| + 2\|t\| + 1,$$

which shows finiteness of $\omega_\delta(st)$ and $\omega_\delta(t)$, and likewise that

$$\|stzt^{-1}s^{-1}\| \le 2\|s\| + \|tzt^{-1}\| \le \omega_\delta(t) + 2\|s\|.$$

Hence for all $\delta > 0$

$$\omega(st) \le \omega_\delta(st) \le \omega_\delta(t) + 2\|s\|.$$

Passing to the limit, one has

$$\omega(st) \le \omega(t) + 2\|s\|.$$

From here

$$\omega(t) = \omega(s^{-1}st) \le \omega(st) + 2\|s^{-1}\|,$$

i.e.

$$\omega(t) - 2\|s\| \le \omega(st).$$

Also since $\omega(e_X) = 0$, the substitution $t = e_X$ gives $\omega(s) \le \omega(e_X) + 2\|s\|$, the final inequality.
Now, working in the right norm topology, let $\varepsilon > 0$ and put $\delta = \varepsilon/2$. Fix x and consider $y \in B_\delta(x) = B_\delta(e_X)x$. Write $y = wx$ with $\|w\| \le \delta$; then taking $s = w$ and $t = x$ we have

$$\omega(x) - 2\delta \le \omega(y) \le \omega(x) + 2\delta,$$

i.e.

$$|\omega(y) - \omega(x)| \le \varepsilon, \text{ for all } y \in B_{\varepsilon/2}(x).$$

■

Thus the oscillation as a function from X to the additive reals $\mathbb{R}$ is bounded in the sense of the application discussed after Prop. 2.15.

Our final group of results and later comments rely on density ideas and on the following definition.

Definition. A point x is said to be in the *topological centre* $Z_\Gamma(X)$ of a normed group X if γ_x is continuous (at e_X, say).

The theorem below shows that an equivalent definition could refer to x such that λ_x is continuous in (X, d_R) (cf. [HS] Def. 2.4 in the context of semigroups, where one does not have inverses); we favour a definition introducing the concept in terms of the norm, rather than one of the associated metrics.

Proposition 3.43. *The topological centre Z_Γ of a normed group X is a closed subsemigroup; it comprises the set of t such that λ_t is continuous under d_R. Furthermore, if X is separable and topologically complete the topological centre is a closed subgroup.*

Proof. Since $\gamma_{xy} = \gamma_x \circ \gamma_y$, the centre is a subsemigroup. Since $\gamma_t = \lambda_t \circ \rho_{t^{-1}}$ and $\lambda_t = \rho_t \circ \gamma_t$, we have $t \in Z_\Gamma$ iff λ_t is continuous. As for its being closed, suppose that $x_n \to_R x$ with $x_n \in Z_\Gamma$, $z_n \to e$, and $\varepsilon > 0$. It is enough to prove that λ_x is continuous at e_X (as $d_R(xt_n, xt) = d_R(xz_n, x)$ and $z_n := t_n t^{-1} \to e$ iff $t_n \to_R t$). There is M such that for $m > M$, $d_R(x, x_M) < \varepsilon/2$, and N such that $d_R(x_M z_n, e) < \varepsilon/2$, for $n > N$. So for $n > N$ we have

$$d_R(xz_n, e) \leq d_R(xz_n, x_M z_n) + d_R(x_M z_n, e) \leq d_R(x, x_M) + d_R(x_M z_n, e) < \varepsilon.$$

Thus $xz_n \to x$ for each null z_n. Thus λ_x is continuous at e_X and hence continuous.
Now suppose that X is completely metrizable and separable. For $t \in Z_\Gamma$ the homomorphism γ_t is continuous, so has a closed graph Φ. But Φ may be viewed as the graph of the inverse homomorphism $(\gamma_t)^{-1} = \gamma_{t^{-1}}$, so by the Souslin Graph Theorem (Th. 11.12) $\gamma_{t^{-1}}$ is continuous, i.e. $t^{-1} \in Z_\Gamma$. ∎

The next two results stand in contrast to the possible pathology, as summarized in Th. 3.50 below. We show in Th. 3.49 that if a normed group is topological just 'near e' (in no matter how small a neighbourhood), then it is topological globally. In fact being topological just 'somewhere' is enough (Th. 3.50). This necessitates an appeal to the Subgroup Dichotomy Theorem for Normed groups, a version of the Banach-Kuratowski Theorem which we discuss much later in Th. 6.13.

Theorem 3.44. *In a normed group X, connected and Baire under the right norm topology, if $\omega = 0$ in a neighbourhood of e_X, then X is a topological group.*

Proof. If $\omega = 0$ in a neighbourhood of e, then e is an interior point of Z_Γ, so let $V := B_\varepsilon(e) \subseteq Z_\Gamma$, for some $\varepsilon > 0$. Then $V^{-1} = V$, and so, by the semigroup property of Z_Γ (Th. 3.43), $U := \bigcup_{n\in\mathbb{N}} V^n$ is an open subgroup of Z_Γ. As U is Baire and non-meagre, by Th. 6.13 it is clopen and so is the whole of X (in view of connectedness). So $X = Z_\Gamma$ and again by Th. 3.4 X is a topological group. ∎

THEOREM 3.45. *In a topologically complete, separable, connected normed group X, if the topological centre is non-meagre, then X is a topological group.*

Proof. The centre Z_Γ is a closed, hence Baire, subgroup. If it is non-meagre, by Th. 6.13 it is clopen and hence the whole of X (by connectedness). Again by Th. 3.4, X is a topological group. ∎

REMARK. Suppose the normed group X is topologically complete and connected. Under the circumstances, by the Squared Pettis Theorem (Th. 5.8), since Z_Γ is closed and so Baire, if non-meagre it contains e_X as an interior point of $(Z_\Gamma Z_\Gamma^{-1})^2$; then Z_Γ generates the whole of X. But as Z_Γ is only a semigroup, we cannot deduce that X is a topological group.

We now focus on conditions which yield 'topological group' behaviour at least 'somewhere'. Our analysis via 'oscillation' sharpens Montgomery's result concerning 'separate implies joint continuity'.

A *semitopological* metric group X is a group with a metric that is not necessarily invariant but with right-shifts $\rho_y(x) = xy$ and left-shifts $\lambda_x(y) = xy$ continuous (so that multiplication is separately continuous). Montgomery [Mon2] proves that, in a semitopological metric group, joint continuity is implied by completeness. From our perspective, we may disaggregate his result into three steps: a simple initial observation, a category argument (Prop. 3.46), and an appeal to oscillation. For a general metric d which defines the context of the first of these, we must interpret $||z||$ as $d(z,e)$ and $\Omega(\varepsilon)$ as $\{t : (\lim_{\delta\searrow 0} \sup_{||z||\leq\delta} d(tz,t)) < \varepsilon\}$. The latter set refers to left shifts, so the language of the initial observation corresponds to left-shift continuity.

INITIAL OBSERVATION. *In a Baire, left topological (in particular a semitopological) metric group, for each non-empty open set W and $\varepsilon > 0$, the set $\Omega(\varepsilon) \cap W$ is non-meagre.*

Proof. Let $\varepsilon > 0$. On taking d in place of d_R, this follows from (cover), since for $t \in W$, λ_t is continuous at e and so there is $\delta > 0$ such that $t \in \Lambda_\delta(\varepsilon/2) \cap W \subseteq \Omega(\varepsilon) \cap W$. The latter set is thus non-empty and open, so non-meagre. ∎

The rest of his argument, using a general metric d, relies on the weaker property embodied in the Initial Observation, that each set $\Omega(\varepsilon)$ is non-meagre in any neighbourhood. So we may interpret his arguments in a normed group context to yield two interesting results. (The first may be viewed as defining a 'local metric admissability condition', compare Prop. 2.14 and the 'uniform continuity' of Lemma 3.5.) In Th. 3.46 below we are able to relax the hypothesis of Montgomery's Theorem (Th. 3.47).

Proposition 3.46 (Montgomery's Uniformity Lemma, [Mon2, Lemma]). *For a normed groupX under its right norm topology, Baire in this topology, and* $\varepsilon > 0$, *if* $\Omega(\varepsilon) \cap U$ *is non-meagre for some open set* U, *then there are* $\delta = \delta(\varepsilon) > 0$ *and an open* $V \subset U$ *such that* $V \subseteq \Lambda_\delta(\varepsilon)$, *i.e.,* $d_R(t, tz) \le \varepsilon$ *for all* $\|z\| \le \delta$ *and* $t \in V$.
In particular, if in an open set U *the oscillation is less than* ε *at points of a non-meagre set, then it is at most* ε *at all points of some non-empty open subset of* U.

Proof. As $\Omega_\delta(\varepsilon) \subseteq \Lambda_\delta(\varepsilon)$ (with $d = d_R$), we have

$$\Omega(\varepsilon) \cap U \subset \bigcup\nolimits_{1/\delta \in \mathbb{N}} \Lambda_\delta(\varepsilon) \cap U.$$

So if $U \cap \Omega(\varepsilon)$ is non-meagre, then $U \cap \Lambda_\delta(\varepsilon)$ is non-meagre for some $\delta > 0$ and so, by Baire's Theorem, dense in some open V with $\mathrm{cl}V \subset H$. But $\Lambda_\delta(\varepsilon)$ is closed, so $V \subset \Lambda_\delta(\varepsilon)$. Thus $d(t, tz) \le \varepsilon$ for all $\|z\| \le \delta$ and $t \in V$. ∎

Proposition 3.47 (Montgomery's Joint Continuity Theorem, [Mon2, Th. 1]). *Let* X *be a normed group, locally complete in the right norm topology, and* W *a non-empty open set* W. *If* $\Omega(\varepsilon) \cap W$ *is non-meagre for each* $\varepsilon > 0$, *then there is* $w \in W$ *with* γ_w *continuous. So if* $\Omega(\varepsilon) \cap U$ *is non-meagre for each* $\varepsilon > 0$ *and each non-empty* $U \subseteq W$, *then* $W \cap Z_\Gamma$ *is dense in* W.
In particular, if $\Omega(\varepsilon) \cap U$ *is non-meagre for each* $\varepsilon > 0$ *and every open set* U, *then* X *is a topological group.*
More generally, if for some open W *and all* $\varepsilon > 0$ *the set* $\Omega(\varepsilon) \cap W$ *is non-meagre in* W, *and* X *is separable and connected, then* X *is a topological group.*

Proof. Working in the right topology, and by Prop. 3.46 taking successively $\varepsilon(n) = 2^{-n}$ for ε, we may choose inductively $\delta(n)$ and open sets U_n with $U_{n+1} \subseteq U_n$ such that $U_{n+1} \subseteq \Lambda_{\delta(n)}(\varepsilon(n))$. So if $w \in \bigcap U_n$, then for each n we have $\omega_{\delta(n)}(w) \le \varepsilon(n)$, so that $\omega(w) = 0$.
The final assertion follows by Prop. 3.43, since now the centre Z_Γ is dense in the space. ∎

The preceding result, already a sharpening of Montgomery's original result, says that if X is not a topological group then the oscillation is bounded away from zero on a co-meagre set. But we can improve on this. It will be convenient (cf. Th. 3.48 below) to make the following

Definition. Working in the right norm topology (X, d_R), call t an ε-*shifting point* (on the left) if there is $\delta > 0$ such that for $\|z\| \le \delta$

$$d_R(t, tz) < \varepsilon,$$

equivalently, in oscillation function terms, $\omega_\delta(t) \le \varepsilon$ (since $\|tzt^{-1}\| \le \varepsilon$ for $\|z\| \le \delta$).

Remarks. 1. A sequential version may be formulated: call t an ε-shifting point for the null sequence z_n if there exists $N(\varepsilon)$ such that for $m > N(\varepsilon)$

$$d_R(t, tz_m) < \varepsilon.$$

Then t is an ε-shifting point iff t is ε-shifting for every null sequence. Indeed, if t is not an ε-shifting point, then for each $\delta = 1/n$ there is z_n with $\|z_n\| < 1/n$ such that $d_R(t, tz_n) \geq \varepsilon$, so t is not ε-shifting for this null sequence.
2. In the notation associated with oscillation, t is ε-shifting for $\{z_n\}$ if

$$t \in H_\varepsilon(\{z_n\}) := \bigcup\nolimits_n G_\varepsilon(z_n).$$

3. Evidently, if t is an ε-shifting point for each $\varepsilon > 0$, then γ_t is continuous (being continuous at e) and so a member of the topological centre $Z_\Gamma(X)$ of the normed group.
4. The sequential version is motivated by the Kestelman-Borwein-Ditor Theorem of Section 1 (Th. 1.2) which, roughly speaking, says that $tz_n \to t$ generically. (See Cor. 3.50.)
5. In referring to this property, the theorem which follows assumes something less than that the centre Z_Γ is dense, only that the open set $H_\varepsilon(\{z_n\})$ is dense for each $\varepsilon > 0$ and each $\{z_n\}$.

Theorem 3.48 (Dense Oscillation Theorem). *In a normed group X*

$$\bigcap\nolimits_{n\in\mathbb{N}} \mathrm{cl}\,[\Omega(1/\mathrm{n})] = \bigcap\nolimits_{\mathrm{n}\in\mathbb{N}} \Omega(1/\mathrm{n}) = \mathrm{Z}_\Gamma.$$

Hence, if for each $\varepsilon > 0$ the ε-shifting points are dense, equivalently $\Omega(\varepsilon) = \{t : \omega(t) < \varepsilon\}$ is dense for each $\varepsilon > 0$, then the normed group is topological.
More generally, if for some open W and all $\varepsilon > 0$ the set $\Omega(\varepsilon) \cap W$ is dense in W, then $\omega = 0$ on W; in particular,
(i) if $e_X \in W$ and X is connected and Baire under its norm topology, then X is a topological group,
(ii) if X is separable, connected and topologically complete in its norm topology, then X is a topological group.

Proof. The opening assertion follows from the continuity of ω. For $\varepsilon > 0$, if $\Omega(\varepsilon)$ is dense on W, then $\mathrm{cl}W \subseteq \mathrm{cl}\Omega(\varepsilon)$. Hence, if $\Omega(\varepsilon)$ is dense on W for all $\varepsilon > 0$, $\mathrm{cl}W \subseteq Z_\Gamma$. So if $W = X = Z_\Gamma$, i.e. γ_s is continuous for all $s \in X$, then the conclusion follows from the Equivalence Theorem (Th. 3.4). For a more general W, the conclusion follows from Th. 3.45. ∎

Remark. It is instructive to see how the density property of the last theorem bestows the ε-shifting property to nearby points. Fix s and $\varepsilon > 0$. For $n > 1/\varepsilon$, let $t \in \Omega(1/n) \cap B_\varepsilon(s)$.Then for some $\delta = \delta(n)$ we have $\omega_\delta(t) \leq 1/n$, equivalently $d(tz, t) \leq 1/n$ for $\|z\| \leq \delta$, and so for such z

$$\begin{aligned} d_R(sz, s) &\leq d_R(sz, tz) + d_R(tz, t) + d_R(t, s) \\ &\leq 2d_R(s, t) + 1/n \leq 3\varepsilon. \end{aligned}$$

Thus $\omega_\delta(s) \leq 3\varepsilon$. (Since $\varepsilon > 0$ was arbitrary, $\omega(s) = 0$, so γ_s is continuous, and so $s \in Z_R$.) We use this idea several times over in the next result.

We now give a necessary and sufficient criterion for a normed group to be topological by refering not to continuity, but to approximation of left-shifts by right-shifts. This turns out to be equivalent both to a commutator condition and to a shifting property condition.

An extended comment on the commutator condition is in order, because the condition finesses the descriptive character of the relation $x = yz$. Proposition 3.49 below employs the following 'commutator oscillation' set (and its density):

$$C(\varepsilon) := \bigcup\nolimits_{n\in\mathbb{N}} C_{1/n}(\varepsilon), \text{ where}$$
$$C_\delta(\varepsilon) := \bigcap\nolimits_{\|z\|\leq\delta}\{y : \|zyz^{-1}y^{-1}\| \leq \varepsilon\ \} = \bigcap\nolimits_{\|z\|\leq\delta}\{y : d_R(zy, yz) \leq \varepsilon\ \}.$$

This is an 'oscillation set', since $\|\gamma_y(z)\| - \|z\| \leq \|[z,y]_L\| \leq \|\gamma_y(z)\| + \|z\|$; indeed one might refer to $\bar{\omega}(y) := \lim_{\delta\searrow 0}\sup_{\|z\|\leq\delta}\|zyz^{-1}y^{-1}\|$, but for the fact that $\bar{\omega}(y) = \omega(y)$. Furthermore, $\bigcap_{n\in\mathbb{N}} C(1/n) = Z_\Gamma$, since for $\delta < \varepsilon$ we have the 'inner regularity of C': $\Lambda_\delta(\varepsilon) \subseteq C_\delta(2\varepsilon)$, and the 'outer regularity of C': $C_\delta(\varepsilon) \subseteq \Lambda_\delta(2\varepsilon)$. So, since density is the vehicle of proof, one may carry over the proof of the Montgomery Theorem (Th. 3.47) with $\mathrm{cl}[C_\delta(\varepsilon)]$ in lieu of $\Lambda_\delta(\varepsilon)$. Note that these inclusions permit use of $C_\delta(\varepsilon)$ even if the latter has poor descriptive character (i.e. we do not need to know anything about the relation $x = yz$). Of course, for X separable and topologically complete, if $\{(y,z) : d_R(yz, zy) \leq \varepsilon\}$ has analytic graph, then the set $C_\delta(\varepsilon)$ is co-analytic (complement of a Souslin-$\mathcal{F}$ set, see Section 11 for background), because

$$y \notin C_\delta(\varepsilon) \iff (\exists z \in B_\delta(e_X))[d_R(yz, zy) \leq \varepsilon].$$

Under these circumstances, $C_\delta(\varepsilon)$ is Baire by Nikodym's Theorem (Th.11.5); but Prop. 3.49 does not need this.

The next result is, for normed groups, a sharpening of the Montgomery Theorem (Th. 3.47), in view of Montgomery's Initial Observation above that, for a semitopological group, each set $\Omega(\varepsilon) \cap W$ is non-meagre for W a non-empty open set (and in particular each set $\Omega(\varepsilon)$ is dense). This arises from our use of $d = d_R$, when Montgomery uses an arbitrary (compatible) metric d in Th. 3.46, and so relegates the implementation of category to the last rather than an earlier step.

Proposition 3.49 (Left-right Approximation Criterion). *For W a non-empty right-open subset of a normed group X, the following are equivalent:*
(a) For each $t \in W$ and each $\eta > 0$, there are y_η and $\delta > 0$ such that $d_R(tz, zy_\eta) \leq \eta$ for all $\|z\| \leq \delta$, i.e. for each $t \in W$ the left-shift λ_t may be locally approximated near the identity by a right-shift ρ_y.
(b) For each $\varepsilon > 0$, the set $C(\varepsilon) = \{y : (\exists\delta > 0)[d_R(yz, zy) \leq \varepsilon$ for all all $\|z\| \leq \delta]\}$ is dense in W – i.e. $C(\varepsilon) \cap W$ is dense in W.
(c) For each $\varepsilon > 0$, the set $\Omega(\varepsilon) = \{t : (\exists\delta > 0)[d_R(tz, t) < \varepsilon$ for all $\|z\| \leq \delta]\}$ is dense in W.
Suppose that for each $t \in W$ the left-shift λ_t may be locally approximated near the identity by a right-shift. Then:

(i) $W \cap Z_\Gamma$ is dense in W;
(ii) $\omega(t) = 0$ for all $t \in W$.
In particular, if $W = X$, then X is topological.
For a general W, as above,
(i) if $e_X \in W$ and X is connected and Baire (under its norm topology), then X is a topological group,
(ii) if X is separable, connected and topologically complete (in its norm topology), then X is a topological group.

Proof. We first verify that (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(a).
Assume (a). Let $\varepsilon > 0$. Consider a nonempty $U \subseteq W$. Pick $t \in U$ and suppose that $B_\eta(t) \subseteq U$ with $\eta < \varepsilon$. By assumption, there is $y_\eta = y_\eta(t)$ such that for some $\delta = \delta(\eta) < \eta$ we have

$$d_R(tz, zy_\eta) \leq \eta/2 \text{ for all } \|z\| \leq \delta.$$

Then in particular $d_R(t, y_\eta) \leq \eta/2 < \eta$, and also for all $\|z\| \leq \delta$

$$d_R(y_\eta z, zy_\eta) \leq d_R(y_\eta z, tz) + d_R(tz, zy_\eta) = d_R(y_\eta, t) + d_R(tz, zy_\eta) \leq \eta < \varepsilon.$$

Thus $y_\eta \in B_\eta(t) \cap C(\varepsilon) \subseteq U \cap C(\varepsilon)$. That is, (b) holds.
Assume (b). Consider a non-empty $U \subseteq W$. Pick $t \in U$ and suppose that $B_\eta(t) \subseteq U$ with $\eta < \varepsilon/3$. By assumption, there is $y_\eta = y_\eta(t) \in B_\eta(t)$ such that for some $\delta = \delta(\eta) < \eta$ we have

$$d_R(y_\eta z, zy_\eta) \leq \eta \text{ for all } \|z\| \leq \delta.$$

We prove that y_η is a 3η-shifting point and so a ε-shifting point, i.e. that

$$d_R(y_\eta z, y_\eta) \leq 3\eta \text{ for all } \|z\| \leq \delta.$$

Indeed, we have

$$\begin{aligned} d_R(y_\eta z, y_\eta) &\leq d_R(y_\eta z, zy_\eta) + d_R(zy_\eta, y_\eta) = d_R(y_\eta z, zy_\eta) + d_R(z, e) \\ &\leq 2\eta + \delta < 3\eta < \varepsilon. \end{aligned}$$

Thus, $y_\eta \in B_\eta(t) \cap \Omega(\varepsilon) \subseteq U \cap \Omega(\varepsilon)$. That is, (c) holds.
Now suppose that (c) holds. Consider $t \in W$ and $\varepsilon > 0$. Suppose that $B_\eta(t) \subseteq W$ with $\eta < \varepsilon/2$. By assumption, there is $y_\eta = y_\eta(t)$ such that for some $\delta = \delta(\eta) < \eta$ we have

$$d_R(y_\eta z, zy_\eta) \leq \eta \text{ for all } \|z\| \leq \delta.$$

Hence

$$d_R(tz, zy_\eta) \leq d_R(tz, y_\eta z) + d_R(y_\eta z, zy_\eta) \leq 2\eta < \varepsilon.$$

So for $y = y_\eta$, we have $d_R(tz, zy) < \varepsilon$ for all $\|z\| \leq \delta$. Thus (a) holds.
Now that we have verified the equivalences, suppose that (a) holds.
From (c), for $t \in W$ and any $\varepsilon > 0$, we have $\omega(t) \leq \omega_{\delta(\varepsilon)}(t) \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary, we have $\omega(t) = 0$. Hence if $W = X$, then $X = Z_\Gamma$ and the group is topological. The other two conclusions follow from Th. 3.44 and 3.45. ∎

Remark. In the penultimate step above with $W = X$, one can take $\varepsilon > 0$; then for $0 < \eta \le \varepsilon$ we have $d(t, y_\eta(t)) \le \eta$, and $\omega(y_\eta) < 3\eta \le 3\varepsilon$, so the points $\{y_\eta(t) : t \in X, 0 < \eta \le \varepsilon\} \subset \Omega_{3\varepsilon}$ are dense in X. Then by Th. 3.48 the group is topological.

In the next result we ask that KBD holds with left-shifts in a right norm topology.

Corollary 3.50. *Let X be a normed group, Baire in its right norm topology. Suppose KBD holds in X in the following form:*
'For each null z_n and each non-meagre, Baire set T, there are $t \in T$ and an infinite $\mathbb{M}_t$ such that $tz_m \in T$ for $m \in \mathbb{M}_t$.'
Then, for each $\varepsilon > 0$, $S(\varepsilon) = X$, i.e. every point is an ε-shifting point for any $\varepsilon > 0$.
In particular, X is a topological group.

Proof. Suppose not. Then there is x and $\varepsilon > 0$ such that x is not ε-shifting, i.e. for each n there is $z_n \in B_{1/n}(x)$ such that

$$d(x, xz_n) \ge \varepsilon.$$

Let $\eta < \varepsilon/4$. Since z_n is null and $B_\eta(x)$ is open (so Baire) and non-meagre, by the assumed KBD there are $t \in B_\eta(x)$ and an infinite $\mathbb{M}_t$ such that $tz_m \in B_\eta(x)$ for $m \in \mathbb{M}_t$. So, since $d(t, tz_m) < 2\eta$, for any $m \in \mathbb{M}_t$

$$\begin{aligned} d_R(x, xz_m) &\le d_R(x, t) + d_R(t, tz_m) + d_R(tz_m, xz_m) \\ &= 2d_R(x, t) + d_R(t, tz_m) < 4\eta < \varepsilon, \end{aligned}$$

a contradiction.
Thus $X = S(\varepsilon)$, for each $\varepsilon > 0$. By Th. 3. 49, X is a topological group. ∎

Theorem. 3.51 is a corollary of the Dense Oscillation Theorem (Th. 3.48) and indicates a 'Darboux-like' pathology when the normed group is not topological.

Theorem 3.51 (Pathology Theorem). *If a normed group X is not a topological group, then there is an open set on which the oscillation function is uniformly bounded away from 0.*

Proof. This follows from the continuity of ω at any point t where $\omega(t) > 0$. This also follows from Th. 3.48, since for some $\varepsilon > 0$, the open set $U := X\backslash\mathrm{cl}[\Omega(\varepsilon)]$ is non-empty, and $\omega(t) \ge \varepsilon$ for $t \in U$, as $t \notin \Omega(\varepsilon)$. ∎

By way of a final clarification of our interest in ε-shifting points, we return to the literature of 'separate implies joint continuity' and in particular to the key notion of quasi-continuity, which we adapt here to a metric context (for further information see e.g. [Bou2]).

Definition. A map $f : X \to Y$ between metric spaces is *quasi-continuous* at x if for $\varepsilon > 0$ there are $a \in B_\varepsilon^X(x)$ and $\delta > 0$ such that

$$f(u) \in B_\varepsilon^Y(f(x)), \text{ for all } u \in B_\delta^X(a).$$

The following result connects quasi-continuity of left-shifts λ_t with ε-shifting.

Theorem 3.52. *Let X be a normed group X.*
(i) The left-shift $\lambda_t(x)$, as a self-map of X under the right norm topology, is quasi-continuous at any point/all points x iff for every $\varepsilon > 0$ there are $y = y(\varepsilon)$ and $\delta = \delta(\varepsilon) > 0$ such that

$$d_L(t, y(\varepsilon)) < \varepsilon, \text{ and } d_R(tz, y(\varepsilon)) < \varepsilon, \text{ for } \|z\| < \delta.$$

(ii) If λ_t is quasi-continuous, then t is an ε-shifting point for each $\varepsilon > 0$.
(iii) In these circumstance, γ_t has zero oscillation, hence γ_t and so λ_t is continuous.

Proof. (i) This is a routine transcription of the last definition, so we omit the details. The point y of the Theorem is obtained from the point a of the definition via $y := ta^{-1}$.
(ii) This conclusion come from taking $z = e$ and applying the triangle inequality to obtain

$$d_R(tz, t) < 2\varepsilon, \text{ for } \|z\| < \delta(\varepsilon).$$

(iii) It follows from (ii) that $\omega(t) = 0$, so that γ_t is continuous at e and hence everywhere; λ_t is then continuous, being a composition of continuous functions, since $tx = \rho_t(\gamma_t(x))$. ∎

Of course in the setting above $t_m := y(1/m)$ converges to t under both norm topologies.

This gives a restatement of a preceding result (Th. 3.46).

Theorem 3.53. *In a normed group X with right norm topology, if for a dense set of t the left-shifts $\lambda_t(x)$ are quasi-continuous, then the normed group is topological.*

Alternatively, note that under the current assumptions the topological centre Z_Γ is dense, and being closed is the whole of X. Our closing comment addresses the opening issue of this subsection – converging subsequences – in terms of subcontinuity. We recall a result of Bouziad, again specialized to our metric context.

Theorem 3.54 ([Bou2, Lemma 2.4]). *For $f : X \to Y$ a quasi-continuous map between metric spaces with X Baire, the set of subcontinuity points of f is a dense subset of X.*

The result confirms that if λ_t is quasi-continuus then it is subcontinuous on a dense set of points, a fortiori at one point, and so at e by the remarks to the definition of subcontinuity.

4. Subadditivity

DEFINITION. Let X be a normed group. A function $p : X \to \mathbb{R}$ is *subadditive* if

$$p(xy) \leq p(x) + p(y).$$

Thus a norm $\|x\|$ and so also any g-conjugate norm $\|x\|_g$ are examples. Recall from [Kucz, p.140] the definitions of *upper and lower hulls* of a function p :

$$M_p(x) = \lim_{r\to 0+} \sup\{p(z) : z \in B_r(x)\},$$
$$m_p(x) = \lim_{r\to 0+} \inf\{p(z) : z \in B_r(x)\}.$$

(Usually these are of interest for convex functions p.) These definitions remain valid for a normed group. (Note that e.g. $\inf\{p(z) : z \in B_r(x)\}$ is a decreasing function of r.) We understand the balls here to be defined by a *right-invariant* metric, i.e.

$$B_r(x) := \{y : d(x, y) < r\} \text{ with } d \text{ right-invariant.}$$

These are subadditive functions if the group G is $\mathbb{R}^d$. We reprove some results from Kuczma [Kucz], thus verifying the extent to which they may be generalized to normed groups. Only our first result appears to need the Klee property (bi-invariance of the metric); fortunately this result is not needed in the sequel. The Main Theorem below concerns the behaviour of $p(x)/\|x\|$.

