THE STEINHAUS THEOREM AND REGULAR VARIATION: De BRUIJN AND AFTER

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Abstract

The contributions of N. G. de Bruijn to regular variation, and recent developments in this field, are discussed. A new version of the Uniform Convergence Theorem is given.

1. Introduction: the Uniform Convergence Theorem of regular variation

The theory of regular variation originates with Jovan Karamata (1902-1967) in 1930 [Kar2], and concerns the extensive consequences of limit relations of the form

$$f(tx)/f(x) \to g(t) \in (0, \infty) \qquad (x \to \infty) \qquad \forall\ t > 0, \tag{RV}$$

written multiplicatively, or

$$h(t + x) - h(x) \to k(t) \in \mathbb{R} \qquad (x \to \infty) \qquad \forall\ t \in \mathbb{R}, \tag{RV_+}$$

written additively.

Karamata worked with continuous functions. The modern era of the subject began with the path-breaking paper by three Dutch mathematicians, Korevaar[1], van Aardenne-Ehrenfest[2] and de Bruijn [KorvAEdB]. The context was broadened to its natural setting, measurability. Here we find the main theorem of the subject, the *Uniform Convergence Theorem* (UCT): under measurability, the convergence in (RV) is uniform on compact t-sets in $(0, \infty)$. This result is so important that many proofs have been given; see [BinGT] (BGT below) 1.2 for six, and [BOst5] for more.

From (RV_+) (it is convenient to work additively in proofs),

$$k(t + u) = k(t) + k(u). \tag{CFE}$$

That is, k is an *additive function*. It satisfies the *Cauchy functional equation*. As is well known (see e.g. [Kuc]), additive functions exhibit a sharp

[1] Jacob (Jaap) Korevaar (1923-), author of the modern classic [Kor] on Tauberian theorems

[2] T. van Aardenne-Ehrenfest (1905-1984); see de Bruijn's memorial article [dB3]

dichotomy: they are either continuous (whence easily seen to be of the obvious form $k(x) \equiv \rho x$ for some ρ), or pathological (in particular, unbounded above and below on any interval, or set of positive measure, or even less – see [BinO10]). Discontinuous additive functions may be manufactured in great profusion from Hamel bases of the reals over the rationals (see e.g. BGT 1.1.4 and the recent [DorFN]). Subject to enough regularity to avoid this *Hamel pathology*, one has $k(x) \equiv \rho x$ in (RV_+), or $g(x) \equiv x^\rho$ in (RV) (*Characterisation Theorem*: BGT, 1.4; cf. §11.6).

As well as analysis and Tauberian theory, other areas of application of regular variation include probability theory (BGT Ch. 8; [Bin1], [Bin2]) and analytic number theory ([dBvL]; BGT Ch. 6).

2. The Representation Theorem

Second only in importance to the UCT is another result of Karamata (continuous case), Korevaar, van Aardenne-Ehrenfest and de Bruijn [KorvAEdB] (measurable case), the *Representation Theorem* (BGT 1.3) for regularly varying functions:

$$h(x) = \rho x + d(x) + \int_0^x e(u)du, \qquad (d(.) \to d, \quad e(.) \to 0)$$

(as is customary, we use two functions here for convenience, with non-uniqueness in representation), or in multiplicative notation

$$f(x) = x^\rho \ell(x), \qquad \ell(x) = c(x) \exp\{\int_1^x \epsilon(u)du/u\},$$

$$c(.) \to c \in (0, \infty), \qquad \epsilon(.) \to 0$$

(here e, ϵ are continuous, but may be taken to be as smooth as we wish – see below). Here $\rho \in \mathbb{R}$ is called the *index* of regular variation; the set of such functions f is R_ρ, the class of functions *regularly varying* with index ρ. So $\ell \in R_0$; ℓ is called *slowly varying* (ℓ for *lente*, or *langsam*). Such *Karamata representations* are inherently non-unique: one can adjust one of $c(.)$ and $\epsilon(.)$ at the cost of compensating adjustment to the other. Since $\epsilon(.)$ appears inside an integral, smoothness is more desirable here than for $c(.)$. Indeed, one is often interested in f only to within asymptotic equivalence; $c(.)$ need not even be measurable, and one can replace it by its limit c. Thus the smoothness of f is governed by that of $c(.)$.