LEMMA 4.1 (cf. [Kucz, L. 1 p. 403]). *For a normed group G with the Klee property, m_p and M_p are subadditive.*

Proof. For $a > m_p(x)$ and $b > m_p(y)$ and $r > 0$, let $d(u, x) < r$ and $d(v, y) < r$ satisfy

$$\inf\{p(z) : z \in B_r(x)\} \leq p(u) < a, \text{ and } \inf\{p(z) : z \in B_r(y)\} \leq p(v) < b.$$

Then, by the Klee property,

$$d(xy, uv) \leq d(x, u) + d(y, v) < 2r.$$

Now

$$\inf\{p(z) : z \in B_{2r}(xy)\} \leq p(uv) \leq p(u) + p(v) < a + b,$$

hence

$$\inf\{p(z) : z \in B_{2r}(xy)\} \leq \inf\{p(z) : z \in B_r(x)\} + \inf\{p(z) : z \in B_r(x)\},$$

and the result follows on taking limits as $r \to 0+$. ■

Lemma 4.2 (cf. [Kucz, L. 2 p. 403]). *For a normed group G, if $p : G \to \mathbb{R}$ is subadditive, then*

$$m_p(x) \leq M_p(x) \text{ and } M_p(x) - m_p(x) \leq M_p(e).$$

Proof. Only the second assertion needs proof. For $a > m_p(x)$ and $b < M_p(x)$, there exist $u, v \in B_r(x)$ with

$$a > p(u) \geq m_p(x), \text{ and } b < p(v) \leq M_p(x).$$

So

$$b - a < p(v) - p(u) \leq p(vu^{-1}u) - p(u) \leq p(vu^{-1}) + p(u) - p(u) = p(vu^{-1}).$$

Now

$$\|vu^{-1}\| \leq \|v\| + \|u\| < 2r,$$

so $vu^{-1} \in B_{2r}(e)$, and hence

$$p(vu^{-1}) \leq \sup\{p(z) : z \in B_{2r}(e)\}.$$

Hence, with r fixed, taking a, b to their respective limits,

$$M_p(x) - m_p(x) \leq \sup\{p(z) : z \in B_{2r}(e)\}.$$

Taking limits as $r \to 0+$, we obtain the second inequality. ■

Lemma 4.3. *For a normed group G and any subadditive function $f : G \to \mathbb{R}$, if f is locally bounded above at a point, then it is locally bounded at every point.*

Proof. We repeat the proof in [Kucz, Th. 2, p.404], thus verifying that it continues to hold in a normed group.
Suppose that p is locally bounded above at t_0 by K. We first show that f is locally bounded above at e. Suppose otherwise that for some $t_n \to e$ we have $p(t_n) \to \infty$. Now $t_n t_0 \to e t_0 = t_0$, and so

$$p(t_n) = p(t_n t_0 t_0^{-1}) \leq p(t_n t_0) + p(t_0^{-1}) \leq K + p(t_0^{-1}),$$

a contradiction. Hence p is locally bounded above at e, i.e. $M_p(e) < \infty$. But $0 \leq M_p(x) - m_p(x) \leq M_p(e)$, hence both $M_p(x)$ and $m_p(x)$ are finite for every x. That is, p is locally bounded above and below at each x. ■

The next result requires that both $f(x)$ and $f(x^{-1})$ be Baire functions; this happens for instance when (i) f is even, i.e. $f(x) = f(x^{-1})$, with $f(x) := \|gxg^{-1}\|$ an example of some interest here (cf. Th. 3.27 and in connection with the oscillation function of Section 3.3), and (ii) both $f(x)$ and $x \to x^{-1}$ are Baire, so that the normed group is a topological group (Th. 3.41).

Proposition 4.4 ([Kucz, Th. 3, p. 404]). *For a topologically complete normed group G and a Baire function $f : G \to \mathbb{R}$ with $x \to f(x^{-1})$ Baire, if f is subadditive, then f is locally bounded.*

Proof. By the Baire assumptions, for some k $H^k := \{x : |f(x)| < k \text{ and } |f(x^{-1})| < k\}$ is non-meagre. Note the symmetry: $x \in H^k$ iff $x^{-1} \in H^k$. Suppose that f is not locally bounded; then it is not locally bounded above at some point u, i.e. there exists $u_n \to u$ with

$$f(u_n) \to +\infty.$$

Put $z_n := u_n u^{-1}$; by the KBD Theorem Th. 1.2 (or Th. 5.1), for some $k \in \omega$, $t, t_m \in H^k$ and an infinite $\mathbb{M}$, we have

$$\{tt_m^{-1}u_n u^{-1}t_m : m \in \mathbb{M}\} \subseteq H^k.$$

By symmetry, for m in $\mathbb{M}$, we have

$$\begin{aligned} f(u_m) &= f(t_m t^{-1}(tt_m^{-1}u_m u^{-1}t_m)t_m^{-1}u) \\ &\leq f(t_m) + f(t^{-1}) + f(tt_m^{-1}u_m u^{-1}t_m) + f(t_m^{-1}) + f(u) \\ &\leq 4k + f(u), \end{aligned}$$

which contradicts $f(u_m) \to +\infty$. ∎

We recall that *vanishingly small word-nets* were defined in Section 3.2.

THEOREM 4.5. *Let G be a normed group with a vanishingly small word-net. Let $p : G \to \mathbb{R}_+$ be Baire, subadditive with*

$$\beta := \limsup_{\|x\|\to 0+} \frac{p(x)}{\|x\|} < \infty.$$

Then

$$\limsup_{\|x\|\to\infty} \frac{p(x)}{\|x\|} \leq \beta < \infty.$$

Proof. Let $\varepsilon > 0$. Let $b = \beta + \varepsilon$. Hence on $B_\delta(e)$ for δ small enough to guarantee the existence of Z_δ and M_δ we have also

$$\frac{p(x)}{\|x\|} \leq b.$$

By Proposition 4.4, we may assume that p is bounded by some constant K in $B_\delta(e)$. Let $\|x\| > M_\delta$.

Choose a word $w(x) = z_0 z_1 ... z_n$ with $\|z_i\| = \delta(1 + \varepsilon_i)$ with $|\varepsilon_i| < \varepsilon$, with

$$p(x_i) < b\|x_i\| = b\delta(1 + \varepsilon_i)$$

and

$$d(x, w(x)) < \delta,$$

i.e.

$$x = w(x)s$$

for some s with $\|s\| < \delta$ and

$$1 - \varepsilon \leq \frac{n(x)\delta}{\|x\|} \leq 1 + \varepsilon.$$

Now

$$\begin{aligned}p(x) = p(ws) &\leq p(w) + p(r) = \sum p(z_i) + p(s) \\ &\leq \sum b\delta(1+\varepsilon_i) + p(s) \\ &= nb\delta(1+\varepsilon) + K.\end{aligned}$$

So

$$\frac{p(x)}{\|x\|} \leq \frac{n\delta}{\|x\|} b(1+\varepsilon) + \frac{M}{\|x\|}.$$

Hence we obtain

$$\frac{p(x)}{\|x\|} \leq b(1+\varepsilon)^2 + \frac{M}{\|x\|}.$$

So in the limit

$$\limsup_{\|x\|\to\infty} \frac{p(x)}{\|x\|} < \beta,$$

as asserted. ∎

We note a related result, which requires the following definition. For p subadditive, put (for this section only)

$$p_*(x) = \liminf_{y\to x} p(y), \qquad p^*(x) := \limsup_{y\to x} p(y).$$

These are subadditive and lower (resp. upper) semicontinuous with $p_*(x) \leq p(x) \leq p^*(x)$.

Theorem 4.6 (Mueller's Theorem – [Mue, Th. 3]). *Let p be subadditive on a locally compact group G and suppose*

$$\liminf_{x\to e} p^*(x) \leq 0.$$

Then p is continuous almost everywhere.

We now return to the proof of Theorem 3.20, delayed from Section 3.2.

Proof of Theorem 3.20. Apply Theorem 4.5 to the subadditive function $p(x) := \|f(x)\|$, which is continuous and so Baire. Thus there is X such that, for $\|x\| \geq X$,

$$\|f(x)\| \leq \beta\|x\|.$$

Taking $\varepsilon = 1$ in the definition of a word-net, there is $\delta > 0$ small enough so that $B_\delta(e)$ is pre-compact and there exists a compact set of generators Z_δ such that for each x there is a word of length $n(x)$ employing generators of Z_δ with $n(x) \leq 2\|x\|/\delta$. Hence if $\|x\| \leq X$ we have $n(x) \leq 2M/\delta$. Let $N := [2M/\delta]$, the least integer greater than $2M/\delta$. Note that $Z_\delta^N := Z_\delta \cdot \ldots \cdot Z_\delta$ (N times) is compact. The set $B_K(e)$ is covered by the compact swelling $K :=\mathrm{cl}[Z_\delta^N B_\delta(e)]$. Hence, we have

$$\sup_{x\in K} \frac{\|f(x)\|}{\|x\|} < \infty,$$

(referring to $\beta_g < \infty$, and continuity of $\|x\|_g/\|x\|$ away from e), and so

$$M \leq \max\{\beta, \sup_{x\in K} \|f(x)\|/\|x\|\} < \infty.$$

5. Generic Dichotomy

In this section we develop the first of several (in fact six) bi-topological approaches to a generalization of the Kestelman-Borwein-Ditor Theorem (KBD) in the introduction (Th.1.1) We will see later just how useful the result can be in several areas: we regard it as a measure-category analogue of the celebrated *probabilistic method* of Erdős (for which see e.g. [AS], [TV], [GRS]), here expanded to a theorem on the generic alternative – a *generic dichotomy* (as defined below). The aproach of this section, inspired by a close reading of [BHW], ultimately rests on one-sided completeness in the underlying normed structure, namely that the right (or, left) norm *topology* be completely metrizable on some non-meagre subspace. (The two choices are equivalent, since (X, d_R) and (X, d_L) are isometric – see Prop. 2.15.) This embraces groups of homeomorphisms that may not be topological groups.

For background on topological group completeness, refer to [Br-1] for a discussion of the three uniformities of a topological group. (There the one-sided completeness is implied by the ambidextrous uniformity being complete, cf. [Kel, Ch. 6 Problem Q].) Compare also Th. 3.9 on ambidextrous refinement. Actually we apparently need only local versions of topological completeness, so we recall Brown's Theorem that if a topological group is locally complete then it is paracompact and topologically complete. (In fact the structure is even more tightly prescribed, see [Br-2].)

Alternative approaches are given in subsequent sections with modified assumptions.

To formulate a first generalization of KBD we will need a pair of definitions. To motivate them recall (see e.g. [Eng, 4.3.23 and 24]) that a metric space A is *completely metrizable* iff it is a $\mathcal{G}_\delta$ subset of its completion (i.e. $A = \bigcap_{n\in\omega} G_n$ with each G_n open in the completion of A), in which case it has an equivalent metric under which it is complete. Thus when (X, d_R) is complete, a $\mathcal{G}_\delta$ subset A of X has a metric $\rho = \rho_A$, equivalent to d_R, under which (A, ρ) is complete. (So for each $a \in A$ and $\varepsilon > 0$ there is $\delta > 0$ such that $B_\delta(a) \subseteq B^\rho_\varepsilon(a)$, where $B_\delta(a)$ refers to d_R, and this enables the construction of ρ-Cauchy sequences.)

With this in mind we may return to Brown's theorem on completeness implied by local completness, to note that in the metrizable context the result follows from a localization principle of Montgomery in [Mont0] asserting in particular that a subspace that is locally a $\mathcal{G}_\delta$ at all its points is itself a $\mathcal{G}_\delta$. (One need only embed a metric space in its own completion.)

Definition. Say that a normed group $(X, \|\cdot\|)$ is *topologically complete* if (X, d_R) is completely metrizable as a metric space; equivalently, one may require that (X, d_L) be topologically complete, as the latter is homeomorphic to (X, d_R) and topological completeness is indeed topological (see [Eng] Th. 4.3.26 taken together with Th. 3.9.1 – there the term Čech-complete is used). In particular, a locally compact normed group is topologically complete.

The last definition places a stringent condition on a normed group: the only subgroups of a topologically complete group which are themselves topologically complete are $\mathcal{G}_\delta$. Our related second definition represents a significant weakening of topological completeness, as non-meagre Borel subspaces will have this property (by Th. 5.2 below, since they have the Baire property). The format allows us to capture a feature of measure-category duality: both exhibit $\mathcal{G}_\delta$ *inner-regularity* modulo sets which we are prepared to neglect. This generalizes a definition given in [BOst-KCC] for the case of the real line.

Definition. For X a normed group call $A \subset X$ *almost complete* in category/measure, if
(i) (A is non-meagre and) there is a meagre set N such that $A\backslash N$ is a $\mathcal{G}_\delta$ completely metrizable, or, respectively,
(ii) X is a locally compact topological group (hence topologically complete) and for each $\varepsilon > 0$ there is a Haar-measurable set N with $m(N) < \varepsilon$ and $A\backslash N$ a $\mathcal{G}_\delta$.

The term 'almost complete' (in the category sense above) is due to E. Michael (see [Michael]), but the notion was introduced by Frolík in terms of open 'almost covers' (i.e. open families that cover a dense subspace, see [Frol-60] §4) and demonstrated its relation to the existence of dense $\mathcal{G}_\delta$-subspace. It was thus first named 'almost Čech-complete' by Aarts and Lutzer ([AL, Section 4.1.2]; compare [HMc]). For metric spaces our category definition above is equivalent (and more directly connects with completeness). Indeed, on the one hand a completely regular space is almost Čech-complete iff it contains a dense Čech-complete (or topologically complete) subspace, i.e. one that is absolutely $\mathcal{G}_\delta$ (is $\mathcal{G}_\delta$ in some/any compactification). On the other hand a metrizable Baire space X contains a dense completely-metrizable $\mathcal{G}_\delta$-subset iff X is a completely metrizable $\mathcal{G}_\delta$-set up to a meagre set. (A metrizable subspace is absolutely $\mathcal{G}_\delta$ iff it embeds as a $\mathcal{G}_\delta$ in its completion – cf. [Eng, Th. 4.3.24].)

We comment further on the definition once we have stated its primary purpose, which is to give the weakest hypothesis under which the classical KBD Theorem may be generalized.

Theorem 5.1 (Kestelman-Borwein-Ditor Theorem – [BOst-Funct]). *Suppose X is an almost complete normed group (e.g. completely metrizable), or in particular a locally compact topological group. Let $\{z_n\} \to e_X$ be a null sequence. If $T \subseteq X$ is non-meagre Baire under d_R^X (or resp. non-null Haar-measurable), then there are $t, t_m \in T$ with $t_m \to_R t$ and an infinite set $\mathbb{M}_t$ such that*

$$\{t t_m^{-1} z_m t_m : m \in \mathbb{M}_t\} \subseteq T.$$

If further X is a topological group, then for generically all $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{t z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Returning to the critical notion of almost completeness, we note that A almost complete is Baire resp. measurable. A bounded non-null measurable subset A is almost complete: for each $\varepsilon > 0$ there is a compact (so $\mathcal{G}_\delta$) subset K with $|A\backslash K| < \varepsilon$, so we may take $N = A\backslash K$. Likewise a Baire non-meagre set in a complete metric space is almost complete – this is in effect a restatement of Baire's Theorem:

Theorem 5.2 (Baire's Theorem – almost completeness of Baire sets). *For a completely metrizable space X and $A \subseteq X$ Baire non-meagre, there is a meagre set M such that $A\backslash M$ is completely metrizable and so A is almost complete.*
Hence, in a metrizable almost complete space a subset B is Baire iff the subspace B is almost complete.

Proof. For $A \subseteq X$ Baire non-meagre we have $A \cup M_1 = U\backslash M_0$ with M_i meagre and U a non-empty open set. Now $M_0 = \bigcup_{n\in\omega} N_n$ with N_n nowhere dense; the closure $F_n := \bar{N}_n$ is also nowhere dense (and the complement $E_n = X\backslash F_n$ is dense, open). The set $M_0' = \bigcup_{n\in\omega} F_n$ is also meagre, so $A_0 := U\backslash M_0' = \bigcap_{n\in\omega} U \cap E_n \subseteq A$. Taking $G_n := U \cap E_n$, we see that A_0 is completely metrizable.
If X is almost complete, then any subspace of X that is almost complete is a Baire set, since an absolute $\mathcal{G}_\delta$ has the Baire property in X. As to the converse, for a Baire set $B \subseteq X$ with X almost complete, write $X = H_X \cup N_X$ with N_X meagre and H_X an absolute $\mathcal{G}_\delta$ and $B = (U\backslash M_B) \cup N_B$ with U open and M_B, N_B meagre. We have just seen that without loss of generality M_B may be taken to be a meagre $\mathcal{F}_\sigma$ subset of U (otherwise choose F_B a meagre $\mathcal{F}_\sigma$ containing M_B and let F_B and $N_B \cup (F_B\backslash M_B)$ replace M_B and N_B respectively). Intersecting the representations of X and B, one has $B = H_B \cup N_B'$ for $H_B := H_X \cap (U\backslash F_B)$, an absolute $\mathcal{G}_\delta$, and some meagre $N_B' \subseteq N_B \cup N_X$. So, B is almost complete. ∎

Th. 5.2 says that, in a complete space, a set which is almost open is almost complete. More generally, even if the space is not complete, any non-meagre separable analytic set (for definition of which see Section 11) is almost complete – a result observed by S. Levi in [Levi]. (More in fact is true – see [Ost-AH] Cor. 2 and [Ost-AB].) In an almost complete space the distinction between the two notions of *Baire property* and *Baire subspace* is blurred, the two being indistinguishable. Almost completely metrizable spaces may be characterized in a useful fashion by reference to a less demanding absoluteness condition than topological completeness (we recall the latter is equivalent to being an absolute $\mathcal{G}_\delta$ – see above). It may be shown that a non-meagre normed group is almost complete iff it is almost absolutely analytic (see [Ost-AB],[Ost-LB3]).

The KBD Theorem is a generic assertion about embedding into target sets. We address first the source of this genericity, which is that a property inheritable by supersets either holds generically or fails outright. This is now made precise.

Definition. For X a Baire space (e.g. $\mathbb{R}_+$ with the Euclidean or density topology) denote by $\mathcal{B}a(X)$, or just $\mathcal{B}a$, the Baire sets of the space X, and recall these form a σ-algebra. Say that a correspondence $F : \mathcal{B}a \to \mathcal{B}a$ is *monotonic* if $F(S) \subseteq F(T)$ for $S \subseteq T$.

The nub is the following simple result, which we call the Generic Dichotomy Principle.

Theorem 5.3 (Generic Dichotomy Principle). *For X Baire and $F : \mathcal{B}a \to \mathcal{B}a$ monotonic: either*
(i) there is a non-meagre $S \in \mathcal{B}a$ with $S \cap F(S) = \emptyset$, or,
(ii) for every non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi almost all of T.
Equivalently: the existence condition that $S \cap F(S) \neq \emptyset$ should hold for all non-meagre $S \in \mathcal{B}a$, implies the genericity condition that, for each non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi almost all of T.

Proof. Suppose that (i) fails. Then $S \cap F(S) \neq \varnothing$ for every non-meagre $S \in \mathcal{B}a$. We show that (ii) holds. Suppose otherwise; thus for some T non-meagre in $\mathcal{B}a$, the set $T \cap F(T)$ is not almost all of T. Then the set $U := T \backslash F(T) \subseteq T$ is non-meagre (it is in $\mathcal{B}a$ as T and $F(T)$ are) and so

$$\begin{aligned} \emptyset \neq U \cap F(U) &\qquad (S \cap F(S) \neq \emptyset \text{ for every non-meagre } S) \\ \subseteq U \cap F(T) &\qquad (U \subseteq T \text{ and } F \text{ monotonic}). \end{aligned}$$

But as $U := T \backslash F(T)$, $U \cap F(T) = \emptyset$, a contradiction.
The final assertion simply rephrases the dichotomy as an implication. ∎

The following corollary permits the onus of verifying the existence condition of Theorem 5.3 to be transferred to topological completeness.

Theorem 5.4 (Generic Completeness Principle). *For X Baire and $F : \mathcal{B}a \to \mathcal{B}a$ monotonic, if $W \cap F(W) \neq \emptyset$ for all non-meagre $W \in \mathcal{G}_\delta$, then, for each non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi almost all of T.*
That is, either
(i) there is a non-meagre $S \in \mathcal{G}_\delta$ with $S \cap F(S) = \emptyset$, or,
(ii) for every non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi almost all of T.

Proof. From Theorem 5.2, for S non-meagre in $\mathcal{B}a$ there is a non-meagre $W \subseteq S$ with $W \in \mathcal{G}_\delta$. So $W \cap F(W) \neq \emptyset$ and thus $\emptyset \neq W \cap F(W) \subseteq S \cap F(S)$, by monotonicity. By Theorem 5.3 for every non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi almost all of T. ∎

Examples. Here are three examples of monotonic correspondences with X the reals. The first two relate to standard results. The following one is canonical for the current section. Each correspondence F below gives rise to a correspondence $\Phi(A) := F(A) \cap A$ which is a 'lower density' (or 'upper') and plays a role in the theory of *liftings* ([IT1], [IT]) and category measures ([Oxt2, Th. 22.4]) and so gives rise to a fine topology on the real line. See also [LMZ] Section 6F on lifting topologies.

1. Here we apply Theorem 5.4 to the real line with the density topology, in which the meagre sets are the null sets. Let $\mathcal{B}$ denote a countable basis of intervals for the usual (Euclidean) topology. For any set T and $0 < \alpha < 1$ put

$$\mathcal{B}_\alpha(T) := \{I \in \mathcal{B} : |I \cap T| > \alpha|I|\},$$

which is countable, and

$$F(T) := \bigcap\nolimits_{\alpha \in \mathbb{Q} \cap (0,1)} \bigcup \{I : I \in \mathcal{B}_\alpha(T)\}.$$

Thus F is monotone in T, $F(T)$ is measurable (even if T is not) and $x \in F(T)$ iff x is a density point of T. If T is measurable, the set of points x in T for which $x \in I \in \mathcal{B}$ implies that $|I \cap T| < \alpha|I|$ is null (see [Oxt2, Th 3.20]). Hence any non-null measurable set contains a density point. It follows that almost all points of a measurable set T are density points. This is the Lebesgue Density Theorem ([Oxt2, Th 3.20], or [Kucz, Section 3.5]).

2. In [PWW, Th. 2] a category analogue of the Lebesgue's Density Theorem is established. This follows more simply from our Theorem 5.4.

3. For KBD, let $z_n \to 0$ and put $F(T) := \bigcap_{n \in \omega} \bigcup_{m > n} (T - z_m)$. Thus $F(T) \in \mathcal{B}a$ for $T \in \mathcal{B}a$ and F is monotonic. Here $t \in F(T)$ iff there is an infinite $\mathbb{M}_t$ such that $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$. Let us call such a t a *translator* (for $\{z_n\}$ into T). The Generic Dichotomy Principle asserts that once we have proved (for which see Theorem 5.6 below) that an arbitrary non-meagre Baire set T contains a translator, then quasi all elements of T are translators.

Theorem 5.5B (Displacements Lemma – Baire Case). *In a normed group X which is Baire under the right norm topology, for A Baire and non-empty in X and $a \in A$ there is $r = \varepsilon(A, a) > 0$ such that*

$$A \cap A(a^{-1}xa) \text{ is non-meagre for any } x \text{ with } \|x\| < r.$$

If X is a topological group there is $r = \delta(A) > 0$ such that

$$A \cap Ax \text{ is non-meagre for any } x \text{ with } \|x\| < r.$$

Proof. We work first in a normed group under its right norm topology. Thus right-shifts $\rho_t(x) := xt$ and their inverses $\rho_{t^{-1}}(x)$ are uniformly continuous. Hence, for any t, the set A is Baire iff its shift At is Baire. Since the conclusion of the lemma is inherited by supersets, we may assume without loss of generality that $A = U \backslash M$ with M meagre and U open and non-empty. Suppose that $a \in A$. Taking $y = a^{-1}$, we have $e = ay \in Ay = Uy \backslash My = V \backslash N$ where $V = Uy$ and $N = My$, which are respectively open and meagre (since ρ_y is a

homeomorphism). Now for some $r > 0$, $V \supseteq B_r(e_X)$.
Thus for $x \in B_r(e_X)\backslash N$ we have $\|x\| < r$ and $x \notin N$ and, as $e \in V\backslash N$ and $B_r(x) = B_r(e_X)x$, we have

$$\begin{aligned}(B_r(e_X)\backslash N) \cap (B_r(x)\backslash Nx) &\subseteq (V\backslash N) \cap (Vx\backslash Nx) = (Uy\backslash My) \cap (Uyx\backslash Myx) \\ &\subseteq Ay \cap Ayx.\end{aligned}$$

Moreover if the intersection $L := B_r(e_X)\backslash N \cap B_r(x)\backslash Nx$ is meagre, then, for $s < \min\{\|x\|, r - \|x\|\}$ we have, as $s + \|x\| < r$ that

$$B_s(e_X) \subseteq B_r(e_X) \cap B_r(x) \subseteq L \cup N \cup Nx,$$

so that $B_s(e_X)$ is meagre, a contradiction. Thus $Ay \cap Ayx$ is Baire non-meagre, for any x with $\|x\| < r$, and hence also $A \cap Aa^{-1}xa$.
We now suppose that X is a topological group and deduce the final assertion. Fix $a \in A$. The automorphism $x \to a^{-1}xa$ and its inverse $y \to aya^{-1}$ are now continuous at e (Theorem 3.4); so for some $\delta > 0$, if $\|y\| \leq \delta$, putting $x = aya^{-1}$ we have $\|x\| \leq \varepsilon(A, a)$ and so

$$A \cap Ay \text{ is non-meagre for any } y \text{ with } \|y\| < \delta.$$

■

Theorem 5.5M (Displacements Lemma – measure case; [Kem] Th. 2.1 in $\mathbb{R}^d$ with $B_i = E$, $a_i = t$, [WKh]). *In a locally compact metric group with right-invariant Haar measure μ, if E is non-null Borel, then $f(x) := \mu[E \cap (E + x)]$ is continuous at $x = e_X$, and so for some $\varepsilon = \varepsilon(E) > 0$*

$$E \cap (Ex) \text{ is non-null, for } \|x\| < \varepsilon.$$

Proof. Apply Theorem 61.A of [Hal-M, Ch. XII, p. 266], which asserts that $f(x)$ is continuous. ■

Theorem 5.6 (Generalized BHW Lemma – Existence of sequence embedding; cf. [BHW, Lemma 2.2]). *In a normed group (resp. locally compact metrizable topological group) X, for A almost complete Baire non-meagre (resp. non-null measurable) and a null sequence $z_n \to e_X$, there exist $t \in A$, an infinite $\mathbb{M}_t$ and points $t_m \in A$ such that $t_m \to t$ and*

$$\{tt_m^{-1}z_m t_m : m \in \mathbb{M}_t\} \subseteq A.$$

If X is a topological group, then there exist $t \in A$ and an infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq A.$$

Proof. The result is upward hereditary, so without loss of generality we may assume that A is topologically complete Baire non-meagre (resp. measurable non-null) and completely metrizable, say under a metric $\rho = \rho_A$. (For A measurable non-null we may pass down to a compact non-null subset, and for A Baire non-meagre we simply take away a meagre set to leave a Baire non-meagre $\mathcal{G}_\delta$ subset; then A as a metrizable space is complete –

cf. [Eng, 4.3.23].) Since this is an equivalent metric, for each $a \in A$ and $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that $B_\delta(a) \subseteq B^\rho_\varepsilon(a)$, where $B_\delta(a)$ refers to the metric d^X_R.) Thus, by taking $\varepsilon = 2^{-n-1}$ the δ-ball $B_\delta(a)$ has ρ-diameter less than 2^{-n}.
Working inductively in a normed-group setting, we define non-empty open subsets of A (of possible translators) B_n of ρ-diameter less than 2^{-n} as follows; they are of course Baire subsets of X. With $n = 0$, we take $B_0 = A$. Given n and B_n open in A, choose $b_n \in B_n$ and N such that $\|z_k\| < \min\{\frac{1}{2}\|x_n\|, \varepsilon(B_n)\}$, for all $k > N$. Let $x_n := z_N \in Z$; then by the Displacements Lemma $B_n \cap (B_n b_n^{-1} x_n^{-1} b_n)$ is non-empty (and open). We may now choose a non-empty subset B_{n+1} of A which is open in A with ρ-diameter less than 2^{-n-1} such that $\mathrm{cl}_A B_{n+1} \subset B_n \cap (B_n b_n^{-1} x_n^{-1} b_n) \subseteq B_n$. By completeness, the intersection $\bigcap_{n\in\mathbb{N}} B_n$ is non-empty. Let

$$t \in \bigcap\nolimits_{n\in\mathbb{N}} B_n \subset A.$$

Now $tb_n^{-1}x_n b_n \in B_n \subset A$, as $t \in B_{n+1} \subset B_n b_n^{-1} x_n^{-1} b_n$, for each n. Hence $\mathbb{M} := \{m : z_m = x_n \text{ for some } n \in \mathbb{N}\}$ is infinite. Now $b_n \in B_n$ so $b_n \to_R t$, so $w_n := b_n t^{-1} \to e$. Thus $tb_n^{-1}x_n b_n = w_n^{-1} x_n w_n t$, as $b_n = w_n t$. Moreover, if $z_m = x_n$, then adjusting the notation we may write either

$$\{tt_m^{-1} z_m t_m : m \in \mathbb{M}_t\} \subseteq A,$$

or

$$\{w_m^{-1} z_m w_m t : m \in \mathbb{M}_t\} \subseteq A.$$

The latter shows that the right-shift ρ_t underlies the conclusion of the theorem and not a left-shift.
As for the topological group setting, the Displacements Lemma shows that we may pass to the final conclusion by substituting e for b_n to obtain

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq A.$$

■

We now apply Theorem 5.3 (Generic Dichotomy) to extend Theorem 5.6 from an existence to a genericity statement, thus completing the proof of Theorem 5.1.

THEOREM 5.7 (Genericity of sequence embedding). *In a normed topological group (resp. locally compact metric toplogical group) X, for $T \subseteq X$ almost complete in category (resp. measure) and $z_n \to e_X$, for generically all $t \in T$ there exists an infinite $\mathbb{M}_t$ such that*

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Working as usual in d^X_R, the correspondence

$$F(T) := \bigcap\nolimits_{n\in\omega} \bigcup\nolimits_{m>n} (Tz_m^{-1})$$

takes Baire sets T to Baire sets and is monotonic. Here $t \in F(T)$ iff there exists an infinite $\mathbb{M}_t$ such that$\{tz_m : m \in \mathbb{M}_t\} \subseteq T$. By Theorem 5.6 $F(T) \cap T \neq \emptyset$, for T Baire non-meagre, so by Generic Dichotomy $F(T) \cap T$ is quasi all of T (cf. Example 1 above). ■

Remark. For a similar approach to work in the normed group setting we would need to know that the monotone correspondence

$$G(T) := T \cap \bigcap\nolimits_{n\in\omega} \bigcup\nolimits_{m>n} T \cdot g_m(T), \text{ where } g_m(t) := t^{-1} z_m^{-1} t,$$

takes Baire sets to Baire sets. Of course $t \in G(T)$ iff $t \in T$ and $t = t'_m t_m^{-1} z_m^{-1} t_m \in T$ for some $t_m, t'_m \in T$ and so $t t_m^{-1} z_m t_m = t'_m \in T$. To see the difficulty, write $t_m = w_m t$ and compute that

$$\begin{aligned} &t \in G(T) \iff (\forall n)(\exists m > n)\exists\{w_n\}(\exists u, s, t')(\forall k) \\ &\quad \left[t' \in T \ \&\ s \in T \ \&\ d_R(w_k, e) \le 1/k \ \&\ s = w_m t \ \&\ t = t'u \ \&\ u = w_m z_m^{-1} w_m^{-1}\right]. \end{aligned}$$

If the graph of the relation $t = xy$ were analytic, we could deduce that $G(T)$ is analytic (see section 11 for definition) for T a $\mathcal{G}_\delta$ set (all that is needed for Th. 5.4); that in turn guarantees that $G(T)$ is Baire. However, if even the relation $e = xy$ were analytic, this would imply that inversion is continuous and so the normed group would be topological (see Th. 3.41). We can nevertheless say a little more about $G(T)$.

Theorem 5.7A (Non-meagreness of sequence embedding – normed groups). *In a normed group X, for $T \subseteq X$ almost complete in category, U open with $T \cap U$ non-meagre, and $z_n \to e_X$, the set S_U of $t \in T \cap U$ for which there exist points $t_m \in T$ with $t_m \to_R t$ and an infinite $\mathbb{M}_t$ with*

$$\{t t_m^{-1} z_m t_m : m \in \mathbb{M}_t\} \subseteq T$$

is non-meagre.

Proof. Suppose not; then there is an open set U such that S_U is meagre. Letting H be a meagre $\mathcal{F}_\sigma$ cover of S_U, the set $T' := (T\backslash H) \cap U$ is Baire and non-meagre. But then by Th. 5.6 there exists points $t, t_m \in T'$ and infinite set $\mathbb{M}_t$ such that

$$\{t t_m^{-1} z_m t_m : m \in \mathbb{M}_t\} \subseteq T' \subseteq T \cap U,$$

a contradiction. ■

Theorem 5.8 (Squared Pettis Theorem). *Let X be a topologically complete normed group and A Baire non-meagre under the right norm topology. Then e_X is an interior point of $(AA^{-1})^2$.*

Proof. Suppose not. Then we may select $z_n \in B_{1/n}(e)\backslash(AA^{-1})^2$. As $z_n \to e$, we apply Th. 5.6 to A, to find $t \in A$, $\mathbb{M}_t$ infinite and $t_m \in A$ for $m \in \mathbb{M}_t$ such that $t t_m^{-1} z_m t_m \in A$ for all $m \in M_t$. So for $m \in \mathbb{M}_t$

$$z_m \in AA^{-1}AA^{-1} = (AA^{-1})^2,$$

a contradiction. ■

Remarks. 1. See [Fol] for an early use of a similar, doubled 'difference set' and [Hen] for the consequences of higher order versions in connection with uniform boundedness.
2. One might have assumed less and required that A be almost complete; but we have fairly general applications in mind. In fact one may assume almost completeness of X in place of topological completeness. The proof above merely needs the Baire non-meagre set A to contain an almost complete subset, but that turns out to be equivalent to X being almost complete. (See [Ost-LB3] Th. 2 for the separable case and [Ost-AB] for the non-separable case).
3. This one-sided result will be used in Section 11 (Th. 11.11) to show that Borel homomorphisms of topologically complete normed separable groups are continuous. When X is a topological group, there is no need to square (and the order AA^{-1} may be commuted to $A^{-1}A$, since A is then Baire non-meagre iff A is); this follows from Th. 5.6, but we delay this derivation to an alternative bi-topological space setting.

We close this section with a KBD-like result for normed groups. Thereafter we shall be concerned mostly (though not exclusively) with topological normed groups. The result is striking, since under a weak assumption it permits some non-trivial 'left-right transfer'. We do not know whether this assumption implies that the normed group is topological. We need a definition.

Definition. Say that a group-norm is *density-preserving* if under one (or other) of the norm topologies, for each dense set D, the set $\gamma_g(D)$ is dense for each conjugacy γ_g. See [Ost-AB] for an application.

Note that D is dense in X under d_R iff D^{-1} is dense in X under d_L, since $d_R(x,d) = d_L(x^{-1}, d^{-1})$. Thus density preservation under d_R is equivalent to density preservation under d_L.

Proposition 5.9. *If the group-norm on X is density-preserving, then under the right norm topology the left-shift gD of any dense set D is dense. Likewise for the left norm density and right-shifts.*

Proof. Fix a dense set D, a point g, and $\varepsilon > 0$. For any $x \in X$, put $y = xg^{-1}$. Since $\gamma_g(D)$ is dense we may find $d \in D$ such that $d_R(y, gdg^{-1}) < \varepsilon$; then $d_R(x, gd) = d_R(yg, gd) = d_R(y, gdg^{-1}) < \varepsilon$. Thus gD is dense. ∎

Remarks. 1. The result shows that density preservation can be defined equivalently by reference to appropriate shifts.
2. If D^{-1} is dense under d_L, then so is aD^{-1} (since $\lambda_a(t)$ is a homeomorphism). However, this does not mean that aD is dense under d_R, so the definition of density preservation asks for more.

In the theorem below D is dense under d_R; this means that D^{-1}, and so each xD^{-1}, are dense under d_L. The surprise is that, for quasi all x, xD^{-1} is also dense under d_R.

Theorem 5.10 (Generic Density Theorem, [HJ, Th.2.3.7]). *Let X be Baire under its right norm topology with a density-preserving norm. For A co-meagre in X and D countable and dense under d_R*

$$\{x \in A : (xD^{-1}) \cap A \textit{ is dense in } X\}$$

is co-meagre in X.

Proof. For each x, the set Ax is co-meagre as $\rho_x(t) = tx$ is a homeomorphism. Hence

$$B = A \cap \bigcap\nolimits_{d \in D} Ad$$

is co-meagre, as D is countable. Thus for $x \in B$ and $d \in D$ we have $xd^{-1} \in A$. Now let V be open with $a \in B_r(a) = B_r(e)a \subset V$. Let $x \in X$. We claim that there is $d \in D$ such that $x \in B_r(a)d$. By assumption aD is dense, so there is $ad \in B_r(x) = B_r(e)x$. Put $ad = zx$ with $\|z\| < r$. Then $x = z^{-1}ad \in B_r(a)d$, as claimed. Thus $v := xd^{-1} = z^{-1}a \in V$ and so for $x \in B$ we have

$$v = xd^{-1} \in (xD^{-1}) \cap A \cap V.$$

That is, $(xD^{-1}) \cap A$ is dense in X, for x in the co-meagre set B. ■

6. Steinhaus theory and Dichotomy

If ψ_n converges to the identity, then, for large n, each ψ_n is almost an isometry. Indeed, as we shall see in Section 12, by Proposition 12.5, we have

$$d(x, y) - 2\|\psi_n\| \le d(\psi_n(x), \psi_n(y)) \le d(x, y) + 2\|\psi_n\|.$$

This motivates our next result; we need to recall a definition and the Category Embedding Theorem from [BOst-LBII], whose proof we reproduce here for completeness.

Definition (Weak category convergence). A sequence of homeomorphisms ψ_n satisfies the *weak category convergence* condition (wcc) if:
For any non-empty open set U, there is a non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap\nolimits_{n \ge k} V \backslash \psi_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \ge k)\ \psi_n(t) \in V.$$

For this 'convergence to the identity' form, see [BOst-LBII].

Theorem 6.1 (Category Embedding Theorem, CET). *Let X be a topological space. Suppose given homeomorphisms $\psi_n : X \to X$ for which the weak category convergence condition (wcc) is met. Then, for any non-meagre Baire set T, for quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{\psi_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Take T Baire and non-meagre. We may assume that $T = U\backslash M$ with U non-empty and open and M meagre. Let $V \subseteq U$ satisfy (wcc). Since the functions h_n are homeomorphisms, the set

$$M' := M \cup \bigcup\nolimits_n h_n^{-1}(M)$$

is meagre. Writing 'i.o.' for 'infinitely often', put

$$\begin{aligned} W = \mathbf{h}(V) &:= \bigcap\nolimits_{k\in\omega} \bigcup\nolimits_{n\geq k} V \cap h_n^{-1}(V) = \limsup[h_n^{-1}(V) \cap V] \\ &= \{x : x \in h_n^{-1}(V) \cap V \text{ i.o.}\} \subseteq V \subseteq U. \end{aligned}$$

So for $t \in W$ we have $t \in V$ and

$$v_m := h_m(t) \in V, \tag{*}$$

for infinitely many m – for $m \in \mathbb{M}_t$, say. Now W is co-meagre in V. Indeed

$$V\backslash W = \bigcup\nolimits_{k\in\omega} \bigcap\nolimits_{n\geq k} V\backslash h_n^{-1}(V),$$

which by (wcc) is meagre.

Take $t \in W\backslash M' \subseteq U\backslash M = T$, as $V \subseteq U$ and $M \subseteq M'$. Thus $t \in T$. For $m \in \mathbb{M}_t$, we have $t \notin h_m^{-1}(M)$, since $t \notin M'$ and $h_m^{-1}(M) \subseteq M'$; but $v_m = h_m(t)$ so $v_m \notin M$. By (*), $v_m \in V\backslash M \subseteq U\backslash M = T$. Thus $\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T$ for t in a co-meagre set.

To deduce that quasi-all $t \in T$ satisfy the conclusion of the theorem, put $S := T\backslash \mathbf{h}(T)$; then S is Baire and $S \cap \mathbf{h}(T) = \emptyset$. If S is non-meagre, then by the preceeding argument there are $s \in S$ and an infinite $\mathbb{M}_s$ such that $\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S$, i.e. $s \in \mathbf{h}(S) \subseteq \mathbf{h}(T)$, a contradiction. (This last step is an implicit appeal to a generic dichotomy – see Th. 5.4.) ■

Examples. In $\mathbb{R}$ we may consider $\psi_n(t) = t + z_n$ with $z_n \to z_0 := 0$. It is shown in [BOst-LBII] that for this sequence the condition (wcc) is satisfied in both the usual topology and the density topology on $\mathbb{R}$. This remains true in $\mathbb{R}^d$, where the specific instance of the theorem is referred to as the Kestelman-Borwein-Ditor Theorem; see the next section ([Kes], [BoDi]; compare also the Oxtoby-Hoffmann-Jørgensen zero-one law for Baire groups, [HJ, p. 356], [Oxt1, p. 85], cf. [RR-01]). In fact in any metrizable group X with right-invariant metric d^X, for a null sequence tending to the identity $z_n \to z_0 := e_X$, the mapping defined by $\psi_n(x) = z_n x$ converges to the identity (see [BOst-TRI], Corollary to Ford's Theorem); here too (wcc) holds. This follows from the next result, which extends the proof of [BOst-LBII]; cf. Theorem 7.5.

THEOREM 6.2 (First Verification Theorem for weak category convergence). *For (X, d) a metric space, if ψ_n converges to the identity under $\hat{d} = \hat{d}^{\mathcal{H}}$, then ψ_n satisfies the weak category convergence condition (wcc).*

Proof. It is more convenient to prove the equivalent statement that ψ_n^{-1} satisfies the category convergence condition.
Put $z_n = \psi_n(z_0)$, so that $z_n \to z_0$. Let k be given.
Suppose that $y \in B_\varepsilon(z_0)$, i.e. $r = d(y, z_0) < \varepsilon$. For some $N > k$, we have $\varepsilon_n = d(\psi_n, id) < \frac{1}{3}(\varepsilon - r)$, for all $n \geq N$. Now

$$\begin{aligned} d(y, z_n) &\leq d(y, z_0) + d(z_0, z_n) \\ &= d(y, z_0) + d(z_0, \psi_n(z_0)) \leq r + \varepsilon_n. \end{aligned}$$

For $y = \psi_n(x)$ and $n \geq N$,

$$\begin{aligned} d(z_0, x) &\leq d(z_0, z_n) + d(z_n, y) + d(y, x) \\ &= d(z_0, z_n) + d(z_n, y) + d(x, \psi_n(x)) \\ &\leq \varepsilon_n + (r + \varepsilon_n) + \varepsilon_n < \varepsilon. \end{aligned}$$

So $x \in B_\varepsilon(z_0)$, giving $y \in \psi_n(B_\varepsilon(z_0))$. Thus

$$y \notin \bigcap\nolimits_{n \geq N} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)) \supseteq \bigcap\nolimits_{n \geq k} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)).$$

It now follows that

$$\bigcap\nolimits_{n \geq k} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)) = \emptyset.$$

■

Our next result serves a cautionary purpose: the subsequent Remark shows that an application of the Category Embedding Theorem (Th. 6.1) to shifts under the norm topology needs X to be a topological group, rather than a normed group.

THEOREM 6.3. *Let X be a normed group.*
(i) Under the right norm topology of d_R the homeomorphisms $\rho_n(x) := xz_n$ converge under $\hat{d}_R$ to the identity for all $z_n \to e$ iff X is a topological group.
(ii) The commutator condition that for any x and any null sequence z_n, $[z_n, x]_R := z_n x z_n^{-1} x^{-1} \to e$ as $n \to \infty$, implies that X is a topological group.

Proof. (i) The right-shifts $\rho_n(x) := xz_n$ are continuous, as $d_R(xz_n, yz_n) = d_R(x, y)$. Now

$$\|\rho_n\| \to 0 \text{ iff } \sup\nolimits_g d_R(gz_n, g) \to 0 \text{ iff } \|g_n z_n g_n^{-1}\| \to 0 \text{ for any } g_n.$$

Thus in particular, if ρ_n converges to the identity for each null sequence $z_n \to e$, we have $gz_n g^{-1} \to e$ for each g, i.e. each conjugacy is continuous; thus X is a topological group by Theorem 3.4 (Equivalence Theorem).
(ii) This is immediate from the corollary on commutators to Th. 3.4 (via Lemma 2.21), but may also be proved directly as follows. Let $x \in X$ and let $z_n \to e_X$. Since inversion

is continuous at the identity, the commutator condition has the equivalent formulation that $(xz_nx^{-1})z_n^{-1} \to e$, and this combined with the triangle inequality

$$\|xz_nx^{-1}\| = \|xz_nx^{-1}z_n^{-1}z_n\| \leq \|xz_nx^{-1}z_n^{-1}\| + \|z_n^{-1}\|$$

implies that $\gamma_x(z)$ is continuous at $z = e$. As x is arbitrary Theorem 3.4 again implies that X is a topological group. ■

Remark. Let X be given the right norm topology and let $z_n \to e$. For the homeomorphism $\psi_n(x) = \rho_n(x) = xz_n$ one has

$$\bigcap\nolimits_{n\geq k} B_\varepsilon(e_X)\backslash\psi_n(B_\varepsilon(e)) = \emptyset,\ \ k = 1, 2, ...$$

Nevertheless, we cannot deduce from here that

$$\bigcap\nolimits_{n\geq k} B_\varepsilon(x)\backslash\psi_n(B_\varepsilon(x)) = \emptyset.$$

The obstruction is that

$$\psi_n(B_\varepsilon(x)) = B_\varepsilon(x)z_n = B_\varepsilon(e)xz_n \neq B_\varepsilon(e)z_nx.$$

A natural argument that fails is to say that for $z \in B_\varepsilon(x)$ with $z = yx$ one has $d_R(yx, x) = \|y\| < \varepsilon$ and so for n large

$$d_R(yx, z_nx) \leq d_R(yx, x) + d_R(x, z_nx) = \|y\| + \|z_n\| < \varepsilon.$$

But this gives only that $z = yx \in B_\varepsilon(z_nx) = B_\varepsilon(e)z_nx$ rather than in $z \in B_\varepsilon(e)xz_n$. Thus one is tempted to finesse this difficulty by requiring additionally that, for fixed x, $z_nxz_n^{-1}x^{-1} \to e$ as $n \to \infty$, on the grounds that for large n

$$d_R(yx, xz_n) \leq d_R(yx, x) + d_R(x, z_nx) + d_R(z_nx, xz_n) < \varepsilon.$$

This does indeed yields $z = yx \in B_\varepsilon(xz_n) = B_\varepsilon(x)z_n = \psi_n(B_\varepsilon(x))$, as desired. However, the assertion (ii) shows that we have appealed to a topological group structure.
We mention that the expected modification of the above argument becomes valid under the ambidextrous topology generated by $d_S := \max\{d_R, d_L\}$. However, shifts are not then guaranteed to be continuous.

As a first corollary we have the following topological result; we deduce later, also as corollaries, measure-theoretic versions in Theorems 7.6 and 11.14. Here in the left-sided category variant we refer to the left-shifts $\psi_n(t) = z_nt$ which converge to the identity under a right-invariant metric, but, as we also need these shifts to be homeomorphisms (so right-to-right continuous in the sense of Section 3), it is necessary to require the normed group to be a topological group – by the last Remark. We thus obtain here a weakened result. (Note that 'normed topological group' is synonymous with 'metric group'.)

Corollary 6.4 (Topological Kestelman-Borwein-Ditor Theorem). *In a normed topological group X let $\{z_n\} \to e_X$ be a null sequence. If T is a Baire subset of X, then for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{z_mt : m \in \mathbb{M}_t\} \subseteq T.$$

Likewise, for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Apply Th. 6.2, taking for d a right-invariant metric, d_R^X say; the continuous maps $\psi_n(t) = z_n t$ satisfy $d_R^X(z_n t, t) = \|z_n\|_{\mathcal{H}} \to 0$, so converge to the identity. Likewise taking for d a left-invariant metric d_L^X say, the continuous maps $\psi_n(t) = tz_n$ satisfy $d_R^X(tz_n, t) = \|z_n\|_{\mathcal{H}} \to 0$, so again converge to the identity. ∎

As a corollary of the KBD Theorem of Section 5 (Th. 5.1) we have the following important result known for topological groups (see [RR-TG], Rogers [Jay-Rog, p. 48], and [Kom1] for the topological vector space setting) and here proved in the metric setting.

Theorem 6.5 (Piccard-Pettis Theorem – Piccard [Pic1], [Pic2], Pettis [Pet1], [RR-TG] cf. [BOst-TRII]). *In a normed topological group whose norm topology is Baire: for A Baire and non-meagre (in the norm topology), the sets AA^{-1} and $A^{-1}A$ both have non-empty interior.*

Proof. Suppose otherwise. We work first with the right-invariant metric $d_R(x, y) = \|xy^{-1}\|$ and assume A^{-1} is Baire non-meagre in the right-norm topology. Consider the set $A^{-1}A$. Suppose the conclusion fails for $A^{-1}A$, for each integer $n = 1, 2, ...$ there is $z_n \in B_{1/n}(e) \backslash A^{-1}A$; hence $z_n \to z_0 = e$. Applying either the KBD Theorem for topologically complete normed groups or its variant for topological groups, there is $a \in A$ such that for infinitely many n

$$az_n \in A, \text{ or } z_n \in A^{-1}A,$$

a contradiction. Thus, for some n, the open ball $B_{1/n}(e)$ is contained in $A^{-1}A$. We next consider the set AA^{-1}. As the inversion mapping $x \to x^{-1}$ is a homeomorphism (in fact an isometry, see Prop. 2.3) from the right- to the left-norm topology, the set A is Baire non-meagre in the left-norm topology iff A^{-1} is Baire non-meagre in the right-norm toplogy. But the inversion mapping carries the ball $B_{1/n}(e)$ into itself, and so we may now conclude that AA^{-1} contains an open ball $B_{1/n}(e)$, as $(A^{-1})^{-1} = A$. ∎

One says that a set A is *thick* if e is an interior point of AA^{-1} (see e.g. [HJ, Section 3.4]). The next result (proved essentially by the same means) applied to the additive group $\mathbb{R}$ implies the Kestelman-Borwein-Ditor ([BOst-LBII]) theorem on the line. The name used here refers to a similar (weaker) property studied in Probability Theory (in the context of probabilities regarded as a semigroup under convolution, for which see [PRV], or [Par, 3.2 and 3.5], [BlHe], [Hey]). We need a definition.

Definition (cf.[BOst-StOstr]). In a normed topological group G, say that a set A is *(properly) right-shift compact, resp. strongly right-shift compact* if, for any sequence of points a_n in A, (resp. in G) there is a point t and a subsequence $\{a_n : n \in \mathbb{M}_t\}$ such that $a_n t$ lies entirely in A and converges through $\mathbb{M}_t$ to a point $a_0 t$ in A; similarly for *left-shift* *compact.* Evidently, finite Cartesian products of shift-compact sets are shift-compact. Thus a right-shift compact set A is precompact. (If the subsequence $a_m t$ converges to $a_0 t$, for m in $\mathbb{M}_t$, then likewise a_m converges to a_0, for m in $\mathbb{M}_t$.)

PROPOSITION 6.6. *In a normed topological group, if a subgroup S is locally right-shift compact, then S is closed and locally compact. Conversely, a closed, locally compact subgroup is locally right-shift compact.*

Proof. Suppose that $a_n \to a_0$ with $a_n \in S$. If $a_m t \to a_0 t \in S$ down a subset M, then $a_0 t(a_m t)^{-1} = a_0 a_m^{-1} \in S$ for $m \in M$. Hence also $a_0 = a_0 a_m^{-1} a_m \in S$ for $m \in M$. Thus S is closed. ∎

EXAMPLE. In the additive group $\mathbb{R}$, the subgroup $\mathbb{Z}$ is closed and locally compact, so shift-compact. Of course, $\mathbb{Z}$ is too small to contain shifts of arbitrary null sequences. We return to this matter in the remarks after Th. 7.7, where we distinguish between proper shift-compactness as here (so that we are concerned only with sequences in a given set) and null-shift compactness where we are concerned with shifting subsequences of arbitrary sequences into a given set.

EXAMPLE. Note that $A \subseteq \mathbb{R}$ is density-open (open in the density topology) iff each point of A is a density point of A. Suppose a_0 is a limit point (in the usual topology) of such a set A; then, for any $\varepsilon > 0$, we may find a point $\alpha \in A$ within $\varepsilon/2$ of a_0 and hence some $t \in A$ within $\varepsilon/2$ of the point α such that some subsequence $t + a_m$ is included in A, with limit $t + a_0$ and with $|t| < \varepsilon$. That is, a density-open set is strongly shift-compact.

REMARK. Suppose that $a_n = (a_n^i) \in A = \prod A_i$. Pick t^i and inductively infinite $\mathbb{M}_i \subseteq \mathbb{M}_{i-1}$ so that $a_n^i t^i \to a_0^i t^i$ along $n \in \mathbb{M}_i$ with $a_n^i t^i \in A_i$ for $n \in \omega$. Diagonalize $\mathbb{M}_i$ by setting $\mathbb{M} := \{m_i\}$, where $m_{n+1} = \min\{m \in \mathbb{M}_{n+1} : m > m_n\}$. Then the subsequence $\{a_m : m \in \mathbb{M}\}$ satisfies, for each J finite,

$$\mathrm{pr}_J a_m t \subseteq \prod\nolimits_{j \in J} A_j \text{ for eventually all } m \in \mathbb{M},$$

and so in the product topology $a_m t \to a_0 t$ through $\mathbb{M}$, where $(a^i)(t^i)$ is defined to be $(a^i t^i)$.

THEOREM 6.7 (Shift-Compactness Theorem). *In a normed topological group G, for A precompact, Baire and non-meagre, the set A is properly right-shift compact, i.e., for any sequence $a_n \in A$, there are $t \in G$ and $a \in A$ such that $a_n t \in A$ and $a_n t \to a$ down a subsequence. Likewise the set A is left-shift compact.*

Proof. First suppose $a_n \in A \subseteq \bar{A}$ with $\bar{A}$ compact. Without loss of generality $a_n \to a_0 \in \bar{A}$. Hence $z_n := a_n a_0^{-1} \to e_G$. By Theorem 6.2 (the First Verification Theorem), $\psi_n(x) := z_n x$ converges to the identity. Hence, for some $a \in A$ and infinite $\mathbb{M}$, we have $\{z_m a : m \in \mathbb{M}\} \subseteq A$. Taking $t = a_0^{-1} a$, we thus have $a_n t \in A$ and $a_n t \to a \in A$ along $\mathbb{M}$. Replace A by A^{-1} to obtain the other-handed result. ∎

The following theorem asserts that a 'covering property modulo shift' is satisfied by bounded shift-compact sets. It will be convenient to make the following

Definitions. 1. Say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *shift-covers* X, or is a *shifted-cover* of X if, for some $d_1, ..., d_h$ in G,

$$(D_1 - d_1) \cup ... \cup (D_h - d_h) = X.$$

Say that X is *compactly shift-covered* if every open cover $\mathcal{U}$ of X contains a finite subfamily $\mathcal{D}$ which shift-covers X.

2. For N a neighbourhood of e_X say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *N-strongly shift-covers* A, or is an *N-strong shifted-cover* of A, if there are $d_1, ..., d_h$ in N such that

$$(D_1 - d_1) \cup ... \cup (D_h - d_h) \supseteq A.$$

Say that A is *compactly strongly shift-covered*, or *compactly shift-covered with arbitrarily small shifts* if every open cover $\mathcal{U}$ of A contains for each neighbourhood N of e_X a finite subfamily $\mathcal{D}$ which N-strongly shift-covers A.

Theorem 6.8 (Compactness Theorem – modulo shift, [BOst-StOstr]). *Let A be a right-shift compact subset of a separable normed topological group G. Then A is compactly shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, and for each member of $\mathcal{V}$ a translator, such that the corresponding translates of $\mathcal{V}$ cover A.*

Proof. Let $\mathcal{U}$ be an open cover of A. Since G is second-countable we may assume that $\mathcal{U}$ is a countable family. Write $\mathcal{U} = \{U_i : i \in \omega\}$. Let $Q = \{q_j : j \in \omega\}$ enumerate a dense subset of G. Suppose, contrary to the assertion, that there is no finite subset $\mathcal{V}$ of $\mathcal{U}$ such that elements of $\mathcal{V}$, translated each by a corresponding member of Q, cover A. For each n, choose $a_n \in A$ not covered by $\{U_i - q_j : i, j < n\}$. As noted earlier, A is precompact, so we may assume, by passing to a subsequence (if necessary), that a_n converges to some point a_0, and also that, for some t, the sequence $a_n t$ lies entirely in A. Let U_i in $\mathcal{U}$ cover $a_0 t$. Without loss of generality we may assume that $a_n t \in U_i$ for all n. Thus $a_n \in U_i t^{-1}$ for all n. Thus we may select $V := U_i q_j$ to be a translation of U_i such that $a_n \in V = U_i q_j$ for all n. But this is a contradiction, since a_n is not covered by $\{U_{i'} q_{j'} : i', j' < n\}$ for $n > \max\{i, j\}$. ∎

The above proof of the compactness theorem for shift-covering may be improved to strong shift-covering, with only a minor modification (replacing Q with a set $Q^\varepsilon = \{q_j^\varepsilon :$

$j \in \omega\}$ which enumerates, for given $\varepsilon > 0$, a dense subset of the ε ball about e), yielding the following.

THEOREM 6.9 (Strong Compactness Theorem – modulo shift, cf. [BOst-StOstr]). *Let A be a strongly right-shift compact subset of a separable normed topological group G. Then A is compactly strongly shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A, and any neighbourhood of e_X there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, and for each member of $\mathcal{V}$ a translator in N such that the corresponding translates of $\mathcal{V}$ cover A.*

Next we turn to the Steinhaus theorem, which we will derive in Section 8 (Th. 8.3) more directly as a corollary of the Category Embedding Theorem. For completeness we recall in the proof below its connection with the Weil topology introduced in [We].

DEFINITIONS ([Hal-M, Section 72, p. 257 and 273]). 1. A *measurable group* $(X, \mathcal{S}, m)$ is a σ-finite measure space with X a group and m a non-trivial measure such that both $\mathcal{S}$ and m are left-invariant and the mapping $x \to (x, xy)$ is measurability preserving.
2. A measurable group X is *separated* if for each $x \neq e_X$ in X, there is a measurable $E \subset X$ of finite, positive measure such that $\mu(E \triangle xE) > 0$.

THEOREM 6.10 (Steinhaus Theorem – cf. Comfort [Com, Th. 4.6 p. 1175]). *Let X be a locally compact topological group which is separated under its Haar measure. For measurable A having positive finite Haar measure, the sets AA^{-1} and $A^{-1}A$ have non-empty interior.*

Proof. For X separated, we recall (see [Hal-M, Sect. 62] and [We]) that the Weil topology on X, under which X is a topological group, is generated by the neighbourhood base at e_X comprising sets of the form $N_{E,\varepsilon} := \{x \in X : \mu(E \triangle xE) < \varepsilon\}$, with $\epsilon > 0$ and E measurable and of finite positive measure. Recall from [Hal-M, Sect. 62] the following results: (Th. F) a measurable set with non-empty interior has positive measure; (Th. A) a set of positive measure contains a set of the form GG^{-1}, with G measurable and of finite, positive measure; and (Th. B) for such G, $N_{G\varepsilon} \subseteq GG^{-1}$ for all small enough $\varepsilon > 0$. Thus a measurable set has positive measure iff it is non-meagre in the Weil topology. Thus if A is measurable and has positive measure it is non-meagre in the Weil topology. Moreover, by [Hal-M] Sect 61, Sect. 62 Ths. A and B, the metric open sets of X are generated by sets of the form $N_{E,\varepsilon}$ for some Borelian-($\mathcal{K}$) set E of positive, finite measure. By the Piccard-Pettis Theorem, Th. 6.3 (from the Category Embedding Theorem, Th. 6.1) AA^{-1} contains a non-empty Weil neighbourhood $N_{E,\varepsilon}$. ■

REMARK. See Section 7 below for an alternative proof via the density topology drawing on Mueller's Haar-measure density theorem [Mue] and a category-measure theorem of Martin [Mar] (and also for extensions to products AB). The following theorem has two versions: a normed topological group version, immediately following, and a normed group version given in Theorem 6.13; the proofs are rather different.