For most purposes, one can take $\epsilon(.)$ to be as smooth as one wishes – C^∞, for example. For the resulting theory of *smooth variation*, see BGT 1.8. This rests on work of another trio of Dutch mathematicians, Balkema, Geluk and de Haan [BalGdH], which in turn uses work by de Bruijn [dB2] of 1959 (in recognition of which [BinO5] refers to 'de Bruijn's representation theorem'). De Bruijn was concerned with the *inversion of asymptotic relations* – if

$$f(g(x)) \sim x \qquad (x \to \infty),$$

how to pass between asymptotic properties of f to those of g. He also obtained Tauberian theorems of exponential type (see e.g. BGT 7.12, where the three cases are called the Tauberian theorems of Kohlbecker, Kasahara and de Bruijn – though in fact de Bruijn [dB2] considered all three cases; cf. [dB1], Ch. 4). These occur in various areas, including duality theory in convex analysis. But one of the most important is in probability theory, in connection with the theory of *large deviations*; see [Bin3] and the references cited there.

3. Category-measure duality

In measure theory, the null sets are the small sets. In topology, one encounters the *Baire category theorem*. Here, the small sets are the meagre sets (sets of the first category). For a monograph treatment of the extensive links and parallels between measure and category, see the classic – Oxtoby [Ox].

Matuszewska [Mat] showed in 1964 that one could develop a theory of regular variation, imposing a topological restriction – that functions have the Baire property (briefly, be 'Baire') – rather than the measure-theoretic restriction of measurability. One can obtain the UCT, Representation, Characterisation Theorem and the other main results of the theory, but in a topological setting, with meagre sets playing the role of null sets. In BGT, the two theories are developed in parallel, with the measurable case treated as the principal one, following the historical development. But note that neither of the measurable and Baire cases contains the other.

It turns out that in fact it is the category theory which is the principal one. One can develop the two theories as one, working bi-topologically, with the Baire case when one imposes the usual (Euclidean) topology, and the measurable one when one imposes the *density topology*. This is the topology obtained by calling a set open if all its points are density points (in the sense of the Lebesgue density theorem – that almost all points of a measurable set are density points). That such a development might be possible is suggested by results such as that a set has the Baire property under the density topology iff it is (Lebesgue) measurable (see e.g. Kechris [Kec], (17.47.iv) p.119).

This insight is the basis of the theory of *topological regular variation*, developed extensively in recent years by the present authors. See in particular [BinO1], [BinO7] for category-measure aspects.

4. The theorems of Steinhaus and Ostrowski

The theorem of Steinhaus (from 1920; BGT Th. 1.1.1) – the foundation stone of regular variation – states that for a measurable set A of positive measure, the difference set $A - A := \{a_1 - a_2 : a_i \in A\}$ contains a neighbourhood of the origin. This has been much generalised – to the Baire case (Piccard, 1939), by Pettis (1950, 1951), and many others. From this, one obtains the theorem of Ostrowski (from 1929; BGT Th. 1.1.7, Th. 1.1.8; the Baire case is due to Mehdi in 1964), which gives the above-mentioned dichotomy for solutions to the Cauchy functional equation. From these, the theory may be developed as in BGT. From Th. 1.1.8 a discontinuous additive function is neither Lebesgue-measurable nor has the property of Baire. But, under assumptions consistent with ZFC, it may be Marczewski-measurable [DorFN]. The broader context here is 'negligibility': see [BreL] for σ-ideals motivated by forcing, and for topological aspects [Ost3].

In the Steinhaus and Piccard theorems, the relevant dichotomy is that the difference set is either topologically small (has empty interior), or topologically large (contains a neighbourhood of the origin). The general context is that of topological groups ([BajK]; cf. [Bal]). One can also work with normed groups [BinO6]. Here the dichotomy takes the form: normed groups are either topological or pathological. We note that the real line under the density topology is *not* a topological group.