Theorem 6.11 (Subgroup Dichotomy Theorem – normed topological groups, Banach-Kuratowski Theorem – [Ban-G, Satz 1], [Kur-1, Ch. VI. 13. XII]; cf. [Kel, Ch. 6 Pblm P] ; cf. [BGT, Cor. 1.1.4] and also [BCS] and [Be] for the measure variant). *Let X be a normed topological group which is non-meagre and A any Baire subgroup. Then A is either meagre or clopen in X.*

Proof. Suppose that A is non-meagre. We show that e is an interior point of A, from which it follows that A is open. Suppose otherwise. Then there is a sequence $z_n \to e$ with $z_n \in B_{1/n}(e)\backslash A$. Now for some $a \in A$ and infinite M we have $z_n a \in A$ for all $n \in M$. But A is a subgroup, hence $z_n = z_n aa^{-1} \in A$ for $n \in M$, a contradiction.
Now suppose that A is not closed. Let a_n be a sequence in A with limit x. Then $a_n x^{-1} \to e$. Now for some $a \in A$ and infinite M we have $z_n x^{-1} a \in A$ for all $n \in M$. But A is a subgroup, so z_n^{-1} and a^{-1} are in A and hence, for all $n \in M$, we have $x^{-1} = z_n^{-1} z_n x^{-1} aa^{-1} \in A$. Hence $x \in A$, as A is a subgroup. ∎

Remark. Banach's proof is purely topological, so applies to topological groups (even though originally stated for metric groups), and relies on the mapping $x \to ax$ being a homeomorphism, likewise Kuratowski's proof, which proceeds via another dichotomy as detailed below. We refer to McShane's proof, cited below, as it yields a slightly more general version.

Theorem 6.12 (Kuratowski-McShane Dichotomy – [Kur-B], [Kur-1], [McSh, Cor. 1]). *Suppose $H \subseteq Auth(X)$ acts transitively on the topological space X, and $Z \subseteq X$ is Baire and has the property that for each $h \in H$*

$$Z = h(Z) \text{ or } Z \cap h(Z) = \emptyset,$$

i.e. under each $h \in H$, either Z is invariant or Z and its image are disjoint. Then, either Z is meagre or it is clopen.

Theorem 6.13 (Subgroup Dichotomy Theorem – normed groups). *In a normed group X, Baire under its norm topology, a Baire non-meagre subgroup is clopen.*

Proof. We work under the right norm topology and denote the subgroup in question S . Let $H := \{\rho_x : x \in X\} \subseteq Auth(X)$. Then as S is a subgroup, for $x \in S$, $\rho_x(S) = S$, and, for $x \notin S$, $\rho_x(S) \cap S = \emptyset$. Hence, by the Kuratowski-McShane Dichotomy (Th. 6.12), as S is non-meagre, it is clopen. ∎

The result below generalizes the category version of the Steinhaus Theorem [St] of 1920, first stated explicitly by Piccard [Pic1] in 1939, and restated in [Pet1] in 1950; in the current form it may be regarded as a 'localized-refinement' of [RR-TG]. We need a definition which extends sequential convergence to continuous convergence.

Definition (cf. [Mon2]). Let $\{\psi_u : u \in I\}$ for I an open interval in $\mathbb{R}$ be a family of homeomorphisms in $\mathcal{H}(X)$. Let $u_0 \in I$. Say that ψ_u *converges to the identity as* $u \to u_0$ if

$$\lim_{u\to u_0} \|\psi_u\| = 0.$$

The setting of the next theorem is quite general: homogeneity (relative to $\mathcal{H}(X)$), i.e. all we require is that any point may be transformed to another by a bounded homeomorphism of (X, d).

Theorem 6.14 (Generalized Piccard-Pettis Theorem: [Pic1], [Pic2], [Pet1], [Pet2], [BGT, Th. 1.1.1], [BOst-StOstr], [RR-TG], cf. [Kel, Ch. 6 Prb. P]). *Let X be a homogenous space. Suppose that the homeomorphisms ψ_u converge to the identity as $u \to u_0$, and that A is Baire and non-meagre. Then, for some $\delta > 0$, we have*

$$A \cap \psi_u(A) \neq \emptyset, \text{ for all } u \text{ with } d(u, u_0) < \delta,$$

or, equivalently, for some $\delta > 0$

$$A \cap \psi_u^{-1}(A) \neq \emptyset, \text{ for all } u \text{ with } d(u, u_0) < \delta.$$

Proof. We may suppose that $A = V\backslash M$ with M meagre and V open. Hence, for any $v \in V\backslash M$, there is some $\varepsilon > 0$ with

$$B_\varepsilon(v) \subseteq U.$$

As $\psi_u \to id$, there is $\delta > 0$ such that, for u with $d(u, u_0) < \delta$, we have

$$\hat{d}(\psi_u, id) < \varepsilon/2.$$

Hence, for any such u and any y in $B_{\varepsilon/2}(v)$, we have

$$d(\psi_u(y), y) < \varepsilon/2.$$

So

$$W := \psi_u(B_{\varepsilon/2}(z_0)) \cap B_{\varepsilon/2}(z_0) \neq \emptyset,$$

and

$$W' := \psi_u^{-1}(B_{\varepsilon/2}(z_0)) \cap B_{\varepsilon/2}(z_0) \neq \emptyset.$$

For fixed u with $d(u, u_0) < \delta$, the set

$$M' := M \cup \psi_u(M) \cup \psi_u^{-1}(M)$$

is meagre. Let $w \in W\backslash M'$ (or $w \in W'\backslash M'$, as the case may be). Since $w \in B_\varepsilon(z_0)\backslash M \subseteq V\backslash M$, we have

$$w \in V\backslash M \subseteq A.$$

Similarly, $w \in \psi_u(B_\varepsilon(z_0))\backslash\psi_u(M) \subseteq \psi_u(V)\backslash\psi_u(M)$. Hence

$$\psi_u^{-1}(w) \in V\backslash M \subseteq A.$$

In this case, as asserted,

$$A \cap \psi_u^{-1}(A) \neq \emptyset.$$

In the other case ($w \in W' \backslash M'$), one obtains similarly

$$\psi_u(w) \in V \backslash M \subseteq A.$$

Here too

$$A \cap \psi_u^{-1}(A) \neq \emptyset.$$

■

Remarks. 1. In the theorem above it is possible to work with a weaker condition, namely local convergence at z_0, where one demands that for some neighbourhood $B_\eta(z_0)$ and some K,

$$d(\psi_u(z), z) \leq K d(u, u_0), \text{ for } z \in B_\eta(z_0).$$

This implies that, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for $z \in B_\delta(z_0)$,

$$d(\psi_u(z), z) < \varepsilon, \text{ for } z \in B_\delta(z_0).$$

2. The Piccard-Pettis Theorem for topological groups (named by Kelley, [Kel, Ch. 6 Pblm P-(b)], the Banach-Kuratowski-Pettis Theorem, say BKPT for short) asserts the category version of the Steinhaus Theorem [St] that, for A Baire and non-meagre, the set $A^{-1}A$ is a neighbourhood of the identity; our version of the Piccard theorem as stated implies this albeit only in the context of metric groups. Let d^X be a right-invariant metric on X and take $\psi_u(x) = ux$ and $u_0 = e$. Then ψ_u converges to the identity (since $\|\psi_u\| := \sup_x d(ux, x) = d(u, e) = \|u\|$), and so the theorem implies that $B_\delta(e) \subseteq A^{-1}A$ for some $\delta > 0$; indeed $a' \in A \cap \psi_u(A)$ for $u \in B_\delta(e)$ means that $a' \in A$ and, for some $a \in A$, also $ua = a'$ so that $u = a^{-1}a' \in A^{-1}A$. It is more correct to name the following important and immediate corollary the BKPT, since it appears in this formulation in [Ban-G], [Kur-1], derived by different means, and was used by Pettis in [Pet1] to deduce his Steinhaus-type theorem.

Theorem 6.15 (McShane's Interior Points Theorem – [McSh, Cor. 3]). *For X a topological space, let $T : X^2 \to X$ be such that $T_a(x) := T(x, a)$ is a self-homeomorphism for each $a \in X$ such that for each pair (x_0, y_0) there is a self-homeomorphism φ with $y_0 = \varphi(x_0)$ satisfying*

$$T(x, \varphi(x)) = T(x_0, y_0), \text{ for all } x \in X.$$

Let A and B be second category with B Baire. Then the image $T(A, B)$ has interior points and there are $A_0 \subseteq A, B_0 \subseteq B$, with $A \backslash A_0$ and $B \backslash B_0$ meagre and $T(A_0, B_0)$ open.

Remark. Despite it very general appearance, Th. 6.15 has little to offer in the normed group context. Indeed for a normed group X with right topology and with $T(x, y) = xy^{-1}$ each section T_a is a homeomorphism, being the right-shift $\rho_{a^{-1}}$, which is a homeomorphism. However, the equation $xy^{-1} = c$ has solution function $y = \varphi(x) = c^{-1}x$, a left-shift, not in general even continuous. The alternative $T(x,y) = xy^{-1}xy^{-1}$ introduces shift operations to the left of the second x.

7. The Kestelman-Borwein-Ditor Theorem: a bitopological approach

In this section we develop a bi-topological approach to a generalization of the KBD Theorem (Th. 1.1). An alternative approach is given in the next section. Let $(X, \mathcal{S}, m)$ be a probability space which is totally finite. Let m^* denote the outer measure

$$m^*(E) := \inf\{m(F) : E \subset F \in \mathcal{S}\}.$$

Let the family $\{K_n(x) : x \in X\} \subset \mathcal{S}$ satisfy (i) $x \in K_n(x)$, (ii) $m(K_n(x)) \to 0$. Relative to a fixed family $\{K_n(x) : x \in X\}$ define the upper and lower (outer) density at x of any set E by

$$\overline{D}^*(E,x) = \sup \limsup_n m^*(E \cap K_n(x))/m(K_n(x)),$$
$$\underline{D}^*(E,x) = \inf \liminf_n m^*(E \cap K_n(x))/m(K_n(x)).$$

By definition $\overline{D}^*(E,x) \geq \underline{D}^*(E,x)$. When equality holds, one says that the density of E exists at x, and the common value is denoted by $D^*(E,x)$. If E is measurable the star associated with the outer measure m^* is omitted. If the density is 1 at x, then x is a *density point*; if the density is 0 at x, then x is a *dispersion point* of E. Say that a *(weak) density theorem* holds for $\{K_n(x) : x \in X\}$ if for every set (every measurable set) A almost every point of A is a density (an outer density) point of A. Martin [Mar] shows that the family

$$\mathcal{U} = \{U : \overline{D}^*(X \backslash U, x) = 0, \text{ for all } x \in U\}$$

forms a topology, the density topology on X, with the following property.

THEOREM 7.1 (Density Topology Theorem). *If a density theorem holds for $\{K_n(x) : x \in X\}$ and U is d-open, then every point of U is a density point of U and so U is measurable. Furthermore, a measurable set such that each point is a density point is d-open.*

We note that the idea of a density topology was introduced slightly earlier by Goffman ([GoWa],[GNN]); see also Tall [T]. It can be traced to the work of Denjoy [Den] in 1915. Recall that a function is approximately continuous in the sense of Denjoy iff it is continuous under the density topology: [LMZ, p. 1].

THEOREM 7.2 (Category-Measure Theorem – [Mar, Th. 4.11]). *Suppose X is a probability space and a density theorem holds for $\{K_n(x) : x \in X\}$. A necessary and sufficient condition that a set be nowhere dense in the d-topology is that it have measure zero. Hence a necessary and sufficient condition that a set be meagre is that it have measure zero. In particular the topological space $(X, \mathcal{U})$ is a Baire space.*

We now see that the preceding theorem is applicable to a Haar measure on a locally compact group X by reference to the following result. Here bounded means pre-compact (covered by a compact set).

THEOREM 7.3 (Haar measure density theorem – [Mue]; cf. [Hal-M, p. 268]). *Let A be a σ-bounded subset and μ a left-invariant Haar measure of a locally compact topological group X. Then there exists a sequence U_n of bounded measurable neigbourhoods of e_X such that $m^*(A \cap U_n x)/m^*(U_n x) \to 1$ for almost all x out of a measurable cover of A.*

COROLLARY 7.4. *In the setting of Theorem 6.4 with A of positive, totally-finite Haar measure, let $(A, \mathcal{S}_A, m_A)$ be the induced probability subspace of X with $m_A(T) = m(S \cap A)/m(A)$ for $T = S \cap A \in \mathcal{S}_A$. Then the density theorem holds in A.*

We now offer a generalization of a result from [BOst-LBII]; cf. Theorem 6.2.

THEOREM 7.5 (Second Verification Theorem for weak category convergence). *Let X be a locally compact topological group with left-invariant Haar measure m. Let V be m-measurable and non-null. For any null sequence $\{z_n\} \to e$ and each $k \in \omega$,*

$$H_k = \bigcap\nolimits_{n \geq k} V \backslash (V \cdot z_n) \text{ is of } m\text{-measure zero, so meagre in the } d\text{-topology.}$$

That is, the sequence $h_n(x) := x z_n^{-1}$ satisfies the weak category convergence condition (wcc)

Proof. Suppose otherwise. We write Vz_n for $V \cdot z_n$, etc. Now, for some k, $m(H_k) > 0$. Write H for H_k. Since $H \subseteq V$, we have, for $n \geq k$, that $\emptyset = H \cap h_n^{-1}(V) = H \cap (Vz_n)$ and so a fortiori $\emptyset = H \cap (Hz_n)$. Let u be a metric density point of H. Thus, for some bounded (Borel) neighbourhood $U_\nu u$ we have

$$m[H \cap U_\nu u] > \frac{3}{4} m[U_\nu u].$$

Fix ν and put

$$\delta = m[U_\nu u].$$

Let $E = H \cap U_\nu u$. For any z_n, we have $m[(Ez_n) \cap U_\nu u z_n] = m[E] > \frac{3}{4}\delta$. By Theorem A of [Hal-M, p. 266], for all large enough n, we have

$$m(U_\nu u \triangle U_\nu u z_n) < \delta/4.$$

Hence, for all n large enough we have $m[(Ez_n) \backslash U_\nu u] \leq \delta/4$. Put $F = (Ez_n) \cap U_\nu u$; then $m[F] > \delta/2$.
But $\delta \geq m[E \cup F] = m[E] + m[F] - m[E \cap F] \geq \frac{3}{4}\delta + \frac{1}{2}\delta - m[E \cap F]$. So

$$m[H \cap (Hz_n)] \geq m[E \cap F] \geq \frac{1}{4}\delta,$$

contradicting $\emptyset = H \cap (Hz_n)$. This establishes the claim. ■

As a corollary of the Category Embedding Theorem, Theorem 6.5 and its Corollary now yield the following result (compare also Th. 10.11).

Theorem 7.6 (First Generalized Kestelman-Borwein Ditor Theorem – Measurable Case). *Let X be a normed locally compact group, $\{z_n\} \to e_X$ be a null sequence in X. If T is Haar measurable and non-null (resp. Baire and non-meagre), then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq T.$$

This theorem in turn yields an important conclusion.

Theorem 7.7 (Kodaira's Theorem – [Kod] Corollary to Satz 18. p. 98, cf. [Com, Th. 4.17 p.1182]). *Let X be a normed locally compact group and $f : X \to Y$ a homorphism into a separable normed group Y. Then f is Haar-measurable iff f is Baire under the density topology iff f is continuous under the norm topology.*

Proof. Suppose that f is measurable. Then under the d-topology f is a Baire function. Hence by the classical Baire Continuity Theorem (see, e.g. Section 11 below, especially Th.11.8), since Y is second-countable, f is continuous on some co-meagre set T. Now suppose that f is not continuous at e_X. Hence, for some $\varepsilon > 0$ and some $z_n \to z_0 = e_X$ (in the sense of the norm on X), we have $\|f(z_n)\| > \varepsilon$, for all n. By the Kestelman-Borwein-Ditor Theorem (Th. 6.1), there is $t \in T$ and an infinite $\mathbb{M}_t$ such that $tz_n \to t = tz_0 \in T$. Hence, for n in $\mathbb{M}_t$, we have

$$f(t)f(z_n) = f(tz_n) \to f(tz_0) = f(t),$$

i.e. $f(z_n) \to e_Y$, a contradiction. ■

Remark. 1. Comfort [Com, Th. 4.17] proves this result for both X and Y locally compact, with the hypothesis that Y is σ-compact and f measurable with respect to the two Haar measures on X and Y. That proof employs Steinhaus' Theorem and the Weil topology. (Under the density topology, Y will not be second-countable.) When Y is metrizable this implies that Y is separable; of course if f is a continuous surjection, Y will be locally compact (cf.[Eng, Th.3.1.10], [Kel, Ch. V Th. 8]).
2. The theorem reduces measurability to the Baire property and in so doing resolves a long-standing issue in the foundations of regular variation; hitherto the theory was established on two alternative foundations employing either measurable functions or Baire functions for its scope, with historical preference for measurable functions in connection with integration. We refer to [BGT] for an exposition of the theory, which characterizes regularly varying functions of either type by a reduction to an underlying homomorphism of the corresponding type relying on its continuity, and then represents either type by very well-behaved functions. Kodaira's Theorem shows that the broader topological class

may be given priority. See in particular [BGT, p. 5,11] and [BOst-LBII].
3. The Kestelman-Borwein-Ditor Theorem inspires the following definitions, which we will find useful in the next section.

Definitions. In a topological group G, following [BOst-FRV], call a set T *subuniversal,* or *null-shift-precompact* as in the more recent paper [BOst-StOstr], if for any null sequence $z_n \to e_G$ there is $t \in G$ and infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T.$$

Call a set T *generically subuniversal* ([BOst-FRV]), or *null-shift-compact* (cf. [BOst-StOstr]), if for any null sequence $z_n \to e_G$ there is $t \in G$ and infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \text{ and } t \in T.$$

Thus the Kestelman-Borwein-Ditor Theorem asserts that a set T which is Baire non-meagre, or measurable non-null, is (generically) subuniversal. The term subuniversal is coined from Kestelman's definition of a set being 'universal for null sequences' ([Kes, Th. 2]) , which required $\mathbb{M}_t$ above to be co-finite rather than infinite. By Theorem 6.7 (Shift-compactness Theorem), a generically subuniversal (null-shift-compact) subset of a normed group is shift-compact. (The definition of 'shift-compact' refers to arbitrary sequences – see Section 6.)

Our final results follow from the First Generalized KBD Theorem (Th. 7.6 above) and are motivated by the literature of *extended* regular variation in which one assumes only that for a function $h : \mathbb{R}_+ \to \mathbb{R}_+$

$$h^*(u) := \limsup_{\|x\|\to\infty} h(ux)h(x)^{-1}$$

is finite on a 'large enough' domain set (see [BOst-RVWL], [BGT] Ch. 2,3 for the classical context of $\mathbb{R}_+^*$). We need the following definitions generalizing their $\mathbb{R}$ counterparts (in [BOst-RVWL]) to the normed group context.

Definitions. 1. Say that $\mathbf{NT}^*(\{T_k\})$ holds, in words No Trumps holds *generically*, if for any null sequence $z_n \to e_X$ there is $k \in \omega$ and an infinite $\mathbb{M}$ such that

$$\{tz_m : m \in \mathbb{M}\} \subset T_k \text{ and } t \in T_k.$$

For the definition of $\mathbf{NT}$ see [BOst-FRV], [BOst-LBI] where bounded, rather than null, sequences z_n appear and the location of the translator t need not be in T_k. [Of course $\mathbf{NT}^*(\{T_k : k \in \omega\})$ implies $\mathbf{NT}(\{T_k : k \in \omega\})$.]
2. For X a normed group, $h : T \to Y$ or $\mathbb{R}_+$, with Y a normed group, put according to context:

$$h^*(u) := \limsup_{\|x\|\to\infty} h(ux)h(x)^{-1} \text{ or } h_Y^*(u) := \limsup_{\|x\|\to\infty} \|h(ux)h(x)^{-1}\|_Y.$$

3. For X a normed group, $h : T \to Y$ or $\mathbb{R}_+$, with $T \subset X$, where Y is a normed group and $\mathbb{R}_+$ refers to the set of positive reals, for $x = \{x_n\}$ with $\|x_n\| \to \infty$, put

$$T_k(x) := \bigcap\nolimits_{n>k} \{t \in T : h(tx_n)h(x_n)^{-1} < n\}$$

or

$$T_k^Y(x) := \bigcap\nolimits_{n>k} \{t \in T : \|h(tx_n)h(x_n)^{-1}\|_Y < n\},$$

according to whether h takes values in $\mathbb{R}_+$ or Y.
Let us say that h is $\mathbf{NT}^*$ on T if for any $x_n \to \infty$ and any null sequence $z_n \to 0$, $\mathbf{NT}^*(\{T_k(x)\})$, resp. $\mathbf{NT}^*(\{T_k^Y(x)\})$, holds.

Theorem 7.8A (Generic No Trumps Theorem or No Trumps* Theorem). *For X a normed topological group, T Baire non-meagre (resp. measurable non-null) and $h : X \to \mathbb{R}_+$ Baire/measurable with $h^*(t) < +\infty$ on T, h is $\mathbf{NT}^*$ on T.*

Proof. The sets $T_k(x)$ are Baire/measurable. Fix $t \in T$. Since $h^*(t) < \infty$ suppose that $h^*(t) < k \in \mathbb{N}$. Then without loss of generality, for all $n > k$, we have $h(tx_n)h(x_n)^{-1} < n$ and so $t \in T_k(x)$. Thus

$$T = \bigcup\nolimits_k T_k(x),$$

and so for some k, the set $T_k(x)$ is Baire non-meagre/measurable non-null. The result now follows from the topological or measurable Kestelman-Borwein-Ditor Theorem (Cor 6.4 or Th. 7.6). ■

The same proof gives:

Theorem 7.8B (Generic No Trumps Theorem or No Trumps* Theorem). *For X, Y normed topological groups, T Baire non-meagre (resp. measurable non-null) and $h : X \to Y$ Baire/measurable with $h_Y^*(t) < +\infty$ on T, h is $\mathbf{NT}^*$ on T.*

We now have two variant generalizations of Theorem 7 of [BOst-RVWL].

Theorem 7.9A (Combinatorial Uniform Boundedness Theorem, cf [Ost-knit]). *In a normed topological group X, for $h : X \to \mathbb{R}_+$ suppose that $h^*(t) < \infty$ on a set T on which h is $\mathbf{NT}^*$. Then for compact $K \subset T$*

$$\limsup\nolimits_{\|x\|\to\infty} \sup\nolimits_{u\in K} h(ux)h(x)^{-1} < \infty.$$

Proof. Suppose not: then for some $\{u_n\} \subset K \subset T$ and $\|x_n\|$ unbounded we have, for all n,

$$h(u_n x_n)h(x_n)^{-1} > n^3. \tag{**}$$

Without loss of generality $u_n \to u \in K$. Now $\|ux_n\| \to \infty$, as $\|x_n\| - \|u\| \le \|ux_n\|$, by the triangle inequality. Thus we may put $y = \{y_n\}$ where $y_n := ux_n$; then

$$T_k(y) := \bigcap\nolimits_{n>k} \{t \in T : h(tux_n)h(ux_n)^{-1} < n\},$$

and $\mathbf{NT}^*(T_k(y))$ holds. Now $z_n := u_n u^{-1}$ is null. So for some $k \in \omega$, $t \in T_k(y)$ and infinite $\mathbb{M}$,

$$\{t(u_m u^{-1}) : m \in \mathbb{M}\} \in T_k(y).$$

So

$$h(tu_m u^{-1}ux_m)h(ux_m)^{-1} < m \text{ and } t \in T.$$

Now $\|u_n x_n\| \to \infty$, as $\|x_n\| - \|u_n\| \le \|u_n x_n\|$ and $\|u_n\|$ is bounded. But $t \in T$ so, as before since $h^*(t) < \infty$, for all n large enough

$$h(tu_n x_n)h(u_n x_n)^{-1} < n.$$

Now also $u \in K \subset T$. So for all n large enough

$$h(ux_n)h(x_n)^{-1} < n.$$

But

$$h(u_n x_n)h(x_n)^{-1} = h(u_n x_n)h(tu_n x_n)^{-1} \times h(tu_n x_n)h(ux_n)^{-1} \times h(ux_n)h(x_n)^{-1}.$$

Then for m large enough and in $\mathbb{M}_t$ we have

$$h(u_m x_m)h(x_m)^{-1} < m^3,$$

a contradiction for such m to (**). ■

We note a generalization with an almost verbatim proof (requiring, mutatis mutandis, the replacement of $h(ux)h(x)^{-1}$ by $\|h(ux)h(x)^{-1}\|$). Note that one cannot deduce Th. 6.7A from this variant by referring to the normed group $Y = \mathbb{R}_+^*$, because the natural norm on $\mathbb{R}_+^*$ is $\|x\|_Y = |\log x|$ (cf. Remarks to Corollary 2.9).

Theorem 7.9B (Combinatorial Uniform Boundedness Theorem). *For $h : X \to Y$ a mapping between normed topological groups and $h_Y^*(u)$ as above, suppose that $h^*(t) < \infty$ on a set T on which h is $\mathbf{NT}^*$. Then for compact $K \subset T$*

$$\limsup\nolimits_{\|x\| \to \infty} \sup\nolimits_{u \in K} \|h(ux)h(x)^{-1}\| < \infty.$$

We may now deduce the result referred to in the remarks to Corollary 2.9, regarding $\pi : X \to Y$ a group homomorphism, by reference to the case $h(x) = \pi(x)$ treated in the Lemma below.

Theorem 7.10 ($\mathbf{NT}^*$ property of quasi-isometry). *If X is a Baire normed topological group and $\pi : X \to Y$ a group homomorphism, where $\|.\|_Y$ is $(\mu$-$\gamma)$-quasi-isometric to $\|.\|_X$ under the mapping π, then for any non-meagre Baire set T, π is $\boldsymbol{NT}^*$ on T.*

Proof. Note that

$$\|h(tx_n)h(x_n)^{-1}\| = \|\pi(tx_n)\pi(x_n)^{-1}\| = \|\pi(t)\|.$$

Hence, as $\pi(e) = e$ (see Examples A4 of Section 2),

$$\{t \in T : h(tx_n)h(x_n)^{-1} < n\} = \{t \in T : \|\pi(t)\| < n\} = B_n^\pi(e),$$

and so

$$\bigcap\nolimits_{n\geq k} T_n(x_n) = \{t \in T : \|\pi(t)\| < k\} = B_k^\pi(e).$$

Now

$$\frac{1}{\mu}\|t\|_X - \gamma \leq \|\pi(t)\|_Y \leq \frac{1}{\mu}\|t\|_X + \gamma,$$

hence $B_n^\pi(e)$ is approximated from above and below by the closed sets $T_n^\pm$:

$$T_n^+ := \{t \in T : \frac{1}{\mu}\|t\|_X + \gamma \leq n\} \subset T(x_n) = B_n^\pi(e) \subset T_n^- := \{t \in T : \frac{1}{\mu}\|t\|_X - \gamma \leq n\},$$

which yields the equivalent approximation:

$$\begin{aligned}\bar{B}_{\mu(k-\gamma)} \cap T = \{t \in T : \|t\|_X \leq \mu(k-\gamma)\} = \bigcap\nolimits_{n\geq k} T_n^+ \\ \subset T_k(x) \subset \bigcap\nolimits_{n\geq k} T_n^- = \{t \in T : \|t\|_X \leq \mu(k+\gamma)\} = T \cap \bar{B}_{\mu(k+\gamma)}.\end{aligned}$$

Hence,

$$T = \bigcup\nolimits_k T_k(x) = \bigcup\nolimits_k T \cap \bar{B}_{\mu(k+\gamma)}.$$

Hence, by the Baire Category Theorem, for some k the set $T_k(x)$ contains a Baire non-meagre set $\bar{B}_{\mu(k-\gamma)} \cap T$ and the proof of Th. 7.8 applies. Indeed if $T \cap \bar{B}_{\mu(k'+\gamma)}$ is non-meagre for some k', then so is $T \cap \bar{B}_{\mu(k'+\gamma)}$ for $k \geq k' + 2\gamma$ and hence also $T_k(x)$ is so. ■

Theorem 7.11 (Global bounds at infinity – Global Bounds Theorem). *Let X be a locally compact topological group with with norm having a vanishingly small global word-net. For $h : X \to \mathbb{R}_+$, if h^* is globally bounded, i.e.*

$$h^*(u) = \limsup\nolimits_{\|x\|\to\infty} h(ux)h(x)^{-1} < B \qquad (u \in X)$$

for some positive constant B, independent of u, then there exist constants K, L, M such that

$$h(ux)h(x)^{-1} < \|u\|^K \qquad (u \geq L, \|x\| \geq M).$$

Hence h is bounded away from ∞ on compact sets sufficiently far from the identity.

Proof. As X is locally compact, it is a Baire space (see e.g. [Eng, Section 3.9]). Thus, by Th. 7.8, the Combinatorial Uniform Boundedness Theorem Th. 7.9A may be applied with $T = X$ to a compact closed neighbourhood $K = \bar{B}_\varepsilon(e_X)$ of the identity e_X, where without loss of generality $0 < \varepsilon < 1$; hence we have

$$\limsup\nolimits_{\|x\|\to\infty} \sup\nolimits_{u\in K} h(ux)h(x)^{-1} < \infty.$$

Now we argue as in [BGT] page 62-3, though with a normed group as the domain. Choose X_1 and $\kappa > \max\{M, 1\}$ such that

$$h(ux)h(x)^{-1} < \kappa \qquad (u \in K,\ \|x\| \geq X_1).$$

Fix v. Now there is some word $w(v) = w_1...w_{m(v)}$ using generators in the compact set Z_δ with $\|w_i\| = \delta(1+\varepsilon_i) < 2\delta$, as $|\varepsilon_i| < 1$ (so $\|w_i\| < 2\delta < \varepsilon$), where

$$d(v, w(v)) < \delta$$

and

$$1 - \varepsilon \leq \frac{m(v)\delta}{\|x\|} \leq 1 + \varepsilon,$$

and so

$$m + 1 < 2\frac{\|v\|}{\delta} + 1 < A\|v\| + 1, \text{ where } A = 2/\delta.$$

Put $w_{m+1} = w^{-1}v$, $v_0 = e$, and for $k = 1, ...m+1$,

$$v_k = w_1...w_k,$$

so that $v_{m+1} = v$. Now $(v_{k+1}x)(v_k x)^{-1} = w_{k+1} \in K$. So for $\|x\| \geq X_1$ we have

$$\begin{aligned} h(vx)h(x)^{-1} &= \prod\nolimits_{k=1}^{m+1} [h(v_k x)h(v_{k-1}x)]^{-1} \\ &\leq \kappa^{m+1} \leq \|v\|^K \end{aligned}$$

(for large enough $\|v\|$), where

$$K = (A \log \kappa + 1).$$

Indeed, for $\|v\| > \log \kappa$, we have

$$(m+1)\log\kappa < (A\|v\|+1)\log\kappa < \|v\|(A\log\kappa + (\log\kappa)\|v\|^{-1}) < \log\|v\|(A\log\kappa+1).$$

For x_1 with $\|x_1\| \geq M$ and with t such that $\|tx_1^{-1}\| > L$, take $u = tx_1^{-1}$; then since $\|u\| > L$ we have

$$h(ux_1)h(x_1)^{-1} = h(t)h(x_1)^{-1} \leq \|u\|^K = \|tx_1^{-1}\|^K,$$

i.e.

$$h(t) \leq \|tx_1^{-1}\|^K h(x_1),$$

so that $h(t)$ is bounded away from ∞ on compact t-sets sufficiently far from the identity. ■

Remarks. 1. The one-sided result in Th. 6.11 can be refined to a two-sided one (as in [BGT, Cor. 2.0.5]): taking $s = t^{-1}$, $g(x) = h(x)^{-1}$ for $h : X \to \mathbb{R}_+$, and using the substitution $y = tx$, yields

$$g^*(s) = \sup\nolimits_{\|y\|\to\infty} g(sy)g(y)^{-1} = \inf\nolimits_{\|x\|\to\infty} h(tx)h(x)^{-1} = h_*(s).$$

2. A variant of Th. 7.11 holds with $\|h(ux)h(x)^{-1}\|_Y$ replacing $h(ux)h(x)^{-1}$.
3. Generalizations of Th. 7.11 along the lines of [BGT] Theorem 2.0.1 may be given for h^* *finite* on a 'large set' (rather than globally *bounded*), by use of the Semigroup Theorem (Th. 9.5).

Taking $h(x) := \|\pi(x)\|_Y$, Cor. 2.9, Th. 7.10 and Th. 7.11 together immediately imply the following.

Corollary 7.12. *If X is a Baire normed group and $\pi : X \to Y$ a group homomorphism, where $\|.\|_Y$ is (μ-γ)-quasi-isometric to $\|.\|_X$ under the mapping π, then there exist constants K, L, M such that*

$$\|\pi(ux)\|_Y / \|\pi(x)\|_Y < \|u\|_X^K \qquad (u \geq L, \|x\|_X \geq M).$$

8. The Subgroup Theorem

In this section G is a normed locally compact topological group with left-invariant Haar measure. We shall be concerned with two topologies on G : the norm topology and the density topology. Under the latter the *binary* group operation need not be jointly continuous (see Heath and Poerio [HePo]); nevertheless a right-shift $x \to xa$, for a constant, is continuous, and so we may say that the density topology is *right-invariant*. We note that if S is measurable and non-null then S^{-1} is measurable and non-null under the corresponding right-invariant Haar and hence also under the original left-invariant measure. We may thus say that both the norm and the density topologies are *inversion-invariant*. Likewise the First and Second Verification Theorems (Theorems 6.2 and 7.5) assert that under both these topologies shift homeomorphisms satisfy (wcc). This motivates a theorem that embraces both topologies as two instances.

Theorem 8.1 (Topological, or Category, Interior Point Theorem). *Let G be given a right-invariant and inversion-invariant topology τ, under which it is a Baire space and suppose that the shift homeomorphisms $h_n(x) = xz_n$ satisfy (wcc) for any null sequence $\{z_n\} \to e$ (in the norm topology). For S Baire and non-meagre in τ, the difference set $S^{-1}S$, and likewise SS^{-1}, is an open neighbourhood of e in the norm topology.*

Proof. Suppose otherwise. Then for each positive integer n we may select

$$z_n \in B_{1/n}(e) \backslash (S^{-1}S).$$

Since $\{z_n\} \to e$ (in the norm topology), the Category Embedding Theorem (Th. 6.1) applies, and gives an $s \in S$ and an infinite $\mathbb{M}_s$ such that

$$\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S.$$

Then for any $m \in \mathbb{M}_s$,

$$sz_m \in S \text{ , i.e. } z_m \in S^{-1}S,$$

a contradiction. Replacing S by S^{-1} we obtain the corresponding result for SS^{-1}. ■

One thus has again.

Corollary 8.2 (Piccard Theorem, [Pic1], [Pic2]). *For S Baire and non-meagre in the norm topology, the difference sets SS^{-1} and $S^{-1}S$ have e as interior point.*

First Proof. Apply the preceding Theorem, since by the First Verification Theorem (Th. 6.2), the condition (wcc) holds. ■

Second Proof. Suppose otherwise. Then, as before, for each positive integer n we may select $z_n \in B_{1/n}(e)\backslash(S^{-1}S)$. Since $z_n \to e$, by the Kestelman-Borwein-Ditor Theorem (Cor. 6.4), for quasi all $s \in S$ there is an infinite $\mathbb{M}_s$ such that $\{sz_m : m \in \mathbb{M}_s\} \subseteq S$. Then for any $m \in \mathbb{M}_s$, $sz_m \in S$, i.e. $z_m \in SS^{-1}$, a contradiction. ■

Corollary 8.3 (Steinhaus Theorem, [St], [We]; cf. Comfort [Com, Th. 4.6 p. 1175] , Beck et al. [BCS]). *In a normed locally compact group, for S of positive measure, the difference sets $S^{-1}S$ and SS^{-1} have e as interior point.*

Proof. Arguing as in the first proof above, by the Second Verification Theorem (Th. 7.5), the condition (wcc) holds and S, in the density topology, is Baire and non-meagre (by the Category-Measure Theorem, Th. 7.2). The measure-theoretic form of the second proof above also applies. ■

The following corollary to the Steinhaus Theorem Th. 6.10 (and its Baire category version) have important consequences in the Euclidean case. We will say that the group G is (weakly) *Archimedean* if for each $r > 0$ and each $g \in G$ there is $n = n(g)$ such that $g \in B^n$ where $B := \{x : \|x\| < r\}$ is the r-ball.

Theorem 8.4 (Category (Measure) Subgroup Theorem). *For a Baire (resp. measurable) subgroup S of a weakly Archimedean locally compact group G, the following are equivalent:*
(i) $S = G$,
(ii) S is Baire non-meagre (resp. measurable non-null).

Proof. By Th. 8.1, for some r-ball B,

$$B \subseteq SS^{-1} \subseteq S,$$

and hence $G = \bigcup_n B^n = S$. ■

We will see in the next section a generalization of the Pettis extension of Piccard's result asserting that, for S, T Baire non-meagre, the product ST contains interior points. As our approach will continue to be bitopological, we will deduce also the Steinhaus result that, for S, T non-null and measurable, ST contains interior points.

9. The Semigroup Theorem

This section, just as the preceding one, is focussed on metrizable locally compact *topological* groups. Since a locally compact *normed* group possesses an invariant Haar-measure, much of the theory developed there and here goes over to locally compact normed groups – for details see [Ost-LB3]. In this section G is again a normed locally compact topological group. The aim here is to prove a generalization to the normed group setting of the following classical result due to Hille and Phillips [H-P, Th. 7.3.2] (cf. Beck et al. [BCS, Th. 2], [Be]) in the measurable case, and to Bingham and Goldie [BG] in the Baire case; see [BGT, Cor. 1.1.5].

Theorem 9.1 (Category (Measure) Semigroup Theorem). *For an additive Baire (resp. measurable) subsemigroup S of $\mathbb{R}_+$, the following are equivalent:*
(i) S contains an interval,
(ii) $S \supseteq (s, \infty)$, for some s,
(iii) S is non-meagre (resp. non-null).

We will need a strengthening of the Kestelman-Borwein-Ditor Theorem, Th. 1.1. involving two sets. First we capture a key similarity (their topological 'common basis', adapting a term from logic) between the Baire and measure cases. Recall ([Rog2, p. 460]) the usage in logic, whereby a set B is a basis for a class $\mathcal{C}$ of sets whenever any member of $\mathcal{C}$ contains a point in B.

Theorem 9.2 (Common Basis Theorem). *For V, W Baire non-meagre in a group G equipped with either the norm or the density topology, there is $a \in G$ such that $V \cap (aW)$ contains a non-empty open set modulo meagre sets common to both, up to translation. In fact, in both cases, up to translation, the two sets share a norm $\mathcal{G}_\delta$ subset which is non-meagre in the norm case and non-null in the density case.*

Proof. In the norm topology case if V, W are Baire non-meagre, we may suppose that $V = I \backslash M_0 \cup N_0$ and $W = J \backslash M_1 \cup N_1$, where I, J are open sets. Take $V_0 = I \backslash M_0$ and $W_0 = J \backslash M_1$. If v and w are points of V_0 and W_0, put $a := vw^{-1}$. Thus $v \in I \cap (aJ)$. So $I \cap (aJ)$ differs from $V \cap (aW)$ by a meagre set. Since $M_0 \cup N_0$ may be expanded to a meagre $\mathcal{F}_\sigma$ set M, we deduce that $I \backslash M$ and $J \backslash M$ are non-meagre $\mathcal{G}_\delta$-sets.
In the density topology case, if V, W are measurable non-null let V_0 and W_0 be the sets of density points of V and W. If v and w are points of V_0 and W_0, put $a := vw^{-1}$. Then $v \in T := V_0 \cap (aW_0)$ and so T is non-null and v is a density point of T. Hence if T_0 comprises the density points of T, then $T \backslash T_0$ is null, and so T_0 differs from $V \cap (aW)$ by a null set. Evidently T_0 contains a non-null closed, hence $\mathcal{G}_\delta$-subset (as T_0 is measurable non-null, by regularity of Haar measure). ■

Theorem 9.3 (Conjunction Theorem). *For V, W Baire non-meagre (resp. measurable non-null) in a group G equipped with either the norm or the density topology, there is $a \in G$ such that $V \cap (aW)$ is Baire non-meagre (resp. measurable non-null) and for any null sequence $z_n \to e_G$ and quasi all (almost all) $t \in V \cap (aW)$ there exists an infinite $\mathbb{M}_t$ such that*

$$\{tz_m : m \in \mathbb{M}_t\} \subset V \cap (aW).$$

Proof. In either case applying Theorem 9.2, for some a the set $T := V \cap (aW)$ is Baire non-meagre (resp. measurable non-null). We may now apply the Kestelman-Borwein-Ditor Theorem to the set T. Thus for almost all $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \subset V \cap (aW).$$

■

See [BOst-KCC] for other forms of countable conjunction theorems. The last result motivates a further strengthening of generic subuniversality (compare Section 6).

Definitions. Let S be generically subuniversal (=null-shift-compact). (See the definitions after Th. 7.7.)
1. Call T *similar* to S if for every null sequence $z_n \to e_G$ there is $t \in S \cap T$ and $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset S \cap T.$$

Thus S is similar to T and both are generically subuniversal.
Call T *weakly similar* to S if if for every null sequence $z_n \to 0$ there is $s \in S$ and $\mathbb{M}_s$ such that

$$\{sz_m : m \in \mathbb{M}_s\} \subset T.$$

Thus again T is subuniversal (=null-shift-precompact).
2. Call S *subuniversally self-similar,* or just *self-similar* (up to inversion-translation), if for some $a \in G$ and some $T \subset S$, S is similar to aT^{-1}.
Call S *weakly self-similar* (up to inversion-translation) if for some $a \in G$ and some $T \subset S$, S is weakly similar to aT^{-1}.

Theorem 9.4 (Self-similarity Theorem). *In a group G equipped with either the norm or the density topology, for S Baire non-meagre (or measurable non-null), S is self-similar.*

Proof. Fix a null sequence $z_n \to 0$. If S is Baire non-meagre (or measurable non-null), then so is S^{-1}; thus we have for some a that $T := S \cap (aS^{-1})$ is likewise Baire non-meagre (or measurable non-null) and so for quasi all (almost all) $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \subset S \cap (aS^{-1}),$$

as required. ■

Theorem 9.5 (Semigroup Theorem – cf. [BCS], [Be]). *In a group G equipped with either the norm or the density topology, if S, T are generically subuniversal (i.e. null-shift-compact) with T (weakly) similar to S, then ST^{-1} contains a ball about the identity e_G. Hence if S is generically subuniversal and (weakly) self-similar, then SS has interior points. Hence for $G = \mathbb{R}^d$, if additionally S is a semigroup, then S contains an open sector.*

Proof. For S, T (weakly) similar, we claim that ST^{-1} contains $B_\delta(e)$ for some $\delta > 0$. Suppose not: then for each positive n there is z_n with

$$z_n \in B_{1/n}(e) \backslash (ST^{-1}).$$

Now z_n^{-1} is null, so there is s in S and infinite $\mathbb{M}_s$ such that

$$\{z_m^{-1}s : m \in \mathbb{M}_t\} \subset T.$$

For any m in $\mathbb{M}_t$ pick $t_m \in T$ so that $z_m^{-1}s = t_m$; then we have

$$z_m^{-1} = t_m s^{-1}, \quad \text{so} \quad z_m = st_m^{-1},$$

a contradiction. Thus for some $\delta > 0$ we have $B_\delta(e) \subset ST^{-1}$.
For S self-similar, say S is similar to $T := aS^{-1}$, for some a, then $B_\delta(e)a \subset ST^{-1}a = S(aS^{-1})^{-1}a = SSa^{-1}a$, i.e. SS has non-empty interior. ■

For information on the structure of semigroups see also [Wr]. For applications see [BOst-RVWL]. By the Common Basis Theorem (Th. 9.2), replacing T by T^{-1}, we obtain as an immediate corollary of Theorem 9.5 a new proof of two classical results, extending the Steinhaus and Piccard Theorem and Kominek's Vector Sum Theorem.

Theorem 9.6 (Product Set Theorem, Steinhaus [St] measure case, Pettis [Pet2] Baire case, cf. [Kom1] and [Jay-Rog, Lemma 2.10.3] in the setting of topological vector spaces and [Be] and [BCS] in the group setting). *In a normed locally compact group, if S, T are Baire non-meagre (resp. measurable non-null), then ST contains interior points.*

10. Convexity

This section begins by developing natural conditions under which the Portmanteau theorem of convex functions (cf. [BOst-Aeq]) remains true when reformulated for a normed group setting, and then deduces generalizations of classical automatic continuity theorems for convex functions on a group.

Definitions. 1. A group G will be called 2-*divisible* (or quadratically closed) if the equation $x^2 = g$ for $g \in G$ always has a unique solution in the group to be denoted $g^{1/2}$. See [Lev] for a proof that any group may be embedded as a subgroup in an overgroup

where the equations over G are soluble (compare also [Lyn1]).
2. In an arbitrary group, say that a subset C is $\frac{1}{2}$-*convex* if, for all x, y

$$x, y \in C \Longrightarrow \sqrt{xy} \in C,$$

where $\sqrt{xy}$ signifies some element z with $z^2 = xy$. We recall the following results.

THEOREM 10.1 (Eberlein-McShane Theorem, [Eb], [McSh, Cor. 10]). *Let X be a 2-divisible topological group of second category. Then any $\frac{1}{2}$-convex non-meagre Baire set has a non-empty interior. If X is abelian and each sequence defined by $x_{n+1}^2 = x_n$ converges to e_X then the interior of a $\frac{1}{2}$-convex set C is dense in C.*

DEFINITION. We say that the function $h : G \to R$ is $\frac{1}{2}$-*convex* on the $\frac{1}{2}$–convex set C if, for $x, y \in C$,

$$h(\sqrt{xy}) \leq \frac{1}{2}\left(h(x) + h(y)\right),$$

with $\sqrt{xy}$ as above.

EXAMPLE. For $G = R_+^*$ the function $h(x) = x$ is $\frac{1}{2}$-convex on G, since

$$2xy \leq x^2 + y^2.$$

THEOREM 10.2 (Convex Minorant Theorem, [McSh]). *Let X be 2-divisible abelian topological group. Let f and g be real-valued functions defined on a non-meagre subset C with f $\frac{1}{2}$-convex and g Baire such that*

$$f(x) \leq g(x), \text{ for } x \in C.$$

Then f is continuous on the interior of C.

LEMMA 10.3 (Averaging Lemma). *In a normed topological group, a non-meagre set T is 'averaging', that is, for any given point $u \in T$ and for any sequence $\{u_n\} \to u$, there are $v \in G$ (a right-averaging translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have*

$$u_n^2 = v_n v.$$

There is likewise a left-averaging translator such that for some $\{w_n\} \subseteq T$ for infinitely many $n \in \omega$, we have

$$u_n^2 = w w_n.$$

Proof. Define null sequences by $z_n = u_n u^{-1}$, and $\tilde{z}_n = u^{-1} u_n$. We are to solve $u_n^2 v^{-1} = v_n \in T$, or

$$u\tilde{z}_n z_n u v^{-1} = v_n \in T, \text{ equivalently } \tilde{z}_n z_n u v^{-1} = u^{-1} v_n \in T' = u^{-1}T.$$

Now put $\psi_n(x) := \tilde{z}_n z_n x$; then

$$d(x, \tilde{z}_n z_n x) = d(e, \tilde{z}_n z_n) = \|\tilde{z}_n z_n\| \leq \|\tilde{z}_n\| + \|z_n\| \to 0.$$

By the Category Embedding Theorem (Th. 6.1), for some $\lambda \in T' = u^{-1}T$, we have with $\lambda = u^{-1}t$ and for infinitely many n

$$u^{-1}v_n \ : = \tilde{z}_n z_n \lambda \in T' = u^{-1}T,$$
$$u\tilde{z}_n z_n \lambda = v_n \in T, \text{ or } u\tilde{z}_n z_n uu^{-1}\lambda = v_n \in T,$$

so

$$u_n^2 u^{-1}\lambda = v_n \in T, \text{ or } u_n^2 = v_n \lambda^{-1} u = v_n v$$

(with $v = \lambda^{-1}u = t^{-1}u^2 \in T^{-1}u^2$).
As for the remaining assertion, note that $u_n^{-1} \to u^{-1}, v_n^{-1} \in T^{-1}$ and

$$u_n^{-2} = v^{-1}v_n^{-1}.$$

Thus noting that T^{-1} is non-meagre (since inversion is a homeomorphism) and replacing T^{-1} by T we obtain the required assertion by a right-averaging translator. ■

Note the connection between the norms of the null sequences is only by way of the conjugate metrics:

$$\|z_n\| = d(e, u_n u^{-1}) = d(u, u_n), \text{ and } \|\tilde{z}_n\| = d(e, u^{-1}u_n) = d(u_n^{-1}, u^{-1}) = \tilde{d}(u_n, u).$$

Whilst we may make no comparisons between them, both norms nevertheless converge to zero.

Definitions. For G, H normed groups, we say that $f : G \to H$ is *locally Lipschitz* at g if, for some neighbourhood N_g of g and for some constants K_g and all x, y in N_g,

$$\left|\left|f(x)f(y)^{-1}\right|\right|_H \leq K_g \|xy^{-1}\|_G.$$

We say that $f : G \to H$ is *locally bi-Lipschitz* at g if, for some neighbourhood N_g of g and for some positive constants K_g, κ_g, and all x, y in N_g,

$$\kappa_g \|xy^{-1}\|_G \leq \left|\left|f(x)f(y)^{-1}\right|\right|_H \leq K_g \|xy^{-1}\|_G.$$

If $f : G \to H$ is invertible, this asserts that both f and its inverse f^{-1} are locally Lipschitz at g and $f(g)$ respectively.
We say that the *norm* on G is *n-Lipschitz* if the function $f_n(x) := x^n$ from G to G is *locally Lipschitz* at all $g \neq e$, i.e. for each there is a neighbourhood N_g of g and positive constants κ_g, K_g so that

$$\kappa_g \|xy^{-1}\|_G \leq \left|\left|x^n y^{-n}\right|\right|_G \leq K_g \|xy^{-1}\|_G.$$

In an abelian context the power function is a homomorphism; we note that [HJ, p. 381] refers to a semigroup being *modular* when each f_n(defined as above) is an injective homomorphism. The condition on the right with $K = n$ is automatic, and so one need require only that for some positive constant κ

$$\kappa \|g\| \leq \|g^n\|.$$

Note that, in the general context of an n-Lipschitz norm, if $x^n = y^n$, then as $(x^n y^{-n}) = e$, we have $\kappa_g \|xy^{-1}\|_G \leq ||x^n y^{-n}||_G = \|e\| = 0$, and so $\|xy^{-1}\|_G = 0$, i.e. the power function is injective. If, moreover, the group is n-divisible, then the power function $f_n(x)$ is an isomorphism.

We note that in the additive group of reals x^2 fails to be locally bi-Lipschitz at the origin (since its derivative there is zero): see [Bart]. However, the following are bi-Lipschitz. 1. In $\mathbb{R}^d$ with additive notation, we have $\|x^2\| := \|2x\| = 2\|x\|$, so the norm is 2-Lipschitz. 2. In $\mathbb{R}_+^*$ we have $\|x^2\| := |\log x^2| = 2|\log x| = 2\|x\|$ and again the norm is 2-Lipschitz. 3. In a Klee group the mapping $f(x) := x^n$ is uniformly (locally) Lipschitz, since

$$\left|\left|x^n y^{-n}\right|\right|_G \leq n\|xy^{-1}\|_G,$$

proved inductively from the Klee property (Th. 2.18) via the observation that

$$\left|\left|x^{n+1}y^{-(n+1)}\right|\right|_G = \left|\left|xx^n y^{-n}y^{-1}\right|\right|_G \leq \left|\left|x^n y^{-n}\right|\right|_G + \left|\left|xy^{-1}\right|\right|_G.$$

Lemma 10.4 (Reflecting Lemma). *Suppose the group-norm is everywhere locally 2-Lipschitz. Then, for T non-meagre, T is reflecting i.e. there are $w \in G$ (a right-reflecting translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have*

$$v_n^2 = u_n w.$$

There is likewise a left-reflecting translator.

Proof. Let $T^2 := \{g : g = t^2 \text{for some } t \in T\}$. By assumption, T^2 is non-meagre. With $u_n = uz_n$, put $S = T^2$ and notice that $u_n w \in S$ iff $uz_n w \in S$ iff $z_n w \in u^{-1}S$. Now $u^{-1}S$ is non-meagre and $\psi_n(x) := z_n x$ as usual converges to the identity, so the existence of $w \in u^{-1}S$ is assured such that $z_n w = u^{-1}v_n^2$. ■

Remarks. 1. Note that the assertion here is

$$u_n^{-1}v_n = wv_n^{-1},$$

so that

$$d(v_n, w) = d(v_n^{-1}, u_n^{-1}) = \tilde{d}(v_n, u_n) \approx \tilde{d}(v_n, u),$$

or

$$d(v_n, w) \approx \tilde{d}(v_n, u),$$

suggesting the terminology of reflection.
2. Boundedness theorems for reflecting and averaging sets follow as in [BOst-Aeq] since the following are true in any group, as we see below.

Theorem 10.5. *In a normed topological group, for f a $\frac{1}{2}$-convex function, if f is locally bounded above at x_0, then it is locally bounded below at x_0 (and hence locally bounded at x_0).*

Proof. Say f is bounded above in $B := B_\delta(x_0)$ by M. Consider $u \in \tilde{B}_\delta(x_0)$. Thus $\tilde{d}(x_0, u) = \|u^{-1}x_0\| < \delta$. Put $t = u^{-1}x_0^2$; then $tx_0^{-1} = u^{-1}x_0$, and so

$$d(t, x_0) = \|tx_0^{-1}\| = \|u^{-1}x_0\| = \tilde{d}(u, x_0) < \delta.$$

Then $t \in B$, and since $x_0^2 = ut$ we have

$$2f(x_0) \leq f(u) + f(t) \leq f(u) + M,$$

or

$$f(u) \geq 2f(x_0) - M.$$

Thus $2f(x_0) - M$ is a lower bound for f on the open set $\tilde{B}_\delta(x_0)$. ■

As a corollary a suitably rephrased Bernstein-Doetsch Theorem ([Kucz], [BOst-Aeq]) is thus true.

Theorem 10.6 (Bernstein-Doetsch Theorem). *In a normed group, for f a $\frac{1}{2}$-convex function, if f is locally bounded above at x_0, then f is continuous at at x_0.*

Proof. We repeat the 'Second proof' of [Kucz, p. 145]. Choose $y_n \to x_0$ with $f(y_n) \to m_f(x_0)$ and $z_n \to x_0$ with $f(z_n) \to M_f(x_0)$. Let $u_n := y_n^2 x_n^{-1}$. Thus $y_n^2 = u_n x_n$ and so

$$2f(y_n) \leq f(u_n) + f(z_n),$$

i.e. $f(u_n) \geq 2f(y_n) - f(z_n)$. Hence in the limit we obtain

$$M_f(x_0) \geq \liminf f(u_n) \geq 2M_f(x_0) - m_f(x_0).$$

One thus has that $M_f(x_0) \leq m_f(x_0)$. But $m_f(x_0) \leq f(x_0) \leq M_f(x_0)$, and both hull values are finite (by the result above). Thus $m_f(x_0) = f(x_0) = M_f(x_0)$, from which continuity follows. ■

We now consider the transferability of upper and lower local boundedness. Our proofs work directly with definitions (so are not modelled after those in Kuczma [Kucz]). We do not however consider domains other than the whole metric group. For clarity of proof structure we give separate proofs for the two cases, first when $G$is abelian and later for general G.

Theorem 10.7 (Local upper boundedness). *In a normed topological group G, for f a $\frac{1}{2}$-convex function defined on G, if f is locally bounded above at some point x_0, then f is locally bounded above at all points.*

Proof. *Case* (i) *The Abelian case.* Say f is bounded above in $B := B_\delta(x_0)$ by M. Given a fixed point t, put $z = z_t := x_0^{-1}t^2$, so that $t^2 = x_0 z$. Consider any $u \in B_{\delta/2}(t)$. Write $u = st$ with $\|s\| < \delta/2$. Now put $y = s^2$; then $\|y\| = \|s^2\| \leq 2\|s\| < \delta$. Hence $yx_0 \in B_\delta(x_0)$. Now

$$u^2 = (st)^2 = s^2t^2 = yx_0z,$$

as the group is abelian. So

$$f(u) \leq \frac{1}{2}f(yx_0) + \frac{1}{2}f(z) \leq \frac{1}{2}M + \frac{1}{2}f(z_t).$$

That is, $\frac{1}{2}(M + f(z_t))$ is an upper bound for f in $B_{\delta/2}(x_0)$.
Case (ii) *The general case.* As before, suppose f is bounded above in $B := B_\delta(x_0)$ by M,and let t be a given a fixed point; put $z = z_t := x_0^{-1}t^2$so that $t^2 = x_0 z$.
For this fixed t the mapping $y \to \alpha(y) := ytyt^{-1}y^{-2}$ is continuous (cf. Th. 3. 7 on commutators) with $\alpha(e) = e$, so $\alpha(y)$ is $o(y)$ as $\|y\| \to 0$. Now

$$sts = [stst^{-1}s^{-2}]s^2t = \alpha(s)s^2t,$$

and we may suppose that, for some $\eta < \delta/2$, we have $\|\alpha(s)\| < \delta/2$, for $\|s\| < \eta$. Note that

$$stst = \alpha(s)s^2t^2.$$

Consider any $u \in B_r(t)$ with $r = \min\{\eta, \delta/2\}$. Write $u = st$ with $\|s\| < r \leq \delta/2$. Now put $y = s^2$. Then $\|y\| = \|s^2\| \leq 2\|s\| < \delta$ and $\|o(s)y\| \leq \eta+\delta/2 < \delta$. Hence $o(s)yx_0 \in B_\delta(x_0)$. Now

$$u^2 = stst = \alpha(s)s^2t^2 = \alpha(s)yx_0z.$$

Hence, by convexity,

$$f(u) \leq \frac{1}{2}f(o(s)yx_0) + \frac{1}{2}f(z) \leq \frac{1}{2}M + \frac{1}{2}f(z_t).$$

■

As an immediate corollary of the last theorem and the Bernstein-Doetsch Theorem (Th. 10.6) we have the following result.

Theorem 10.8 (Dichotomy Theorem for convex functions – [Kucz, p.147]). *In a normed topological group, for $\frac{1}{2}$-convex f (so in particular for additive f) either f is continuous everywhere, or it is discontinuous everywhere.*

The definition below requires continuity of ‘square-rooting’ – taken in the form of an algebraic closure property of degree 2 in a group G, expressed as the solvability of certain ‘quadratic equations’ over the group. Its status is clarified later by reference to Bartle’s Inverse Function Theorem. We recall that a group is n-divisible if $x^n g = e$ is soluble for each $g \in G$. (In the absence of algebraic closure of any degree an extension of G may be constructed in which these equations are solvable – see for instance Levin [Lev].)