The Ostrowski and Baire-Mehdi theorems exhibit the dichotomy above: additive functions are either (continuous and so) linear (ρx) or pathological. As results of this type date back to Darboux in 1875, one may call this the *Darboux dichotomy*. The general context is that of *automatic continuity* (§11.6): here the merest hint of regular behaviour ensures full regularity [BinO10].

The proofs here involve algebraic results concerning additive subgroups of the reals. Such a subgroup is either very small in some sense (in particular, has infinite index), or is the whole of the reals. One may call this the *subgroup dichotomy*.

A comprehensive examination of results of this kind was recently given in

[BinO11], from the point of view of *infinite combinatorics*; see also [BinO6]. In particular, the subgroup dichotomy is relevant to the area of Ramsey theory [BinO8].

5. The foundational question

Shift-compactness derives from probability theory on algebraic and topological structures (see e.g. Parthasarathy [Par, III.2], Heyer [Hey]), the idea being that one may obtain compactness (say, sequential compactness, in a metric-space setting) after suitable shifts. We note that, in the crucially relevant case of the reals under the density topology, translation is continuous, but addition is not. Thus one has good behaviour with one argument, but not with two. This suggests that group actions, rather than topological groups themselves, may be the natural framework here. It is, and this brings in the viewpoint of topological dynamics [Ost2]. Also relevant here are questions of separate versus joint continuity. The prototypical result here is Effros' theorem. For background and references, we refer to Miller and Ostaszewski [MilO] and [Ost5].

The first major question left open in BGT was the *foundational question* (BGT, 1.2.5 p.11): what is the minimal common generalisation of measurability and the Baire property that suffices for the foundations of regular variation – the three principal results, the UCT, the Representation Theorem and the Characterisation Theorem? This question was answered in [BinO4], in terms of the Kestelman-Borwein-Ditor theorem of infinite combinatorics (from results of Kestelman in 1947, Borwein and Ditor in 1978), and the No Trumps property, NT (the term derives from bridge, following on from Ostaszewski's club ♣ [Ost1], itself following on from Jensen's diamond, $\diamondsuit$) – see §6.

The No Trumps property was abstracted from several of the proofs of UCT given in BGT 1.2. These used proof by contradiction, obtaining a sequence witnessing to the contradiction, and extracting from it a suitable subsequence, all of whose members satisfied some condition. The Kestelman-Borwein-Ditor theorem is of this type. It turns out that one can often work 'generically', obtaining the desired property 'quasi-everywhere' – everywhere off a 'negligible' set (meagre in the Baire case, null in the measure case); see [BinO2] for results of this type in a function-space setting. The main result of [BinO2] is there called the Category Embedding Theorem, a tool used in our subsequent papers.

These ideas motivate the theory of shift-compactness, which subsumes them. We refer for detail to [MilO] and [Ost4]. Section 6 below illustrates these with a new result.

6. UCT on the L_1-algebra of a locally compact metric group G

The UCT is the main result in the classical theory of slow and regular variation, and as above many proofs are known. In [BinO4], [BinO5] Parts I & II, the theory is developed in the context of homogeneous spaces, and in particular of topological groups (as homogeneous spaces acting on themselves); there the action is transitive by homogeneity. Here we have G a locally compact metric group and work on $L_1(G)$ with its natural action, but now the action need not be transitive.

The notation in the Introduction defining slow variation in the form (RV) may be easily re-interpreted for a function h with domain a metric space X and values in a topological group H when the multipliers t in (RV) come from a topological group G acting on X, just so long as one has a notion of convergence 'at infinity' giving meaning to

$$h(tx_\nu)h(x_\nu)^{-1} \to g(t) \ (t \in G),$$

or in an abelian context $h(tx_\nu) - h(x_\nu) \to k(t)$. For example, x_ν may run through a 'divergent' sequence x_n in X, or a divergent net (i.e., '$x \to \infty$', rather than $x \to 1$, the group identity when X itself is a group, as in approximate identities – for which see e.g. [Rick], p. 3 and A.3.1). From this perspective, when $X = G$ and both are $\mathbb{R}$, regarded as an additive group, a divergent net is provided by ordering the reals in ascending order.