Definition. We say that the normed group G is *locally convex* at $\lambda = t^2$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that for all g with $\|g\| < \varepsilon$ the equation

$$xtxt = gt^2,$$

equivalently $xtxt^{-1} = g$, has its solutions satisfying $\|x\| < \delta$.

Thus G is locally convex at e if for any $\varepsilon > 0$ there is $\delta > 0$ such that for all g with $\|g\| < \varepsilon$ the equation

$$x^2 = g$$

has its solutions with $\|x\| < \delta$.

Remark. Putting $u = xt$ the local convexity equation reduces to $u^2 = gt^2$, asserting the local existence of square roots (local 2-divisibility). If G is abelian the condition at t reduces to the condition at e.

Theorem 10.9 (Local lower boundedness). *Let G be a locally convex group with a 2-Lipschitz norm, i.e. $g \to g^2$ is a bi-Lipschitz isomorphism such that, for some $\kappa > 0$,*

$$\kappa\|g\| \leq \|g^2\| \leq 2\|g\|.$$

For f a $\frac{1}{2}$-convex function, if f is locally bounded below at some point, then f is locally bounded below at all points.

Proof. Note that by Th. 3.39 the normed group is topological.
Case (i) *The Abelian case.* We change the roles of t and x_0 in the preceeding abelian theorem, treating $t$as a reference point, albeit now for lower boundedness, and x_0 as some arbitrary other fixed point. Suppose that f is bounded below by L on $B_\delta(t)$. Let $yx_0 \in B_{\kappa\delta}(x_0)$, so that $0 < \|y\| < \kappa\delta$. Choose s such that $s^2 = y$. Then,

$$\kappa\|s\| \leq \|y\| < \kappa\delta,$$

so $\|s\| < \delta$. Thus $u = st \in B_\delta(t)$. Now the identity $u^2 = s^2t^2 = yx_0z$ implies that

$$L \leq f(u) \leq \frac{1}{2}f(yx_0) + \frac{1}{2}f(z_t),$$
$$2L - f(z_t) \leq f(yx_0),$$

i.e. that $2L - f(z_t)$ is a lower bound for f on $B_{\kappa\delta}(x_0)$.
Case (ii) *The general case.* Suppose as before that f is bounded below by L on $B_\delta(t)$. Since the map $\alpha(\sigma) := \sigma t\sigma t^{-1}\sigma^{-2}$ is continuous (cf. again Th. 3. 7 on commutators) and $\alpha(e) = e$, we may choose η such that $\|\alpha(\sigma)\| < \kappa\delta/2$, for $\|\sigma\| < \eta$. Now choose $\varepsilon > 0$ such that, for each y with $\|y\| < \varepsilon$, the solution $u = \sigma t$ to

$$u^2 = yt^2$$

has $\|\sigma\| < \eta$. Let $r = \min\{\kappa\delta/2, \varepsilon\}$.
Let $yx_0 \in B_r(x_0)$; then $0 < \|y\| < \kappa\delta/2$ and $\|y\| < \varepsilon$. As before put $z = z_t := x_0^{-1}t^2$ so that $t^2 = x_0z$. Consider $u = \sigma t$ such that $u^2 = yx_0z$; thus we have

$$u^2 = \sigma t\sigma t = yx_0z = yx_0x_0^{-1}t^2 = yt^2.$$

Hence $\|\sigma\| < \eta$ (as $\|y\| < \varepsilon$). Now we write

$$u^2 = \sigma t\sigma t = [\sigma t\sigma t^{-1}\sigma^{-2}]\sigma^2t^2 = \alpha(\sigma)\sigma^2t^2 = yt^2.$$

We compute that

$$y = \alpha(\sigma)\sigma^2$$

and

$$\kappa\delta/2 \geq \|y\| = \|\alpha(\sigma)\sigma^2\| \geq \|\sigma^2\| - \|\alpha(\sigma)\| \geq \kappa\|\sigma\| - \|\alpha(\sigma)\|,$$

so

$$\|\sigma\| \leq \delta/2 + \|\alpha(\sigma)\|/\kappa < \delta/2 + \delta/2 < \delta.$$

Thus $u \in B_\delta(t)$. Now the identity $u^2 = yx_0z$ together with convexity implies as usual that

$$L \leq f(u) \leq \frac{1}{2}f(yx_0) + \frac{1}{2}f(z_t),$$
$$2L - f(z_t) \leq f(yx_0),$$

i.e. $2L - f(z_t)$ is a lower bound for f on $B_{\kappa\delta}(x_0)$. ∎

The local 2-divisibility assumption at t^2 asserts that $f_t(\sigma) := \sigma t\sigma t^{-1}$ is invertible locally at e. Bartle's theorem below guarantees that f_t has uniform local inverse under a smoothness assumption, i.e. that for $\|\sigma\| = \|f_t^{-1}(y)\| < \delta$, for all small enough y, say for $\|y\| < \kappa\delta$. To state the theorem we need some definitions.

Definitions. 1. f is said to have a *derivative* at x_0 if there is a continuous homomorphism $f'(x_0)$ such that

$$\lim_{\|u\|\to 0+} \frac{1}{\|u\|}\|f(ux_0)f(x_0)^{-1}[f'(x_0)(u)]^{-1}\| = 0.$$

2. f is *of class C'* on the open set U if it has a derivative at each point u in U and, for each x_0 and each $\varepsilon > 0$, there is $\delta > 0$ such that, for all x_1, x_2 in $B_\delta(x_0)$ both

$$\|f'(x_1)(u)[f'(x_2)(u)]^{-1}\| < \varepsilon\|u\|$$

and

$$\|f(x_1)f(x_2)^{-1}f'(x_0)(x_1x_2^{-1})^{-1}\| < \varepsilon\|x_1x_2^{-1}\|.$$

The two conditions may be rephrased relative to the right-invariant metric d on the group as

$$d(f'(x_1)(u), f'(x_2)(u)) < \varepsilon\|u\|,$$

and

$$d(f(x_1)f(x_2)^{-1}, f'(x_0)(x_1x_2^{-1}) < \varepsilon d(x_1, x_2).$$

3. Suppose that $y_0 = f(x_0)$. Then f is *smooth* at x_0 if there are positive numbers α, β such that if $0 < d(y, y_0) < \beta$ then there is x such that $y = f(x)$ and $d(x, x_0) \leq \alpha \cdot d(y, y_0)$. If f is invertible, then this asserts that

$$d(f^{-1}(y), f^{-1}(y_0)) \leq \alpha \cdot d(y, y_0).$$

Example. Let $f(x) = tx$ with t fixed. Here f is smooth at x_0 if there are positive numbers α, β such that

$$\|xx_0^{-1}\| \leq \alpha\|tx(tx_0)^{-1}\| = \alpha\|txx_0^{-1}t^{-1}\|.$$

Note that in a Klee group $\|txx_0^{-1}t^{-1}\| = \|t^{-1}txx_0^{-1}\| = \|xx_0^{-1}\|$.

Theorem 10.10 (Bartle's Inverse Function Theorem, [Bart, Th. 2.4]). *In a topologically complete normed group, suppose that*
(i) f is of class C' in the ball $B_r(x_0) = \{x \in G : \|xx_0^{-1}\| < r\}$, for some $r > 0$, and
(ii) $f'(x_0)$ is smooth (at e and so anywhere).
Then f is smooth at x_0 and hence open.
If also the derivative $f'(x_0)$ is an isomorphism, then f has a uniformly continuous local inverse at x_0.

Corollary 10.11. *If $f_t(\sigma) := \sigma t \sigma t^{-1}$ is of class C' on $B_r(e)$ and $f_t'(e)$ is smooth, then G is locally convex at t.*

Proof. Immediate since $f_t(e) = e$. ■

We are now in a position to state generalizations of two results derived in the real line case in [BOst-Aeq].

Proposition 10.12. *Let G be any locally convex group with a 2-Lipschitz norm. If f is $\frac{1}{2}$-convex and bounded below on a reflecting subset S of G, then f is locally bounded below on G.*

Proof. Suppose not. Let T be a reflecting subset of S. Let K be a lower bound on T. If f is not locally bounded from below, then at any point u in $\bar{T}$ there is a sequence $\{u_n\} \to u$ with $\{f(u_n)\} \to -\infty$. For some $w \in G$, we have $v_n^2 = wu_n \in T$, for infinitely many n. Then

$$K \le f(v_n) \le \frac{1}{2}f(w) + \frac{1}{2}f(u_n), \text{ or } 2K - f(w) \le f(u_n),$$

i.e. $f(u_n)$ is bounded from below, a contradiction. ■

Theorem 10.13 (Generalized Mehdi Theorem – cf. [Meh, Th.3]). *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed group, bounded above on an averaging subset S , is continuous on G.*

Proof.

Let T be an averaging core of S. Suppose that f is not continuous, but is bounded above on T by K. Then f is not locally bounded above at some point of $u \in \bar{T}$. Then there is a null sequence $z_n \to e$ with $f(u_n) \to \infty$, where $u_n = uz_n$. Select $\{v_n\}$ and w in G so that, for infinitely many n, we have

$$u_n^2 = wv_n.$$

But for such n,we have

$$f(u_n) \le \frac{1}{2}f(w) + \frac{1}{2}f(v_n) \le \frac{1}{2}f(w) + \frac{1}{2}K,$$

contradicting the unboundedness of $f(u_n)$. ■

The Generalized Mehdi Theorem, together with the Averaging Lemma, implies the classical result below and its generalizations.

THEOREM 10.14 (Császár-Ostrowski Theorem [Csa], [Kucz, p. 210]). *A convex function $f : \mathbb{R} \to \mathbb{R}$ bounded above on a set of positive measure (resp. non-meagre set) is continuous.*

THEOREM 10.15 (Topological Császár-Ostrowski Theorem). *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed topological group, bounded above on a non-meagre subset, is continuous.*

Appeal to the Generalized Borwein-Ditor Theorem yields the following result, which refers to *Radon measures*, for which see Fremlin [Fre-4].

THEOREM 10.16 (Haar-measure Császár-Ostrowski Theorem). *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed topological group carrying a Radon measure, bounded above on a set of positive measure, is continuous.*

11. Automatic continuity: the Jones-Kominek Theorem

This section is dedicated to generalizations to normed groups and to a more general class of topological groups of the following result for the real line. Here we regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and so we say that T is a *spanning subset* of $\mathbb{R}$ if any real number is a finite rational combination of members of T. See below for the definition of an analytic set.

THEOREM 11.1 (Theorems of Jones and Kominek). *Let f be additive on $\mathbb{R}$ and either have a continuous restriction, or a bounded restriction, $f|T$, where T is some analytic set spanning $\mathbb{R}$. Then f is continuous.*

The result follows from the Expansion Lemma and Darboux's Theorem (see below) that an additive function bounded on an interval is continuous. In fact the bounded case above (*Kominek's Theorem,* [Kom2]) implies the continuous case (*Jones's Theorem,* [Jones1], [Jones2]), as was shown in [**?**]. [OC] develops limit theorems for sequences of functionals whose properties are given on various kinds of spanning sets including spanning in the sense of linear rational combinations. Before stating the current generalizations we begin with some preliminaries on analytic subsets of a topological group. We recall ([Jay-Rog, p.11], or [Kech, Ch. III] for the Polish space setting) that in a Hausdorff space X a K*-analytic set* is a set A that is the image under a compact-valued, upper semi-continuous map from $\mathbb{N}^{\mathbb{N}}$; if this mapping takes values that are singletons or empty, the set A is said to be *analytic*. In either case A is Lindelöf. (The topological notion of

K-analyticity was introduced by Choquet, Frolik, Sion and Rogers under variant definitions, eventually found to be equivalent, as a consequence of a theorem of Jayne, see [Jay-Rog, Sect. 2.8 p. 37] for a discussion.) If the space X is a topological group, then the subgroup $\langle A\rangle$ (generated) by an analytic subset A is also analytic and so Lindelöf (for which, see below); note the result due to Loy [Loy] and Christensen [Ch] that an analytic Baire group is Polish (cf. [HJ, Th. 2.3.6 p. 355]). Note that a Lindelöf group need not be metric; see for example the construction due to Oleg Pavlov [Pav]. If additionally the group X is metric, then $\langle A\rangle$ is separable, and so in fact this K-analytic set is analytic (a continuous image of $\mathbb{N}^{\mathbb{N}}$ – see [Jay-Rog, Th. 5.5.1 (b), p. 110]).

Definition. For $\mathcal{H}$ a family of subsets of a space X, we say that a set S is *Souslin-$\mathcal{H}$* if it is of the form

$$S=\bigcup\nolimits_{\alpha\in\omega^\omega}\bigcap\nolimits_{n=1}^{\infty}H(\alpha|n),$$

with each $H(\alpha|n)\in\mathcal{H}$. We will often take $\mathcal{H}$ to be $\mathcal{F}(X)$, the family of closed subsets of the space X.

Definition. Let G be any group. For any positive integer n and for any subset S let $S^{(n)}$, the *n-span* of S, denote the set of S-words of length n. Say that a subset H of G *spans G (*in the sense of group theory), or *generates the group G*, if for any $g\in G$, there are $h_1,...,h_n$ in H such that

$$g=h_1^{\varepsilon_1}\cdot...\cdot h_n^{\varepsilon_n},\text{ with }\varepsilon_i=\pm 1.$$

(If H is *symmetric*, so that $h^{-1}\in H$ iff $h\in H$, there is no need for inverses.)

We begin with results concerning K-analytic groups.

Proposition 11.2. *The span of a K -analytic set is K-analytic; likewise for analytic sets.*

Proof. Since $f(v,w)=vw$ is continuous, $S^{(2)}=f(S\times S)$ is K-analytic by [Jay-Rog, Th 2.5.1 p. 23]. Similarly all the sets $S^{(n)}$ are K-analytic. Hence the span, namely $\bigcup_{n\in\mathbb{N}}S^{(n)}$ is such ([Jay-Rog, Th. 2.5.4 p. 23]). ■

Theorem 11.3 (Intersection Theorem – [Jay-Rog, Th 2.5.3, p. 23]). *The intersection of a K -analytic set with a Souslin-$\mathcal{F}(X)$ in a Hausdorff space X is K-analytic.*

Theorem 11.4 (Projection Theorem – [RW] and . *Jay-Rog] Let X and Y be topological spaces with Y a K-analytic set. Then the projection on X of a Souslin-$\mathcal{F}(X\times Y)$ is Souslin-$\mathcal{F}(X)$.*

Theorem 11.5 (Nikodym's Theorem – [Nik]; [Jay-Rog, p. 42]). *The Baire sets of a space X are closed under the Souslin operation. Hence Souslin-$\mathcal{F}(X)$ sets are Baire.*

We promised examples of Baire sets; we can describe a hierarchy of them.

Examples of Baire sets. By analogy with the projective hierarchy of sets (known also as the *Luzin hierarchy* – see [Kech], p. 313, which may be generated from the closed sets by iterating projection and complementation any finite number of times), we may form the closely associated hierarchy of sets starting with the closed sets and iterating any finite number of times the Souslin operation $\mathbf{S}$ (following the notation of [Jay-Rog]) and complementation, denoted analogously by $\mathbf{C}$ say. Thus in a complete metric space one obtains the family $\mathcal{A}$ of analytic sets, by complementation the family $\mathbf{C}\mathcal{A}$ of co-analytic sets, then $\mathbf{SC}\mathcal{A}$ which contains the previous two classes, and so on. By Nikodym's theorem all these sets have the Baire property. One might call this the *Souslin hierarchy*.
One may go further and form the smallest σ-algebra (with complementation allowed) closed under $\mathbf{S}$ and containing the closed sets; this contains the Souslin hierarchy (implicit through an iteration over the countable ordinals). Members of the latter family are referred to as the *C-sets* – see Nowik and Reardon [NR].

Definitions. 1. Say that a function $f : X \to Y$ between two topological spaces is *$\mathcal{H}$-Baire*, for $\mathcal{H}$ a class of sets in Y, if $f^{-1}(H)$ has the Baire property for each set H in $\mathcal{H}$. Thus f is $\mathcal{F}(Y)$-Baire if $f^{-1}(F)$ is Baire for all closed F in Y. Taking complements, since

$$f^{-1}(Y\backslash H) = X\backslash f^{-1}(H),$$

f is $\mathcal{F}(Y)$-Baire iff it is $\mathcal{G}(Y)$-Baire, when we will simply say that f is *Baire* ('f has the Baire property' is the alternative usage).
2. One must distinguish between functions that are $\mathcal{F}(Y)$-Baire and those that lie in the smallest family of functions closed under pointwise limits of sequences and containing the continuous functions (for a modern treatment see [Jay-Rog, Sect. 6]). We follow tradition in calling these last *Baire-measurable* (originally called by Lebesgue the *analytically representable* functions, a term used in the context of metric spaces in [Kur-1, 2.31.IX, p. 392] ; cf. [Fos]).
3. We will say that a function is *Baire-continuous* if it is continuous when restricted to some co-meagre set. In the real line case and with the density topology, this is Denjoy's approximate continuity ([LMZ, p. 1]); recall ([Kech, 17.47]) that a set is (Lebesgue) measurable iff it has the Baire property under the density topology.

The connections between these concepts are given in the theorems below. See the cited papers for proofs, and for the starting point, Baire's Theorem on the points of discontinuity of a Borel measurable function.

Theorem 11.6 (Discontinuity Set Theorem – [Kur-1, p. 397] or [Kur-A]; [Ba1]; cf. [Ne, I.4]). *(i) For $f : X \to Y$ Baire-measurable, with X, Y metric, the set of discontinuity points is meagre; in particular*
(ii) for $f : X \to Y$ Borel-measurable of class 1, with X, Y metric and Y separable, the set of discontinuity points is meagre.

In regard to (ii) see [Han-71, Th. 10] for the non-separable case. The following theorem, from recent literature, usefully overlaps with the last result.

Theorem 11.7 (Banach-Neeb Theorem – [Ban-T], Th. 4 pg. 35 and Vol I p. 206; [Ne] (i) I. 6 (ii) I.4). *(i) A Borel-measurable $f : X \to Y$ with X, Y metric and Y separable and arcwise connected is Baire-measurable; and*
(ii) a Baire-measurable $f : X \to Y$ with X a Baire space and Y metric is Baire-continuous.

Remark. In fact Banach shows that a Baire-measurable function is Baire-continuous on each perfect set ([Ban-T, Vol. II p. 206]). In (i) if X, Y are completely metrizable, topological groups and f is a homomorphism, Neeb's assumption that Y is arcwise connected becomes unecessary, since, as Pestov [Pes] remarks, the arcwise connectedness may be dropped by referring to a result of Hartman and Mycielski [HM] that a separable metrizable group embeds as a subgroup of an arcwise connected separable metrizable group.

Theorem 11.8 (Baire Continuity Theorem). *A Baire function $f : X \to Y$ is Baire-continuous in the following cases:*
(i) Baire condition (see e.g. [HJ, Th. 2.2.10 p. 346]): Y is a second-countable space;
(ii) Emeryk-Frankiewicz-Kulpa ([EFK]): X is Čech-complete and Y has a base of cardinality not exceeding the continuum;
(iii) Pol condition ([Pol]): f is Borel, X is Borelian-K and Y is metrizable and of non-measurable cardinality;
(iv) Hansell condition ([Han-71]): f is σ-discrete and Y is metric.

We will say that the pair (X, Y) *enables Baire continuity* if the spaces X, Y satisfy either of the two conditions (i) or (ii) above. In the applications below Y is usually the additive group of reals $\mathbb{R}$, so satisfies (i). Building on [EFK], Fremlin ([Frem, Section 10]) characterizes a space X such that every Baire function $f : X \to Y$ is Baire-continuous for all metric Y in the language of 'measurable spaces with negligibles'; reference there is made to disjoint families of negligible sets all of whose subfamilies have a measurable union. For a discussion of discontinuous homomorphisms, especially counterexamples on $C(X)$ with X compact (e.g. employing Stone-Čech compactifications, $X = \beta\mathbb{N}\backslash\mathbb{N}$), see [Dal, Section 9].

Remark. Hansell's condition, requiring the function f to be σ-discrete, is implied by f being analytic when X is absolutely analytic (i.e. Souslin-$\mathcal{F}(X)$ in any complete metric space X into which it embeds). Frankiewicz and Kunen in [FrKu] study the consistency relative to ZFC of the existence of a Baire function failing to have Baire continuity.

The following result provides a criterion for verifying that f is Baire.

Theorem 11.9 (Souslin criterion - for Baire functions). *Let X and Y be Hausdorff topological spaces with Y a K-analytic space. If $f : X \to Y$ has Souslin-$\mathcal{F}(X \times Y)$ graph, then f is Baire.*

Proof. Let $G \subseteq X \times Y$ be the graph of f which is Souslin-$\mathcal{F}(X \times Y)$. For F closed in Y, we have

$$f^{-1}(F) = \mathrm{pr}_X[G \cap (X \times F)],$$

which, by the Intersection Theorem (Th. 11.3), is the projection of a *Souslin-$\mathcal{F}(X \times Y)$* set. By the Projection Theorem (Th. 11.4), $f^{-1}(F)$ is Souslin-$\mathcal{F}(X)$. Closed sets have the Baire property by definition, so by Nikodym's Theorem $f^{-1}(F)$ has the Baire property. ∎

We note that in the realm of separable metric spaces, a *surjective* map f with analytic graph is in fact Borel (since for U open $f^{-1}(U)$ and $f^{-1}(Y \backslash U)$ are complementary analytic and so Borel sets, by Souslin's Theorem (see [Jay-Rog] Th.1.4.1); for the non-separable case compare [Han-71, Th. 4.6(a)]).

Before stating our next theorem we recall a classical result in a sharper form. We are grateful to the referee for the statement and proof of this result in the topological group setting, here amended to the normed group setting.

Theorem 11.10 (Banach-Mehdi Continuity Theorem – [Ban-T, 1.3.4, p. 40], [Meh], [HJ, Th. 2.2.12 p. 348], or [BOst-TRII]). *A Baire-continuous homomorphism $f : X \to Y$ between normed groups, with X Baire in the norm topology, is continuous. In particular this is so for f Borel-measurable and Y separable.*

Proof. We work with the right norm topologies without loss of generality, since inversion is a homomorphism and also an isometry from the left to the right norm topology (Prop. 2.5). We claim that it is enough to prove the following: for any non-empty open G in X and any $\varepsilon > 0$ there is a non-empty open $V \subseteq G$ with $\mathrm{diam}(f(V)) < \varepsilon$. Indeed the claim implies that for each $n \in \mathbb{N}$ the set $W_n := \bigcup\{V : \mathrm{diam}(f(V)) < 1/n$ and V is open and non-empty$\}$ is dense and open in X. Hence, as X is Baire, the intersection $\bigcap_{n\in} W_n$ is a non-empty set containing continuity points of f; but f is a homomorphism so is continuous everywhere.
Now fix G non-empty and open and $\varepsilon > 0$. As f is Baire-continuous, it is continuous when

restricted to $X\backslash M$ for some meagre set M. As M may be included in a countable union of closed nowhere dense sets N, f restricted to some non-meagre $\mathcal{G}_\delta$-set is continuous (e.g. to $X\backslash N$). Passing to a subset, there is a non-meagre $\mathcal{G}_\delta$-set H in X with $B^X_\varepsilon(H) \subseteq G$ such that $\mathrm{diam}_X(H) < \varepsilon/12$ and $\mathrm{diam}_Y(f(H)) < \varepsilon/4$.
Note that $HH^{-1}HH^{-1} \subseteq B^X_\varepsilon(e_X)$ (as $\|h'h^{-1}\| \leq \|h'\| + \|h\|$) and likewise $f(H)f(H)^{-1}f(H)f(H)^{-1} \subseteq B^Y_{\varepsilon/3}(e_Y)$. By the Squared Pettis Theorem (Th. 6.5), there is a non-empty open set U contained in $HH^{-1}HH^{-1}$. Fix $h \in H$ and put $V := Uh$. Then

$$V = Uh \subseteq HH^{-1}HH^{-1}h \subseteq B_\varepsilon(e_X)h = B_\varepsilon(h) \subseteq B^X_\varepsilon(H) \subseteq G,$$

and so $V \subseteq G$; moreover, since f is a homomorphism,

$$f(V) = f(Uh) = f(U)f(h) \subseteq f(H)f(H)^{-1}f(H)f(H)^{-1}f(h) \subseteq B^Y_{\varepsilon/3}(f(h))$$

and so $\mathrm{diam}_Y f(V) < \varepsilon$, as claimed.
The final assertion now follows from the Banach-Neeb Theorem (Th. 11.7). ∎

The Souslin criterion and the next theorem together have as an immediate corollary the classical Souslin-graph Theorem; in this connection recall (see the corollary of [HJ, Th. 2.3.6 p. 355]) that a normed group which is Baire and analytic is Polish. Our proof, which is for normed groups, is inspired by the topological vector space proof in [Jay-Rog, §2.10] of the Souslin-graph theorem; their proof may be construed as having two steps: one establishing the Souslin criterion (Th. 11.9 as above), the other the Baire homomorphism theorem. They state without proof the topological group analogue. (See [Ost-AB] for non-separable analogues.)

Theorem 11.11 (Baire Homomorphism Theorem, cf. [Jay-Rog, §2.10]). *Let X and Y be normed groups with X topologically complete. If $f : X \to Y$ is a Baire homomorphism, then f is continuous. In particular, if f is a homomorphism with a Souslin-$\mathcal{F}(X \times Y)$ graph and Y is in addition a K-analytic space, then f is continuous*

Proof. For $f : X \to Y$ the given homomorphism, it is enough to prove continuity at e_X, i.e. that for any $\varepsilon > 0$ there is $\delta > 0$ such that $B_\delta(e_X) \subseteq f^{-1}[B_\varepsilon(e_X)]$. So let $\varepsilon > 0$. We work with the right norm topology.
Being K-analytic, Y is Lindelöf (cf. [Jay-Rog, Th. 2.7.1, p. 36]) and metric, so separable; so choose a countable dense set $\{y_n\}$ in $f(X)$ and select $a_n \in f^{-1}(y_n)$. Put $T := f^{-1}[B_{\varepsilon/4}(e_Y)]$. Since f is a homomorphism, $f(Ta_n) = f(T)f(a_n) = B_{\varepsilon/4}(e_Y)y_n$. Note also that $f(T^{-1}) = f(T)^{-1}$, so

$$TT^{-1} = f^{-1}[B_{\varepsilon/4}(e_Y)]f^{-1}[B_{\varepsilon/4}(e_Y)^{-1}] = f^{-1}[B_{\varepsilon/4}(e_Y)^2] \subseteq f^{-1}[B_{\varepsilon/2}(e_Y)],$$

by the triangle inequality.
Now

$$f(X) \subseteq \bigcup\nolimits_n B_\varepsilon(e_Y)y_n,$$

so

$$X = f^{-1}(Y) = \bigcup\nolimits_n Ta_n.$$

But X is non-meagre, so for some n the set Ta_n is non-meagre, and so too is T (as right-shifts are homeomorphisms). By assumption f is Baire. Thus T is Baire and non-meagre. By the Squared Pettis Theorem (Th. 5.8), $(TT^{-1})^2$ contains a ball $B_\delta(e_X)$. Thus we have

$$B_\delta(e_X) \subseteq (TT^{-1})^2 \subseteq f^{-1}[B_{\varepsilon/4}(e_Y)^4] = f^{-1}[B_\varepsilon(e_Y)].$$

■

Theorem 11.12 (Souslin-graph Theorem, Schwartz [Schw], cf. [Jay-Rog, p.50]). *Let X and Y be normed groups with Y a K-analytic and X non-meagre. If $f : X \to Y$ is a homomorphism with Souslin-$\mathcal{F}(X \times Y)$ graph, then f is continuous.*

Proof. This follows from Theorems 11.9 and 11.11. ■

Corollary 11.13 (Generalized Jones Theorem: Thinned Souslin-graph Theorem). *Let X and Y be topological groups with X non-meagre and Y a K-analytic set. Let S be a K-analytic set spanning X and $f : X \to Y$ a homomorphism with restriction to S continous on S. Then f is continuous.*

Proof. Since f is continuous on S, the graph $\{(x, y) \in S \times Y : y = f(x)\}$ is closed in $S \times Y$ and so is K-analytic by [Jay-Rog, Th. 2.5.3]. Now $y = f(x)$ iff, for some $n \in \mathbb{N}$, there is $(y_1, ..., y_n) \in Y^n$ and $(s_1, ..., s_n) \in S^n$ such that $x = s_1 \cdot ... \cdot s_n$, $y = y_1 \cdot ... \cdot y_n$, and, for $i = 1, .., n$, $y_i = f(s_i)$. Thus $G := \{(x, y) : y = f(x)\}$ is K-analytic. Formally,

$$G = \mathrm{pr}_{X\times Y}\left[\bigcup\nolimits_{n\in\mathbb{N}}\left[M_n \cap (X \times Y \times S^n \times Y^n) \cap \bigcap\nolimits_{i\le n} G_{i,n}\right]\right],$$

where

$$M_n := \{(x, y, s_1, ..., s_n, y_1, ..., y_n) : y = y_1 \cdot ... \cdot y_n \text{ and } x = s_1 \cdot ... \cdot s_n\},$$

and

$$G_{i,n} := \{(x, y, s_1, ..., s_n, y_1, ..., y_n) \in X \times Y \times X^n \times Y^n : y_i = f(s_i)\}, \text{for } i = 1, ..., n.$$

Here each set M_n is closed and each $G_{i,n}$ is K-analytic. Hence, by the Intersection and Projection Theorems (Th. 11.3 and 11.4), the graph G is K-analytic. By the Souslin-graph theorem f is thus continuous. ■

This is a new proof of the Jones Theorem. We now consider results for the more special normed group context. Here again one should note the corollary of [HJ, Th. 2.3.6 p. 355] that a normed group which is Baire and analytic is Polish. Our next result has a proof which is a minor adaptation of the proof in [BoDi]. We recall that a Hausdorff topological space is *paracompact* ([Eng, Ch. 5], or [Kel, Ch. 6], especially Problem Y) if every open cover has a locally finite open refinement and that (i) Lindelöf spaces and (ii) metrizable spaces are paracompact. Paracompact spaces are normal, hence topological groups need not be paracompact, as exemplified again by the example due to Oleg Pavlov [Pav] quoted earlier or by the example of van Douwen [vD] (see also [Com, Section 9.4 p. 1222]); however, L. G. Brown [Br-2] shows that a locally complete topological group is paracompact (and this includes the locally compact case, cf. [Com, Th. 2.9 p. 1161]). The assumption of paracompactness is thus natural.

Theorem 11.14 (The Second Generalized Kestelman-Borwein-Ditor Theorem: Measurable Case – cf. Th. 7.6). *Let G be a paracompact topological group equipped with a locally finite, inner regular Borel measure m (Radon measure) which is left-invariant, resp. right-invariant (for example, G locally compact, equipped with a Haar measure).*
If A is a (Borel) measurable set with $0 < m(A) < \infty$ and $z_n \to e$, then, for m-almost all $a \in A$, there is an infinite set $\mathbb{M}_a$ such that the corresponding right-translates, resp. left-translates, of z_n are in A, i.e., in the first case

$$\{z_n a : n \in \mathbb{M}_a\} \subseteq A.$$

Proof. Without loss of generality we conside right-translation of the sequence $\{z_n\}$. Since G is paracompact, it suffices to prove the result for A open and of finite measure. By inner-regularity A may be replaced by a σ-compact subset of equal measure. It thus suffices to prove the theorem for K compact with $m(K) > 0$ and $K \subseteq A$. Define a decreasing sequence of compact sets $T_k := \bigcup_{n \geq k} z_n^{-1} K$, and let $T = \bigcap_k T_k$. Thus $x \in T$ iff, for some infinite $\mathbb{M}_x$,

$$z_n x \in K \text{ for } m \in \mathbb{M}_x,$$

so that T is the set of 'translators' x for the sequence $\{z_n\}$. Since K is closed, for $x \in T$, we have $x = \lim_{n \in \mathbb{M}_x} z_n x \in K$; thus $T \subseteq K$. Hence, for each k,

$$m(T_k) \geq m(z_k^{-1} K) = m(K),$$

by left-invariance of the measure. But, for some n, $T_n \subseteq A$. (If $z_n^{-1} k_n \notin A$ on an infinite set $\mathbb{M}$ of n, then since $k_n \to k \in K$ we have $z_n^{-1} k_n \to k \in A$, but $k = \lim z_n^{-1} k_n \notin A$, a contradiction since A is open.) So, for some n, $m(T_n) < \infty$, and thus $m(T_k) \to m(T)$. Hence $m(K) \geq m(T) \geq m(K)$. So $m(K) = m(T)$ and thus almost all points of K are translators. ∎

Remark. It is quite consistent to have the measure left-invariant and the metric right-invariant.

Theorem 11.15 (Analytic Dichotomy Lemma on Spanning). *Let G be a connected, normed group (in the measure case a normed topological group). Suppose that an analytic set $T \subseteq G$ spans a set of positive measure or a non-meagre set. Then T spans G.*

Proof. In the category case, the result follows from the Banach-Kuratowski Dichotomy, Th. 6.13 ([Ban-G, Satz 1], [Kur-1, Ch. VI. 13. XII], [Kel, Ch. 6 Prob. P p. 211]) by

considering S, the subgroup generated by T; since T is analytic, S is analytic and hence Baire, and, being non-meagre, is clopen and hence all of G, as the latter is a connected group.
In the measure case, by the Steinhaus Theorem, Th. 6.10 ([St], [BGT, Th. 1.1.1], [BOst-StOstr]), T^2 has non-empty interior, hence is non-meagre. The result now follows from the category case. ∎

Our next result follows directly from Choquet's Capacitability Theorem [Choq] (see especially [Del2, p. 186], and [Kech, Ch. III 30.C]). For completeness, we include the brief proof. Incidentally, the argument we employ goes back to Choquet's theorem, and indeed further, to [RODav] (see e.g. [Del1, p. 43]).

Theorem 11.16 (Compact Contraction Lemma). *In a normed topological group carrying a Radon measure, for T analytic, if $T \cdot T$ has positive Radon measure, then for some compact subset S of T, $S \cdot S$ has positive measure.*

Proof. We present a direct proof (see below for our original inspiration in Choquet's Theorem). As T^2 is analytic, we may write ([Jay-Rog]) $T^2 = h(H)$, for some continuous h and some $\mathcal{K}_{\sigma\delta}$ subset of the reals, e.g. the set H of the irrationals, so that $H = \bigcap_i \bigcup_j d(i,j)$, where $d(i,j)$ are compact and, without loss of generality, the unions are each increasing: $d(i,j) \subseteq d(i,j+1)$. The map $g(x,y) := xy$ is continuous and hence so is the composition $f = g \circ h$. Thus $T \cdot T = f(H)$ is analytic. Suppose that $T \cdot T$ is of positive measure. Hence, by the capacitability argument for analytic sets ([Choq], or [Si, Th. 4.2 p. 774], or [Rog1, p. 90], there referred to as an 'Increasing sets lemma'), for some compact set A, the set $f(A)$ has positive measure. Indeed if $|f(H)| > \eta > 0$, then the set A may be taken in the form $\bigcap_i d(i,j_i)$, where the indices j_i are chosen inductively, by reference to the increasing union, so that $|f[H \cap \bigcap_{i<k} d(i,j_i)]| > \eta$, for each k. (Thus $A \subseteq H$ and $f(A) = \bigcap_i f[H \cap \bigcap_{i<k} d(i,j_i)]$ has positive measure, cf. [EKR].)
The conclusion follows as $S = h(A)$ is compact and $S \cdot S = g(S) = f(A)$. ∎

Remark. The result may be deduced indirectly from the Choquet Capacitability Theorem by considering the capacity $I : G^2 \to \mathbb{R}$, defined by $I(X) = m(g(X))$, where, as before, $g(x,y) := xy$ is continuous and m denotes a Radon measure on G (on this point see [Del2, Section 1.1.1, p. 186]). Indeed, the set T^2 is analytic ([Rog2, Section 2.8, p. 37-41]), so $I(T^2) = \sup I(K^2)$, where the supremum ranges over compact subsets K of T. Actually, the Capacitability Theorem says only that $I(T^2) = \sup I(K_2)$, where the supremum ranges over compact subsets K_2 of T^2, but such a set may be embedded in K^2 where $K = \pi_1(K) \cup \pi_2(K)$, with π_i the projections onto the axes of the product space.

Corollary 11.17. *For T analytic and $\varepsilon_i \in \{\pm 1\}$, if $T^{\varepsilon_1} \cdot \ldots \cdot T^{\varepsilon_d}$ has positive measure (measure greater than η) or is non-meagre, then for some compact subset S of T, the compact set $K = S^{\varepsilon_1} \cdot \ldots \cdot S^{\varepsilon_d}$ has $K \cdot K$ of positive measure (measure greater than η).*

Proof. In the measure case the same approach may be used based now on the continuous function $g(x_1,...,x_d) := x_1^{\varepsilon_1} \cdot ... \cdot x_d^{\varepsilon_d}$, ensuring that K is of positive measure (measure greater than η). In the category case, if $T' = T^{\varepsilon_1} \cdot ... \cdot T^{\varepsilon_d}$ is non-meagre then, by the Steinhaus Theorem ([St], or [BGT, Cor. 1.1.3]), $T' \cdot T'$ has non-empty interior. The measure case may now be applied to T' in lieu of T. (Alternatively one may apply the Pettis-Piccard Theorem, Th. 6.5, as in the Analytic Dichotomy Lemma, Th. 11.15.) ■

THEOREM 11.18 (Compact Spanning Approximation). *In a connected, normed topological group X, for T analytic in X, if the span of T is non-null or is non-meagre, then there exists a compact subset of T which spans X.*

Proof. If T is non-null or non-meagre, then T spans X (by the Analytic Dichotomy Lemma, Th. 11.15); then for some $\varepsilon_i \in \{\pm 1\}$, $T^{\varepsilon_1} \cdot ... \cdot T^{\varepsilon_d}$ has positive measure/ is non-meagre. Hence for some K compact $K^{\varepsilon_1} \cdot ... \cdot K^{\varepsilon_d}$ has positive measure/ is non-meagre. Hence K spans some and hence all of X. ■

THEOREM 11.19 (Analytic Covering Lemma – [Kucz, p. 227], cf. [Jones2, Th. 11]). *Given normed groups G and H, and T analytic in G, let $f : G \to H$ have continuous restriction $f|T$. Then T is covered by a countable family of bounded analytic sets on each of which f is bounded.*

Proof. For $k \in \omega$ define $T_k := \{x \in T : \|f(x)\| < k\} \cap B_k(e_G)$. Now $\{x \in T : \|f(x)\| < k\}$ is relatively open and so takes the form $T \cap U_k$ for some open subset U_k of G. The Intersection Theorem (Th. 11.3) shows this to be analytic since U_k is an $\mathcal{F}_\sigma$ set and hence Souslin-$\mathcal{F}$. ■

THEOREM 11.20 (Expansion Lemma – [Jones2, Th. 4], [Kom2, Th. 2], and [Kucz, p. 215]). *Suppose that S is Souslin-$\mathcal{H}$, i.e. of the form*

$$S = \bigcup\nolimits_{\alpha \in \omega^\omega} \cap_{n=1}^\infty H(\alpha|n),$$

with each $H(\alpha|n) \in \mathcal{H}$, for some family of analytic sets $\mathcal{H}$ on which f is bounded. If S spans the normed group G, then, for each n, there are sets $H_1, ..., H_k$ each of the form $H(\alpha|n)$, such that for some integers $r_1, ..., r_k$

$$T = H_1 \cdot ... \cdot H_k$$

has positive measure/ is non-meagre, and so $T \cdot T$ has non-empty interior.

Proof. For any $n \in \omega$ we have

$$S \subseteq \bigcup\nolimits_{\alpha \in \omega^\omega} H(\alpha|n).$$

Enumerate the countable family $\{H(\alpha|n) : \alpha \in \omega^n\}$ as $\{T_h : h \in \omega\}$. Since S spans G, we have

$$G = \bigcup_{h\in\omega} \bigcup_{\mathbf{k}\in\mathbb{N}^h} (T_{k_1} \cdot ... \cdot T_{k_h}).$$

As each T_k is analytic, so too is the continuous image

$$T_{k_1} \cdot ... \cdot T_{k_h},$$

which is thus measurable. Hence, for some $h \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^h$ the set

$$T_{k_1} \cdot ... \cdot T_{k_h}$$

has positive measure/ is non-meagre. ■

Definition. We say that S is a *pre-compact set* if its closure is compact. We will say that f is a *pre-compact function* if $f(S)$ is pre-compact for each pre-compact set S.

Theorem 11.21 (Jones-Kominek Analytic Automaticity Theorem for Metric Groups). *Let G be either a non-meagre normed topological group, or a topological group supporting a Radon measure, and let H be K-analytic (hence Lindelöf, and so second countable in our metric setting). Let $h : G \to H$ be a homomorphism between metric groups and let T be an analytic set in G which finitely generates G.*
(i) (Jones condition) If h is continuous on T, then h is continuous.
(ii) (Kominek condition) If h is pre-compact on T, then h is precompact.

Proof. As in the Analytic Covering Lemma (Th. 11.19), write

$$T = \bigcup_{k\in\mathbb{N}} T_k.$$

(i) If h is not continuous, suppose that $x_n \to x_0$ but $h(x_n)$ does not converge to $h(x_0)$. Since

$$G = \bigcup_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} T_k^{(m)},$$

G is a union of analytic sets and hence analytic ([Jay-Rog, Th. 2.5.4 p. 23]). Now, for some m, k the m-span $T_k^{(m)}$ is non-meagre, as is the m-span $S_k^{(m)}$ of some compact subset $S_k \subseteq T_k$. So for some shifted subsequence $tx_n \to tx_0$, where t and x_0 lie in $S_k^{(m)}$. Thus there is an infinite set $\mathbb{M}$ such that, for $n \in \mathbb{M}$,

$$tx_n = t_n^1 ... t_n^m \text{ with } t_n^i \in S_k.$$

Without loss of generality, as S_k is compact,

$$t_n^{(i)} \to t_0^{(i)} \in S_k \subset T,$$

and so

$$tx_n = t_n^1 ... t_n^m \to t_0^1 ... t_0^m = tx_0 \text{ with } t_0^i \in S_k \subset T.$$

Hence, as $t_n^i \to t_0^i \subset T$, we have, for $n \in \mathbb{M}$,

$$\begin{aligned} h(t)h(x_n) &= h(tx_n) = h(t_n^1...t_n^m) = h(t_n^1)...h(t_n^m) \\ &\to h(t_0^1)...h(t_0^m) = h(t_0^1...t_0^m) \\ &= h(tx_0) = h(t)h(x_0). \end{aligned}$$

Thus

$$h(x_n) \to h(x_0),$$

a contradiction.

(ii) If $\{h(x_n)\}$ is not precompact with $\{x_n\}$ precompact, by the same argument, for some $S_k^{(n)}$ and some infinite set $\mathbb{M}$, we have $tx_n = t_n^1...t_n^m$ and $t_n^i \to t_0^i \subset T$, for $n \in \mathbb{M}$. Hence $h(tx_n) = h(t)h(x_n)$ is precompact and so $h(x_n)$ is precompact, a contradiction. ■

The following result connects the preceeding theorem to Darboux's Theorem, that a locally bounded additive function on the reals is continuous ([Dar], or [AD]).

Definition. Say that a homomorphism between normed groups is $\mathbb{N}$-*homogeneous* if $\|f(x^n)\| = n\|f(x)\|$, for any x and $n \in \mathbb{N}$ (cf. Section 2 where $\mathbb{N}$-*homogeneous* norms were considered, for which homomorphisms are automatically $\mathbb{N}$-*homogeneous*). Thus any homomorphism into the additive reals is $\mathbb{N}$-homogeneous. Recall from Section 3.3 that the norm is a *Darboux norm* (or, in this context $\mathbb{N}$-*subhomogeneous*) if there are constants κ_n with $\kappa_n \to \infty$ such that for all elements z of the group

$$\kappa_n\|z\| \le \|z^n\|,$$

or equivalently

$$\|z^{1/n}\| \le \frac{1}{\kappa_n}\|z\|.$$

Thus $z^{1/n} \to e$; a related condition was considered by McShane in [McSh] (cf. the Eberlein-McShane Theorem, Th. 10.1). In keeping with the convention of functional analysis (appropriately to our usage of norm) the next result refers to a locally bounded homomorphism as bounded.

Theorem 11.22 (Generalized Darboux Theorem – [Dar]). *A bounded homomorphism from a normed group to a Darboux normed group ($\mathbb{N}$-subhomogeneous norm) is continuous. In particular, a bounded, additive function on $\mathbb{R}$ is continuous.*

Proof. Suppose that $f : G \to H$ is a homomorphism to a normed $\mathbb{N}$-subhomogeneous group H; thus $\|f(x^n)\| \ge \kappa_n\|f(x)\|$, for any $x \in G$ and $n \in \mathbb{N}$. Suppose that f is bounded by M and, for $\|x\| < \eta$, we have

$$\|f(x)\| < M.$$

Let $\varepsilon > 0$ be given. Choose N such that $\kappa_N > M/\varepsilon$, i.e. $M/\kappa_N < \varepsilon$. Now $x \to x^N$ is continuous, hence there is $\delta = \delta_N(\eta) > 0$ such that, for $\|x\| < \delta$,

$$\|x^N\| < \eta.$$

Consider x with $\|x\| < \delta_N(\eta)$. Then $\kappa_N\|f(x)\| \leq \|f(x)^N\| = \|f(x^N)\| < M$. So for x with $\|x\| < \delta_N(\eta)$ we have

$$\|f(x)\| < M/\kappa_N < \varepsilon,$$

proving continuity at e. ■

Compare [HJ, Th 2.4.9 p. 382]. The Main Theorem of [BOst-Thin] may be given a combinatorial restatement in the group setting. We need some further definitions.

Definition. For G a metric group, let $\mathcal{C}(G) = \mathcal{C}(\mathbb{N}, G) := \{\mathbf{x} \in G^{\mathbb{N}} : \mathbf{x} \text{ is convergent}\}$ denote the *sequence space* of G. For $x \in \mathcal{C}(G)$ we write

$$L(\mathbf{x}) = \lim_n x_n.$$

We make $\mathcal{C}(G)$ into a group by setting

$$\mathbf{x} \cdot \mathbf{y} := \langle x_n y_n : n \in \mathbb{N}\rangle.$$

Thus $\mathbf{e} = \langle e_G\rangle$ and $\mathbf{x}^{-1} = \langle x_n^{-1}\rangle$. We identify G with the subgroup of constant sequences, that is

$$T = \{\langle g : n \in \mathbb{N}\rangle : g \in G\}.$$

The natural action of G or T on $\mathcal{C}(G)$ is then $t\mathbf{x} := \langle tx_n : n \in \mathbb{N}\rangle$. Thus $\langle g\rangle = g\mathbf{e}$, and then $t\mathbf{x} = t\mathbf{e} \cdot \mathbf{x}$.

Definition. For G a group, a set $\mathcal{G}$ of convergent sequences $\mathbf{u} = \langle u_n : n \in \mathbb{N}\rangle$ in $c(G)$ is a *G-ideal in the sequence space* $\mathcal{C}(G)$ if it is a subgroup closed under the mutiplicative action of G, and will be termed *complete* if it is closed under subsequence formation. That is, a complete G-ideal in $\mathcal{C}(G)$ satisfies
(i) $\mathbf{u} \in \mathcal{G}$ implies $t\mathbf{u} = \langle tu_n\rangle \in \mathcal{G}$, for each t in G,
(ii) $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ implies that $\mathbf{u}\mathbf{v}^{-1} \in \mathcal{G}$,
(iii) $\mathbf{u} \in \mathcal{G}$ implies that $\mathbf{u}_{\mathbb{M}} := \{u_m : m \in \mathbb{M}\} \in \mathcal{G}$ for every infinite $\mathbb{M}$.

If $\mathcal{G}$ satisfies (i) and $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ implies only that $\mathbf{uv} \in \mathcal{G}$, we say that $\mathcal{G}$ is a *G-subideal* in $\mathcal{C}(G)$.

Remarks. 0. In the notation of (iii) above, if $\mathcal{G}$ is merely an ideal then $\mathcal{G}^* = \{\mathbf{u}_{\mathbb{M}} : \text{for } \mathbf{u} \in t \text{ and } \mathbb{M} \subset \mathbb{N}\}$ is a complete G-ideal; indeed $t\mathbf{u}_{\mathbb{M}} = (t\mathbf{u})_{\mathbb{M}}$ and $\mathbf{u}_{\mathbb{M}}\mathbf{v}_{\mathbb{M}}^{-1} = (\mathbf{u}\mathbf{v}^{-1})_{\mathbb{M}}$ and $\mathbf{u}_{\mathbb{M}\mathbb{M}'} = \mathbf{u}_{\mathbb{M}'}$ for $\mathbb{M}' \subset \mathbb{M}$.
1. We speak of a Euclidean sequential structure when G is the vector space $\mathbb{R}^d$ regarded as an additive group.
2. The conditions (i) and (ii) assert that $\mathcal{G}$ is similar in structure to a left-ideal, being closed under multiplication by G and a subgroup of $\mathcal{C}(G)$.
3. We refer only to the combinatorial properties of $\mathcal{C}(G)$; but one may give $\mathcal{C}(G)$ a pseudo-norm by setting

$$\|x\|_c := d^G(Lx, e) = \|Lx\|, \text{ where } Lx := \lim x_n.$$

The corresponding pseudo-metric is

$$d(x, y) := \lim d^G(x_n, y_n) = d^G(Lx, Ly).$$

We may take equivalence of sequences with identical limit; then $\mathcal{C}(G)^{\sim}$ becomes a normed group (cf. Th. 3.38). However, in our theorem below we do not wish to refer to such an equivalence.

Definitions. For a family $\mathcal{F}$ of functions from G to H, we denote by $\mathcal{F}(T)$ the family $\{f|T : f \in \mathcal{F}\}$ of functions in $\mathcal{F}$ restricted to $T \subseteq G$. Let us denote a convergent sequence with limit x_0, by $\{x_n\} \to x_0$. We say the property $\mathcal{Q}$ of functions (property being regarded set-theoretically, i.e. as a family of functions from G to H) is *sequential on T* if

$$f \in \mathcal{Q} \text{ iff } (\forall \{x_n : n > 0\} \subseteq T)[(\{x_n\} \to x_0) \implies f|\{x_n : n > 0\} \in \mathcal{Q}(\{x_n : n > 0\})].$$

If we further require the limit point to be enumerated in the sequence, we call $\mathcal{Q}$ *completely sequential on T* if

$$f \in \mathcal{Q} \text{ iff } (\forall \{x_n\} \subseteq T)[(\{x_n\} \to x_0) \implies f|\{x_n\} \in \mathcal{Q}(\{x_n\})].$$

Our interest rests on properties that are completely sequential; our theorem below contains a condition referring to completely sequential properties, that is, the condition is required to hold on convergent sequences with limit included (so on a compact set), rather than on arbitrary sequences.

Note that if $\mathcal{Q}$ is (completely) sequential then $f|\{x_n\} \in \mathcal{Q}(\{x_n\})$ iff $f|\{x_n : n \in \mathbb{M}\} \in \mathcal{Q}(\{x_n : n \in \mathbb{M}\})$, for every infinite $\mathbb{M}$.

Definition. Let $h : G \to H$, with G, H metric groups. Say that a sequence $\mathbf{u} = \{u_n\}$ is *$\mathcal{Q}$-good* for h if

$$h|\{u_n\} \in \mathcal{Q}|\{u_n\},$$

and put

$$\mathcal{G}_{h\mathcal{Q}} = \{\mathbf{u} : h|\{u_n\} \in \mathcal{Q}|\{u_n\}\}.$$

If $\mathcal{Q}$ is completely sequential, then $\mathbf{u}$ is $\mathcal{Q}$-good for h iff every subsequence of $\mathbf{u}$ is $\mathcal{Q}$-good for h, so that $\mathcal{G}_{h\mathcal{Q}}$ is a G-ideal iff it is a complete G-ideal. One then has:

Lemma 11.23. *If $\mathcal{Q}$ is completely sequential and $\mathcal{F}$ preserves $\mathcal{Q}$ under shift and multiplication and division on compacts, then $\mathcal{G}_{h\mathcal{Q}}$ for $h \in \mathcal{F}$ is a G-ideal.*

Theorem 11.24 (Analytic Automaticity Theorem - combinatorial form). *Suppose that functions of $\mathcal{F}$ having $\mathcal{Q}$ on G have $\mathcal{P}$ on G, where $\mathcal{Q}$ is a property of functions from G to H that is completely sequential on G.*

Suppose that, for all $h \in \mathcal{F}$, $\mathcal{G}_{h\mathcal{Q}}$, the family of $\mathcal{Q}$-good sequences is a G-ideal. Then, for any analytic set T spanning G, functions of $\mathcal{F}$ having $\mathcal{Q}$ on T have $\mathcal{P}$ on G.

This theorem is applied with $G = \mathbb{R}^d$ and $H = \mathbb{R}$ in [BOst-Aeq] to subadditive functions, convex functions, and to regularly varying functions (defined on $\mathbb{R}^d$) to derive automatic properties such as automatic continuity, automatic local boundedness and automatic uniform boundedness.

12. Duality in normed groups

In this section – to distinguish two contexts – we use the generic notation of S for a group with metric d^S; recall from Section 3 that $Auth(S)$ denotes the self-homeomorphisms (auto-homeomorphisms) of S; $\mathcal{H}(S)$ denotes the bounded elements of $Auth(S)$. We write $\mathcal{A} \subseteq \mathcal{H}(S)$ for a subgroup of self-homeomorphisms of S. We work in the category of normed groups. However, by specializing to $\mathcal{A} = \mathcal{H}_u(S)$, the homeomorphisms that are bi-uniformly continuous (relative to d^S), we can regard the development as also taking place inside the category of topological groups, by Th. 3.13. We assume that $\mathcal{A}$ is metrized by the supremum metric

$$d^{\mathcal{A}}(t_1, t_2) = \sup\nolimits_{s \in S} d^S(t_1(s), t_2(s)).$$

Note that $e_{\mathcal{A}} = id_S$. The purpose of this notation is to embrace the two cases: (i) $S = X$ and $\mathcal{A} = \mathcal{H}_u(X)$, and (ii) $S = \mathcal{H}_u(X)$ and $\mathcal{A} = \mathcal{H}_u(\mathcal{H}_u(X))$. In what follows, we regard the group $\mathcal{H}_u(X)$ as the topological (uniform) dual of X and verify that (X, d^X) is embedded in the second dual $\mathcal{H}_u(\mathcal{H}_u(X))$. As an application one may use this duality to clarify, in the context of a non-autonomous differential equation with initial conditions, the link between its solutions trajectories and flows of its varying 'coefficient matrix'. See [Se1] and [Se2], which derive the close relationship for a general non-autonomous differential equation $u' = f(u, t)$ with $u(0) = x \in X$, between its trajectories in X and local flows in the function space Φ of translates f_t of f (where $f_t(x, s) = f(x, t + s)$). One may alternatively capture the topological duality as algebraic complementarity – see [Ost-knit] for details. A summary will suffice here. One first considers the commutative diagram below where initially the maps are only *homeomorphisms* (herein $T \subseteq \mathcal{H}_u(X)$ and $\Phi^T(t, x) = (t, tx)$ and $\Phi^X(x, t) = (t, xt)$ are embeddings). Then one extends the diagram to a diagram of *isomorphisms,* a change facilitated by forming the direct product group $G := T \times X$. Thus $G = T_G X_G$ where T_G and X_G are normal subgroups, commuting elementwise, and isomorphic respectively to T and X; moreover, the subgroup T_G, acting multiplicatively on X_G, represents the T-flow on X and simultaneously the multiplicative action of X_G on G represents the X-flow on $T_X = \{t_x : t \in T, x \in X\}$, the group of right-translates of T, where $t_x(u) = \theta_x(t)(u) = t(ux)$. If G has an invariant metric d^G, and T_G and X_G are now regarded as groups of translations on G, then they may be metrized by the supremum metric $\hat{d}^G$, whereupon each is isometric to itself as subgroup of G. Our approach here suffers a loss of elegance, by dispensing with G, but gains analytically by working directly with d^X and $\hat{d}^X$.

$$\begin{tikzcd}
(t,x) \arrow[r, leftrightarrow, "\Phi^T"] \arrow[d, leftrightarrow] & (t,tx) \arrow[d, leftrightarrow] \\
(x,t) \arrow[r, leftrightarrow, "\Phi^X"] & (t,xt)
\end{tikzcd}$$

Here the two vertical maps may, and will, be used as identifications, since $(t,tx) \to (t,x) \to (t,xt)$ are bijections (more in fact is true, see [Ost-knit]).

Definitions. Let X be a topological group with right-invariant metric d^X. We define for $x \in X$ a map $\xi_x : \mathcal{H}(X) \to \mathcal{H}(X)$ by putting

$$\xi_x(s)(z) = s(\lambda_x^{-1}(z)) = s(x^{-1}z), \text{ for } s \in \mathcal{H}_u(X), z \in X.$$

We set

$$\Xi := \{\xi_x : x \in X\}.$$

By restriction we may also write $\xi_x : \mathcal{H}_u(X) \to \mathcal{H}_u(X)$.

Proposition 12.1. *Under composition Ξ is a group of isometries of $\mathcal{H}_u(X)$ isomorphic to X.*

Proof. The identity is given by $e_\Xi = \xi_e$, where $e = e_X$. Note that

$$\xi_x(e_S)(e_X) = x^{-1},$$

so the mapping $x \to \xi_x$ from X to Ξ is bijective. Also, for $s \in \mathcal{H}(X)$,

$$\begin{aligned}(\xi_x \circ \xi_y(s))(z) &= \xi_x(\xi_y(s))(z) = (\xi_y(s))(x^{-1}z)\\ &= s(y^{-1}x^{-1}z) = s((xy)^{-1}z) = \xi_{xy}(s)(z),\end{aligned}$$

so ξ is an isomorphism from X to Ξ and so $\xi_x^{-1} = \xi_{x^{-1}}$.
For x fixed and $s \in \mathcal{H}_u(X)$, note that by Lemma 3.8 and Cor. 3.6 the map $z \to s(x^{-1}z)$ is in $\mathcal{H}_u(X)$. Furthermore

$$d_{\mathcal{H}}(\xi_x(s), \xi_x(t)) = \sup\nolimits_z d^X(s(x^{-1}z), t(x^{-1}z)) = \sup\nolimits_y d^X(s(y), t(y)) = d_{\mathcal{H}}(s,t),$$

so ξ_x is an isometry, and hence is continuous. ξ_x is indeed a self-homeomorphism of $\mathcal{H}_u(X)$, as $\xi_{x^{-1}}$ is the continuous inverse of ξ_x. ■

Remark. The definition above lifts the isomorphism $\lambda : X \to Tr_L(X)$ to $\mathcal{H}_u(X)$. If $T \subseteq \mathcal{H}_u(X)$ is λ-invariant, we may of course restrict λ to operate on T. Indeed, if $T = Tr_L(X)$, we then have $\xi_x(\lambda_y)(z) = \lambda_y\lambda_x^{-1}(z)$, so $\xi_x(\lambda_y) = \lambda_{yx^{-1}}$.

In general it will not be the case that $\xi_x \in \mathcal{H}_u(\mathcal{H}_u(X))$, unless d^X is bounded. Recall that

$$\|x\|_\infty := \sup\nolimits_{s\in\mathcal{H}(X)} \|x\|_s = \sup\nolimits_{s\in\mathcal{H}(X)} d_s^X(x,e) = \sup\nolimits_{s\in\mathcal{H}(X)} d^X(s(x), s(e)).$$

By contrast we have

$$\|f\|_\infty = \sup_z \sup_g d_g^X(f(z), z).$$

However, for $f(z) = \lambda_x(z) := xz$, putting $s = g \circ \rho_z$ brings the the two formulas into alignment, as

$$\|\lambda_x\|_\infty = \sup_z \sup_g d^X(g(xz), g(z)) = \sup_z \sup_g d^X(g(\rho_z(x)), g(\rho_z(e))).$$

This motivates the following result.