We now develop an $L_1(G)$-regular variation theory, then deduce a corresponding version of the UCT. Let G be a locally compact metric group equipped with a (left) Haar measure η. (In the Representation Theorem in §7 below we will specialize to the σ-compact, so separable, case.) Take the domain and range of regularly varying functions to be $X = L_1(G, \eta)$, regarded as the Banach algebra of η-integrable functions $x : G \to R$ under convolution. Thus $||x||_1 = \int |x(g)| d\eta(g)$. The group G defines a natural action on X, namely $* : G \times X \to X$, where

$$(g * x)(t) := x(g^{-1}t).$$

That is, $(gh) * x = g * (h * x)$ and $1_G * x = x$ (cf. [HewR, Ch.5], [Pat, Ch.4]). The map $g \mapsto g * x$ is continuous, since continuity on a compact set implies uniform continuity and because

$$||g * x - h * x||_1 = ||h^{-1}g * x - x||_1.$$

We note the group action is isometric, i.e. $||g * x - g * y||_1 = ||x - y||_1$. We now recall and adapt for the present context some definitions from [BinO4], cf. [BinO6, §7].

Definitions. 1. Call $\mathbf{z} := \{z_n\}$ a *null-sequence* in G if $z_n \to 1_G$.
2. Say the map $h : X \to X$ has the *NT-property* w.r.t. the sequence $\mathbf{x} := \{x_n\}$ if for each $\varepsilon > 0$ the family $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ has the following *shift-compactness* property: for each null sequence $\mathbf{z}$ there are $k \in \omega$, $t \in G$ and $\mathbb{M}$ an infinite set such that:

$$tz_m \in T_k^\varepsilon(\mathbf{x}) \text{ for } m \in \mathbb{M},$$

where

$$T_k^\varepsilon(\mathbf{x}) := \bigcap\nolimits_{n \geq k} \{g : ||h(g * x_n) - h(x_n)||_1 < \varepsilon\}.$$

3. Call the map $h : X \to X$ *slowly varying* (w.r.t. the net $\mathbf{x} := \{x_\delta\}$) if

$$\lim\nolimits_\delta ||h(g * x_\delta) - h(x_\delta)||_1 = 0, \text{ for each } g \in G. \tag{SV}$$

(In applications, the nets may be required to satisfy additional conditions – see the 'regular nets' below. For the connection between slow variation along a net and slow variation with a continuous limit, see e.g. [BinO4, Th. 5 – Equivalence Theorem].)
4. As before, say the map $h : X \to X$ is *Baire* if h^{-1} takes open sets to sets with the Baire property. Say that $h : X \to X$ is *Baire relative to convolution* if the maps $h_x : G \to X$ defined by $h_x(g) := h(g * x)$ are Baire for all x off a meagre set, the *exceptional set* E_h of h.

We will see below that continuous h are Baire relative to convolution (with no exceptional set); Baire functions h are also Baire relative to convolution but with an exceptional set that need not be empty.

Lemma 1. *For $h : X \to X$ continuous, h is Baire relative to convolution with empty exceptional set E_h. In particular, for $\mathbf{x}$ any sequence, the sets $T_k^\varepsilon(\mathbf{x})$ are Baire (have the Baire property).*

Proof. Since $g \mapsto g * x$ is continuous (as noted earlier), the maps $h_x(g) := h(g * x)$ are also continuous. Put $h_n(g) := h(g * x_n)$, again continuous. So for each n, the level set $H_n^\varepsilon(\mathbf{x}) := \{g : ||h(g * x_n) - h(x_n)||_1 < \varepsilon\}$ takes the form $h_n^{-1}[B_{h(x_n)}(\varepsilon)]$ for $B_x(\varepsilon)$ the ball of radius ε centred at x in X, and being open is Baire. Hence so is $T_k^\varepsilon(\mathbf{x}) = \bigcap_{n\geq k} H_n^\varepsilon(\mathbf{x})$. $\square$

Remark. For h continuous, as h_n is continuous, $H_n^\varepsilon(\mathbf{x})$ is open and so in fact each set $T_k^\varepsilon(\mathbf{x})$ is a $\mathcal{G}_\delta$.