Proposition 12.2. *The subgroup $H_X := \{x \in X : \|x\|_\infty < \infty\}$ equipped with the norm $\|x\|_\infty$ embeds isometrically under ξ into $\mathcal{H}_u(\mathcal{H}_u(X))$ as*

$$\Xi_H := \{\xi_x : x \in H_X\}.$$

Proof. Writing $y = x^{-1}z$ or $z = xy$, we have

$$\begin{aligned} d_{\mathcal{H}}(\xi_x(s), s) &= \sup_{z\in X} d^X(s(x^{-1}z), s(z)) = \sup_{y\in X} d^X(s(y), s(xy)) \\ &= \sup_{y\in X} d_s^X(\rho_y e, \rho_y x) = \sup_y d_{s\text{-}y}^X(e, x). \end{aligned}$$

Hence

$$\|\xi_x\|_{\mathcal{H}} = \sup_{s\in\mathcal{H}(X)} d_{\mathcal{H}}(\xi_x(s), s) = \|\lambda_x\|_\infty = \sup_{s\in\mathcal{H}(X)} \sup_{y\in X} d_s^X(y, xy) = \|x\|_\infty.$$

Thus for $x \in H_X$ the map ξ_x is bounded over $\mathcal{H}_u(X)$ and hence is in $\mathcal{H}_u(\mathcal{H}_u(X))$. ∎

The next result adapts ideas of Section 3 on the Lipschitz property in $\mathcal{H}_u$ (Th. 3.22) to the context of ξ_x and refers to the inverse modulus of continuity $\delta(s)$ which we recall:

$$\delta(g) = \delta_1(g) := \sup\{\delta > 0 : d^X(g(z), g(z')) \le 1, \text{ for all } d^X(z, z') \le \delta\}.$$

Proposition 12.3 (Further Lipschitz properties of $\mathcal{H}_u$). *Let X be a normed group with a vanishingly small global word-net. Then for $x, z \in X$ and $s \in \mathcal{H}_u(X)$ the s-z-shifted norm (recalled below) satisfies*

$$\|x\|_{s\text{-}z} := d_{s\text{-}z}^X(x, e) = d^X(s(z), s(xz)) \le 2\|x\|/\delta(s).$$

Hence

$$\|\xi_e\|_{\mathcal{H}(\mathcal{H}_u(X))} = \sup_{s\in\mathcal{H}_u(X)} \sup_{z\in X} \|e\|_{s\text{-}z} = 0,$$

and so $\xi_e \in \mathcal{H}(\mathcal{H}_u(X))$. Furthermore, if $\{\delta(s) : s \in \mathcal{H}_u(X)\}$ is bounded away from 0, then for $x \in X$

$$\begin{aligned} \|\xi_x\|_{\mathcal{H}(\mathcal{H}_u(X))} &= \sup_{s\in\mathcal{H}_u(X)} d_{\mathcal{H}(X)}(\xi_x(s), s) = \sup_{s\in\mathcal{H}_u(X)} \sup_{z\in X} d^X(s(x^{-1}z), s(z)) \\ &\le 2\|x\|/\inf\{\delta(s) : s \in \mathcal{H}_u(X)\}, \end{aligned}$$

and so $\xi_x \in \mathcal{H}(\mathcal{H}_u(X))$.
In particular this is so if in addition X is compact.

Proof. Writing $y = x^{-1}z$ or $z = xy$, we have

$$d_{\mathcal{H}}(\xi_x(s), s) = \sup_{z\in X} d^X(s(x^{-1}z), s(z)) = \sup_{y\in X} d^X(s(y), s(xy)).$$

Fix s. Since s is uniformly continuous, $\delta = \delta(s)$ is well-defined and

$$d(s(z'), s(z)) \leq 1,$$

for z, z' such that $d(z, z') < \delta$. In the definition of the word-net take $\varepsilon < 1$. Now suppose that $w(x) = w_1 ... w_{n(x)}$ with $\|z_i\| = \frac{1}{2}\delta(1 + \varepsilon_i)$ and $|\varepsilon_i| < \varepsilon$, where $n(x) = n(x, \delta)$ satisfies

$$1 - \varepsilon \leq \frac{n(x)\delta}{\|x\|} \leq 1 + \varepsilon.$$

Put $z_0 = z$, for $0 < i < n(x)$

$$z_{i+1} = z_i w_i,$$

and $z_{n(x)+1} = x$; the latter is within δ of x. As

$$d(z_i, z_{i+1}) = d(e, w_i) = \|w_i\| < \delta,$$

we have

$$d(s(z_i), s(z_{i+1})) \leq 1.$$

Hence

$$d(s(z), s(xz)) \leq n(x) + 1 < 2\|x\|/\delta.$$

The final assertion follows from the subadditivity of the Lipschitz norm (cf. Theorem 3.27). ∎

If $\{\delta(s) : s \in \mathcal{H}_u(X)\}$ is unbounded (i.e. the inverse modulus of continuity is unbounded), we cannot develop a duality theory. However, a comparison with the normed vector space context and the metrization of the translations $x \to t(z + x)$ of a linear map $t(z)$ suggests that, in order to metrize Ξ by reference to $\xi_x(t)$, we need to take account of $\|t\|$. Thus a natural metric here is, for any $\varepsilon \geq 0$, the magnification metric

$$d_T^{\varepsilon}(\xi_x, \xi_y) := \sup\nolimits_{\|t\| \leq \varepsilon} d^T(\xi_x(t), \xi_y(t)). \qquad \text{(mag-eps)}$$

By Proposition 2.14 this is a metric; indeed with $t = e_{\mathcal{H}(X)} = id_X$ we have $\|t\| = 0$ and, since d^X is assumed right-invariant, for $x \neq y$, we have with $z_{xy} = e$ that $d^X(x^{-1}z, y^{-1}z) = d^X(x^{-1}, y^{-1}) > 0$. The presence of the case $\varepsilon = 0$ is not fortuitous; see [Ost-knit] for an explanation via an isomorphism theorem. We trace the dependence on $\|t\|$ in Proposition 12.5 below. We refer to Gromov's notion [Gr1], [Gr2] of quasi-isometry under π, in which π is a mapping between spaces. In a first application we take π to be a self-homeomorphism, in particular a left-translation; in the second $\pi(x) = \xi_x(t)$ with t fixed is an evaluation map appropriate to a dual embedding. We begin with a theorem promised in Section 3.

Theorem 12.4 (Uniformity Theorem for Conjugation). *Let $\Gamma : G^2 \to G$ be the conjugation $\Gamma(g, x) := g^{-1}xg$.*
Under a bi-invariant Klee metric, for all a, b, g, h ,

$$d^G(a, b) - 2d^G(g, h) \leq d^G(gag^{-1}, hbh^{-1}) \leq 2d^G(g, h) + d^G(a, b),$$

and hence conjugation is uniformly continuous.

Proof. Referring to the Klee property, via the cyclic property we have

$$\begin{aligned} d^G(gag^{-1}, hbh^{-1}) &= \|gag^{-1}hb^{-1}h^{-1}\| = \|h^{-1}gag^{-1}hb^{-1}\| \\ &\le \|h^{-1}g\| + \|ag^{-1}hb^{-1}\| \\ &\le \|h^{-1}g\| + \|ab^{-1}\| + \|g^{-1}h\|, \end{aligned}$$

for all a, b, yielding the right-hand side inequality. Then substitute $g^{-1}ag$ for a etc., g^{-1} for g etc., to obtain

$$d^G(a,b) \le 2d^G(g^{-1}, h^{-1}) + d^G(gag^{-1}, hbh^{-1}).$$

This yields the left-hand side inequality, as d^G is bi-invariant and so

$$d^G(g^{-1}, h^{-1}) = \tilde{d}^G(g,h) = d^G(g,h).$$

■

Proposition 12.5 (Permutation metric). *For $\pi \in \mathcal{H}(X)$, let $d_\pi(x,y) := d^X(\pi(x), \pi(y))$. Then d_π is a metric, and*

$$d^X(x,y) - 2\|\pi\| \le d_\pi(x,y) \le d^X(x,y) + 2\|\pi\|.$$

In particular, if d^X is right-invariant and $\pi(x)$ is the left-translation $\lambda_z(x) = zx$, then

$$d^X(x,y) - 2\|z\| \le d_z^X(x,y) = d^X(zx, zy) \le d^X(x,y) + 2\|z\|.$$

Proof. By the triangle inequality,

$$d^X(\pi(x), \pi(y)) \le d^X(\pi(x), x) + d^X(x,y) + d^X(y, \pi(y)) \le 2\|\pi\| + d^X(x,y).$$

Likewise,

$$\begin{aligned} d^X(x,y) &\le d^X(x, \pi(x)) + d^X(\pi(x), \pi(y)) + d^X(\pi(y), y) \\ &\le 2\|\pi\| + d^X(\pi(x), \pi(y)). \end{aligned}$$

If $\pi(x) := zx$, then $\|\pi\| = \sup d(zx, x) = \|z\|$ and the result follows. ■

Recall from Proposition 2.2 that for d a metric on a group X, we write $\tilde{d}(x,y) = d(x^{-1}, y^{-1})$ for the *(inversion) conjugate metric*. The conjugate metric $\tilde{d}$ is left-invariant iff the metric d is right-invariant. Under such circumstances both metrics induce the same norm (since $d(e,x) = d(x^{-1}, e)$, as we have seen above). In what follows note that $\xi_x^{-1} = \xi_{x^{-1}}$.

Theorem 12.6 (Quasi-isometric duality). *If the metric d^X on X is right-invariant and $t \in T \subset \mathcal{H}(X)$ is a subgroup, then*

$$\tilde{d}^X(x,y) - 2\|t\|_{\mathcal{H}(X)} \le d^T(\xi_x(t), \xi_y(t)) \le \tilde{d}^X(x,y) + 2\|t\|_{\mathcal{H}(X)},$$

and hence, for each $\varepsilon \ge 0$, the magnification metric (mag-eps) satisfies

$$\tilde{d}^X(x,y) - 2\varepsilon \le d_T^\varepsilon(\xi_x, \xi_y) \le \tilde{d}^X(x,y) + 2\varepsilon.$$

Equivalently, in terms of conjugate metrics,

$$d^X(x,y) - 2\varepsilon \le \tilde{d}_T^\varepsilon(\xi_x, \xi_y) \le d^X(x,y) + 2\varepsilon.$$

Hence,

$$\|x\| - 2\varepsilon \leq \|\xi_x\|_\varepsilon \leq \|x\| + 2\varepsilon,$$

and so $\|x_n\| \to \infty$ *iff* $d^T(\xi_{x(n)}(t), \xi_e(t)) \to \infty$.

Proof. We follow a similar argument to that for the permutation metric. By right-invariance,

$$\begin{aligned} d^X(t(x^{-1}z), t(y^{-1}z)) &\leq d^X(t(x^{-1}z), x^{-1}z) + d^X(x^{-1}z, y^{-1}z) + d^X(y^{-1}z, t(y^{-1}z)) \\ &\leq 2\|t\| + d^X(x^{-1}, y^{-1}), \end{aligned}$$

so

$$d^T(\xi_x(t), \xi_y(t)) = \sup\nolimits_z d^X(t(x^{-1}z), t(y^{-1}z)) \leq 2\|t\| + d^X(x, e_X).$$

Now, again by right-invariance,

$$d^X(x^{-1}, y^{-1}) \leq d(x^{-1}, t(x^{-1})) + d(t(x^{-1}), t(y^{-1})) + d(t(y^{-1}), y^{-1}).$$

But

$$d(t(x^{-1}), t(y^{-1})) \leq \sup\nolimits_z d^X(t(x^{-1}z), t(y^{-1}z)),$$

so

$$d^X(x^{-1}, y^{-1}) \leq 2\|t\| + \sup\nolimits_z d^X(t(x^{-1}z), t(y^{-1}z)) = 2\|t\| + d^T(\xi_x(t), \xi_y(t)),$$

as required. ■

We thus obtain the following result.

THEOREM 12.7 (Topological Quasi-Duality Theorem). *For X a normed group, the second dual Ξ is a normed group isometric to X which, for any $\varepsilon \geq 0$, is ε-quasi-isometric to X in relation to $\tilde{d}_T^\varepsilon(\xi_x, \xi_y)$ and the $\|\cdot\|^\varepsilon$ norm. Here $T = \mathcal{H}_u(X)$.*

Proof. We metrize Ξ by setting $d_\Xi(\xi_x, \xi_y) = d^X(x, y)$. This makes Ξ an isometric copy of X and an ε-quasi-isometric copy in relation to the conjugate metric $\tilde{d}_T^\varepsilon(\xi_x, \xi_y)$ which is given, for any $\varepsilon \geq 0$, by

$$\tilde{d}_T^\varepsilon(\xi_x, \xi_y) := \sup\nolimits_{\|t\| \leq \varepsilon} d^T(\xi_x^{-1}(t), \xi_y^{-1}(t)).$$

In particular for $\varepsilon = 0$ we have

$$d^T(\xi_x^{-1}(e), \xi_y^{-1}(e)) = \sup\nolimits_z d^X(xz, yz) = d(x, y).$$

Assuming d^X is right-invariant, d_Ξ is right-invariant, since

$$d_\Xi(\xi_x\xi_z, \xi_y\xi_z) = d_\Xi(\xi_{xz}, \xi_{yz}) = d^X(xz, yz) = d^X(x, y).$$

■

Remark. Alternatively, working in $Tr_L(X)$ rather than in $\mathcal{H}_u(X)$ and with d_R^X again right-invariant, since $\xi_x(\lambda_y)(z) = \lambda_y\lambda_x^{-1}(z)) = \lambda_{yx^{-1}}(z)$, we have

$$\sup_w d_{\mathcal{H}}(\xi_x(\lambda_w), \xi_e(\lambda_w)) = \sup_v d_v^X(e, x) = \|x\|_\infty^X,$$

possibly infinite. Indeed

$$\begin{aligned}\sup_w d_{\mathcal{H}}(\xi_x(\lambda_w), \xi_y(\lambda_w)) &= \sup_w \sup_z d_R^X(\xi_x(\lambda_w)(z), \xi_y(\lambda_w)(z))\\ &= \sup_w \sup_z d_R^X(wx^{-1}z, wy^{-1}z) = \sup_w d_R^X(vxx^{-1}, vxy^{-1})\\ &= \sup_v d_R^X(vy, vx) = \sup_v d_v^X(y, x).\end{aligned}$$

(Here we have written $w = vx$.)
The refinement metric $\sup_v d^X(vy, vx)$ is left-invariant on the bounded elements (i.e. bounded under the corresponding norm $\|x\| := \sup\{\|vxv^{-1}\| : v \in X\}$; cf. Proposition 2.12). Of course, if d^X were bi-invariant (both right- and left-invariant), we would have

$$\sup_w d_H(\xi_x(\lambda_w), \xi_y(\lambda_w)) = d^X(x, y).$$

13. Divergence in the bounded subgroup

In earlier sections we made on occasion the assumption of a bounded norm. Here we are interested in norms that are unbounded. For S a space and $\mathcal{A}$ a subgroup of $Auth(S)$ equipped with the supremum norm, suppose $\varphi : \mathcal{A} \times S \to S$ is a continuous flow (see Lemma 3.8, for an instance). We will write $\alpha(s) := \varphi^\alpha(s) = \varphi(\alpha, s)$. This is consistent with $\mathcal{A}$ being a subgroup of $Auth(S)$. As explained at the outset of Section 12, we have in mind two pairs $(\mathcal{A}, S)$, as follows.

Example 1. Take $S = X$ to be a normed topological group and $\mathcal{A} = T \subseteq \mathcal{H}(X)$ to be a subgroup of automorphisms of X such that T is a topological group with supremum metric

$$d^T(t_1, t_2) = \sup_x d^X(t_1(x), t_2(x)),$$

e.g. $T = \mathcal{H}_u(X)$. Note that here $e_T = id_X$.

Example 2. $(\mathcal{A}, S) = (\Xi, T) = (X, T)$. Here X is identified with its second dual Ξ (of the preceding section).

Given a flow $\varphi(t, x)$ on $T \times X$, with T closed under translation, the action defined by

$$\varphi(\xi_x, t) := \xi_{x^{-1}}(t)$$

is continuous, hence a flow on $\Xi \times T$, which is identified with $X \times T$. Observe that $t(x) = \xi_{x^{-1}}(t)(e_X)$, i.e. projection onto the e_X coordinate retrieves the T-flow φ. Here, for $\xi = \xi_{x^{-1}}$, writing $x(t)$ for the translate of t, we have

$$\xi(t) := \varphi^\xi(t) = \varphi(\xi, t) = x(t),$$

so that φ may be regarded as a X-flow on T. We now formalize the notion of a sequence converging to the identity and divergent sequence. These are critical to the definition of regular variation [BOst-TRI].

Definition. Let $\psi_n : X \to X$ be self-homeomorphisms.
We say that a sequence ψ_n in $\mathcal{H}(X)$ *converges to the identity* if

$$\|\psi_n\| = \hat{d}(\psi_n, id) := \sup_{t\in X} d(\psi_n(t), t) \to 0.$$

Thus, for all t, we have $z_n(t) := d(\psi_n(t), t) \leq \|\psi_n\|$ and $z_n(t) \to 0$. Thus the sequence $\|\psi_n\|$ is bounded.

Illustrative Examples. In $\mathbb{R}$ we may consider $\psi_n(t) = t + z_n$ with $z_n \to 0$. In a more general context, we note that a natural example of a convergent sequence of homeomorphisms is provided by a flow parametrized by *discrete time* (thus also termed a 'chain') towards a sink. If $\psi : \mathbb{N} \times X \to X$ is a flow and $\psi_n(x) = \psi(n, x)$, then, for each t, the orbit $\{\psi_n(t) : n = 1, 2, ...\}$ is the image of the real null sequence $\{z_n(t) : n = 1, 2, ...\}$.

Proposition 13.1. *(i) For a sequence ψ_n in $\mathcal{H}(X)$, ψ_n converges to the identity iff ψ_n^{-1} converges to the identity.*
(ii) Suppose X has abelian norm. For $h \in \mathcal{H}(X)$, if ψ_n converges to the identity then so does $h^{-1}\psi_n h$.

Proof. Only (ii) requires proof, and that follows from $\left|\left|h^{-1}\psi_n h\right|\right| = \left|\left|hh^{-1}\psi_n\right|\right| = \left|\left|\psi_n\right|\right|$, by the cyclic property. ■

Definitions. 1. Again let $\varphi_n : X \to X$ be self-homeomorphisms. We say that the sequence φ_n in $\mathcal{G}$ *diverges uniformly* if for any $M > 0$ we have, for all large enough n, that

$$d(\varphi_n(t), t) \geq M, \text{ for all } t.$$

Equivalently, putting

$$d_*(h, h') = \inf_{x\in X} d(h(x), h'(x)),$$

$$d_*(\varphi_n, id) \to \infty.$$

2. More generally, let $\mathcal{A} \subseteq \mathcal{H}(S)$ with $\mathcal{A}$ a metrizable topological group. We say that α_n is a *pointwise divergent sequence in* $\mathcal{A}$ if, for each $s \in S$,

$$d_S(\alpha_n(s), s) \to \infty,$$

equivalently, $\alpha_n(s)$ does not contain a bounded subsequence.
3. We say that α_n is a *uniformly divergent sequence in* $\mathcal{A}$ if

$$\|\alpha_n\|_{\mathcal{A}} := d_{\mathcal{A}}(e_{\mathcal{A}}, \alpha_n) \to \infty,$$

equivalently, α_n does not contain a bounded subsequence.

Examples. In $\mathbb{R}$ we may consider $\varphi_n(t) = t + x_n$ where $x_n \to \infty$. In a more general context, a natural example of a uniformly divergent sequence of homeomorphisms is again provided by a flow parametrized by discrete time from a source to infinity. If $\varphi : \mathbb{N} \times X \to X$ is a flow and $\varphi_n(x) = \varphi(n, x)$, then, for each x, the orbit $\{\varphi_n(x) : n = 1, 2, ...\}$ is the image of the divergent real sequence $\{y_n(x) : n = 1, 2, ...\}$, where $y_n(x) := d(\varphi_n(x), x) \geq d_*(\varphi_n, id)$.

Remark. Our aim is to offer analogues of the topological vector space characterization of boundedness: for a bounded sequence of vectors $\{x_n\}$ and scalars $\alpha_n \to 0$ ([Ru, cf. Th. 1.30]), $\alpha_n x_n \to 0$. But here $\alpha_n x_n$ is interpreted in the spirit of duality as $\alpha_n(x_n)$ with the homeomorphisms α_n converging to the identity.

Examples. 1. Evidently, if $S = X$, the pointwise definition reduces to functional divergence in $\mathcal{H}(X)$ defined pointwise:

$$d^X(\alpha_n(x), x) \to \infty.$$

The uniform version corresponds to divergence in the supremum metric in $\mathcal{H}(X)$.
2. If $S = T$ and $\mathcal{A} = X = \Xi$, we have, by the Quasi-isometric Duality Theorem (Th. 12.7), that

$$d^T(\xi_{x(n)}(t), \xi_e(t)) \to \infty$$

iff

$$d^X(x_n, e_X) \to \infty,$$

and the assertion is ordinary divergence in X. Since

$$d_\Xi(\xi_{x(n)}, \xi_e) = d^X(x_n, e_X),$$

the uniform version also asserts that

$$d^X(x_n, e_X) \to \infty.$$

Recall that $\xi_x(s)(z) = s(\lambda_x^{-1}(z)) = s(x^{-1}z)$, so the interpretation of Ξ as having the action of X on T was determined by

$$\varphi(\xi_x, t) = \xi_{x^{-1}}(t)(e) = t(x).$$

One may write

$$\xi_{x(n)}(t) = t(x_n).$$

When interpreting $\xi_{x(n)}$ as x_n in X acting on t, note that

$$d_X(x_n, e_X) \leq d^X(x_n, t(x_n)) + d^X(t(x_n), e_X) \leq \|t\| + d^X(t(x_n), e_X),$$

so, as expected, the divergence of x_n implies the divergence of $t(x_n)$.

The next definition extends our earlier one from sequential to continuous limits.

Definition. Let $\{\psi_u : u \in I\}$ for I an open interval be a family of homeomorphisms (cf. [Mon2]). Let $u_0 \in I$. Say that ψ_u *converges to the identity as* $u \to u_0$ if

$$\lim_{u\to u_0} \|\psi_u\| = 0.$$

This property is preserved under topological conjugacy; more precisely we have the following result, whose proof is routine and hence omitted.

Lemma 13.2. *Let $\sigma \in \mathcal{H}_{unif}(X)$ be a homeomorphism which is uniformly continuous with respect to d^X, and write $u_0 = \sigma z_0$.*
If $\{\psi_z : z \in B_\varepsilon(z_0)\}$ converges to the identity as $z \to z_0$, then as $u \to u_0$ so does the conjugate $\{\psi_u = \sigma\psi_z\sigma^{-1} : u \in B_\varepsilon(u_0), u = \sigma z\}$.

Lemma 13.3. *Suppose that the homeomorphisms $\{\varphi_n\}$ are uniformly divergent, $\{\psi_n\}$ are convergent and σ is bounded, i.e. is in $\mathcal{H}(X)$. Then $\{\varphi_n\sigma\}$ is uniformly divergent and likewise $\{\sigma\varphi_n\}$. In particular $\{\varphi_n\psi_n\}$ is uniformly divergent, and likewise $\{\varphi_n\sigma\psi_n\}$, for any bounded homeomorphism $\sigma \in \mathcal{H}(X)$.*

Proof. Consider $s := \|\sigma\| = \sup d(\sigma(x), x) > 0$. For any M, from some n onwards we have

$$d_*(\varphi_n, id) = \inf_{x\in X} d(\varphi_n(x), x) > M,$$

i.e.

$$d(\varphi_n(x), x) > M,$$

for all x. For such n, we have $d_*(\varphi_n\sigma, id) > M - s$, i.e. for all t we have

$$d(\varphi_n(\sigma(t)), t)) > M - s.$$

Indeed, otherwise at some t this last inequality is reversed, and then

$$\begin{aligned} d(\varphi_n(\sigma(t)), \sigma(t)) &\le d(\varphi_n(\sigma(t)), t) + d(\sigma(t), t) \\ &\le M - s + s = M. \end{aligned}$$

But this contradicts our assumption on φ_n with $x = \sigma(t)$. Hence $d_*(\varphi_n\sigma, id) > M - s$ for all large enough n.
The other cases follow by the same argument, with the interpretation that now $s > 0$ is arbitrary; then we have for all large enough n that $d(\psi_n(x), x) < s$, for all x. ■

Remark. Lemma 13.3 says that the filter of sets (countably) generated from the sets

$$\{\varphi|\varphi : X \to X \text{ is a homeomorphism and } \|\varphi\| \ge n\}$$

is closed under composition with elements of $\mathcal{H}(X)$.

We now return to the notion of divergence.

Definition. We say that pointwise (resp. uniform) divergence is *unconditional divergence* in $\mathcal{A}$ if, for any (pointwise/uniform) divergent sequence α_n,
(i) for any bounded σ, the sequence $\sigma\alpha_n$ is (pointwise/uniform) divergent; and,
(ii) for any ψ_n convergent to the identity, $\psi_n\alpha_n$ is (pointwise/uniform) divergent.

Remark. In clause (ii) each of the functions ψ_n has a bound depending on n. The two clauses could be combined into one by requiring that if the bounded functions ψ_n converge to ψ_0 in the supremum norm, then $\psi_n\alpha_n$ is (pointwise/uniform) divergent.

By Lemma 13.3 uniform divergence in $\mathcal{H}(X)$ is unconditional. We move to other forms of this result.

Proposition 13.4. *If the metric on $\mathcal{A}$ is left- or right-invariant, then uniform divergence is unconditional in $\mathcal{A}$.*

Proof. If the metric $d = d_{\mathcal{A}}$ is left-invariant, then observe that if β_n is a bounded sequence, then so is $\sigma\beta_n$, since

$$d(e, \sigma\beta_n) = d(\sigma^{-1}, \beta_n) \le d(\sigma^{-1}, e) + d(e, \beta_n).$$

Since $\|\beta_n^{-1}\| = \|\beta_n\|$, the same is true for right-invariance. Further, if ψ_n is convergent to the identity, then also $\psi_n\beta_n$ is a bounded sequence, since

$$d(e, \psi_n\beta_n) = d(\psi_n^{-1}, \beta_n) \le d(\psi_n^{-1}, e) + d(e, \beta_n).$$

Here we note that, if ψ_n is convergent to the identity, then so is ψ_n^{-1} by continuity of inversion (or by metric invariance). The same is again true for right-invariance. ∎

The case where the subgroup $\mathcal{A}$ of self-homeomorphisms is the translations Ξ, though immediate, is worth noting.

Theorem 13.5. *(The case $\mathcal{A} = \Xi$) If the metric on the group X is left- or right-invariant, then uniform divergence is unconditional in $\mathcal{A} = \Xi$.*

Proof. We have already noted that Ξ is isometrically isomorphic to X. ∎

Remarks. 1. If the metric is bounded, there may not be any divergent sequences.
2. We already know from Lemma 13.3 that uniform divergence in $\mathcal{A} = \mathcal{H}(X)$ is unconditional.
3. The unconditionality condition (i) corresponds directly to the technical condition placed in [BajKar] on their filter $\mathcal{F}$. In our metric setting, we thus employ a stronger notion of limit to infinity than they do. The filter implied by the pointwise setting is generated by sets of the form

$$\bigcap_{i\in I}\{\alpha : d^X(\alpha_n(x_i), x_i) > M \text{ ultimately}\} \text{ with } I \text{ finite}.$$

However, whilst this is not a countably generated filter, its projection on the x-coordinate:

$$\{\alpha : d^X(\alpha_n(x), x) > M \text{ ultimately}\},$$

is.

4. When the group is locally compact, 'bounded' may be defined as 'pre-compact', and so 'divergent' becomes 'unbounded'. Here divergence is unconditional (because continuity preserves compactness).

THEOREM 13.6. *For $\mathcal{A} \subseteq \mathcal{H}(S)$, pointwise divergence in $\mathcal{A}$ is unconditional.*

Proof. For fixed $s \in S$, $\sigma \in \mathcal{H}(S)$ and $d^X(\alpha_n(s), s))$ unbounded, suppose that $d^X(\sigma\alpha_n(s), s))$ is bounded by K. Then

$$\begin{aligned} d_S(\alpha_n(s), s)) &\leq d_S(\alpha_n(s), \sigma(\alpha_n(s))) + d_S(\sigma(\alpha_n(s)), s) \\ &\leq \|\sigma\|_{\mathcal{H}(S)} + K, \end{aligned}$$

contradicting that $d_S(\alpha_n(s), s))$ is unbounded. Similarly, for ψ_n converging to the identity, if $d_S(\psi_n(\alpha_n(x)), x)$ is bounded by L, then

$$\begin{aligned} d_S(\alpha_n(s), s)) &\leq d_S(\alpha_n(s), \psi_n(\alpha_n(s))) + d_S(\psi_n(\alpha_n(s)), s) \\ &\leq \|\psi_n\|_{\mathcal{H}(S)} + L, \end{aligned}$$

contradicting that $d_S(\alpha_n(s), s))$ is unbounded. ■

COROLLARY 13.7. *Pointwise divergence in $\mathcal{A} \subseteq \mathcal{H}(X)$ is unconditional.*

COROLLARY 13.8. *Pointwise divergence in $\mathcal{A} = \Xi$ is unconditional.*

Proof. In Theorem 13.6, take $\alpha_n = \xi_{x(n)}$. Then unboundedness of $d^T(\xi_{x(n)}(t), t)$ implies unboundedness of $d^T(\sigma\xi_{x(n)}(t), t)$ and of $d^T(\psi_n\xi_{x(n)}(t)), t)$. ■

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