Lemma 2. *If $h : X \to X$ is Baire relative to convolution and slowly varying w.r.t. a sequence $\mathbf{x} := \{x_n\}$, with each $x_n \notin E_h$, then h has the NT-property w.r.t. $\mathbf{x}$.*

In particular, any continuous function $h : X \to X$ that is slowly varying w.r.t. any sequence $\mathbf{x} := \{x_n\}$ has the NT-property w.r.t. $\mathbf{x}$.

Proof. For $\varepsilon > 0$, one has by definition that

$$G := \bigcup_{k\in\mathbb{N}} T_k^\varepsilon(\mathbf{x}).$$

As h is Baire relative to convolution and each $x_n \notin E_h$ each function $h_n(g) := h(g * x_n)$ is Baire and so each $T_k^\varepsilon(\mathbf{x})$ is Baire. So, since G is topologically complete, there is k with $T_k^\varepsilon(\mathbf{x})$ non-meagre and Baire, as h is Baire relative to convolution. It now follows ([BinO6], Cor. 6.4 – *shift-compactness*, in the language of [MilO]) that if $\mathbf{z}$ is null, then for quasi-all $t \in T_k^\varepsilon(\mathbf{x})$ there is an infinite set $\mathbb{M}$ such that

$$tz_m \in T_k^\varepsilon(\mathbf{x}) \text{ for } m \in \mathbb{M}.$$

Specializing to h continuous, by Lemma 1 E_h is empty; so $\mathbf{x}$ may be arbitrary. $\square$

Definition. For V dense open in X, say that $x \in X$ is a *regular point for V* (under the group action) if the (open) set

$$G_x(V) := \{g \in G : g * x \in V\}$$

is dense open in G.

It is possible for G_x to be non-dense. However, this is rare:

Proposition 1. *For G separable and V dense open in X, quasi-all points of X are regular for V.*

Proof. For G separable let $\{d_n : n = 1, 2, ...\}$ be a countable dense subset of G. For $m, n = 1, 2, ...$, put $B_{m,n} := B(d_n, 1/m)$, where $B(d, r)$ is the open ball of radius r in G.

For $B \subseteq G$ put $B * V := \{b * v : b \in B \ \& \ v \in V\}$ and $b * V := \{b\} * V$; so $B * V = \bigcup_{b \in B} b * V$ is dense open for V dense open. Indeed for V dense open in X, $b * V$ is already dense open, since convolution is isometric (see above).

Now G_x, as above, is dense open iff each $B_{m,n}$ meets G_x, i.e. for each m, n some $g \in B_{m,n}$ has the property that $g * x \in V$, equivalently $x \in B_{m,n}^{-1} * V$ for each m, n. Equivalently, G_x is dense open iff

$$x \in R(V) := \bigcap\nolimits_{m,n} B_{m,n}^{-1} * V.$$

Now $R(V)$ is a dense $\mathcal{G}_\delta$ in X and so co-meagre, as $X = L_1(G)$ is complete; all its members are regular for V. $\square$

Proposition 2. *For G separable and $h : X \to X$ Baire, there is a meager set E_h in X such that*
(i) *h_x is Baire for each x in the co-meagre set $R := X \backslash E_h$;*
(ii) *if h is slowly varying w.r.t. a sequence $\mathbf{x} := \{x_n\}$, with each $x_n \in R$, then h has the NT-property w.r.t. $\mathbf{x}$.*

Proof. (i) As $h : X \to X$ is Baire, and X is separable, for some co-meagre Y the function $h_Y := h|Y$ is continuous (see [Oxt], Th. 8.1]). Denoting by ξ_x the map $g \mapsto g * x$, one has

$$G_x(Y) = \xi_x^{-1}(Y) := \{g \in G : g * x \in Y\}.$$

By passing to a subset, if necessary, w.l.o.g. we may assume that $Y = \bigcap_n V_n$ with each V_n dense open in X. For any x in X, the map ξ_x is continuous from G to X and so again each set $G_n := G_x(V_n) = \{g \in G : g * x \in V_n\}$ is open in G. Thus

$$G_x := \{g \in G : g * x \in Y\} = \bigcap\nolimits_n \{g \in G : g * x \in V_n\}$$

is a $\mathcal{G}_\delta$.

Put $R := \bigcap_n R(V_n)$; then R is co-meagre, since each set $R(V_n)$ is co-meagre.

Every $x \in R$ is a regular point point for each V_n. So for $x \in R$ the set G_x is co-meagre. Moreover, for $g \in G_x$, we have $\xi_x(g) \in Y$ so

$$h_x(g) = h(g * x) = h_Y(\xi_x(g)),$$

so $h_x|G_x$ is continuous on G_x, as a composition of continuous functions. As G_x is co-meagre, h_x is Baire (again [Oxt, Th. 8.1]).

By (i) the assertion (ii) follows from Lemma 2. $\square$

Remark. Of course if $h : X \to X$ is Baire and is a homomorphism, then h is continuous.

We now prove the promised uniform convergence in the context of a group action that is is not necessarily transitive. Here the weaker assumption on the action extracts a price: slow variation is defined relative to *regular nets*, i.e. nets consisting of points avoiding a specified meagre set – in order to secure the Baire property for the maps h_x. We are grateful to the Referee for the illuminating example of the horizontal shift action by $\mathbb{R}$ on $\mathbb{R} \times \mathbb{R}$ (that is $t \circ (x, y) = (x + t, y)$) and a Baire self-map of $\mathbb{R} \times \mathbb{R}$ with $h(0, .) : \mathbb{R} \to \mathbb{R}$ wild, and otherwise $h(u, v) = (0, 0)$, for which $h_x(t) = 0$ for all vectors x, except the meagre set of vectors $x = (0, v)$ for which $h_{(0,v)}(t)$ is wild. Recall (BGT §2.9) the occurrence of other exceptional sets in regular variation.

Definition. For Baire $h : X \to X$, say that h is *slowly varying with respect to regular nets* if it is slowly varying w.r.t. nets $\{x_\delta\}$ with all $x_\delta \notin E_h$; here E_h is the meagre set of Proposition 2 corresponding to h.

Theorem (UCT for $L_1(G)$). *For G a locally compact metric group with Haar measure η and $X = L_1(G)$:*

(i) *for G separable (i.e. σ-compact), if $h : X \to X$ is Baire and slowly varying w.r.t. regular nets, then the convergence in (SV) is uniform on compacts;*

(ii) *for general G, uniform convergence in (SV) holds for h continuous and slowly varying w.r.t. arbitary nets.*

Proof. As usual with proofs of the UCT we proceed by contradiction. Suppose that h is Baire and slowly varying but that uniform convergence fails w.r.t. some regular net $\{y_\delta\}$. Then there is a compact set K and $\varepsilon > 0$ such that for each δ there is $\nu = \nu(\delta) \succ \delta$ and a point $g_\nu \in G$ with

$$||h(g_\nu * y_\nu) - h(y_\nu)||_1 > 3\varepsilon.$$

As K is a compact, the subnet $\{g_{\nu(\delta)}\}$ has a cluster-point u. As G is metric, there is a sequence $u_n = g_{\nu(\delta_n)}$ converging to u. Put $x_n := y_{\nu(\delta_n)}$ and $z_n :=$

$u^{-1}u_n$; then $\{z_n\}$ is null and $u_n = uz_n$. Thus, for each n, one has

$$||h(u_n * x_n) - h(x_n)||_1 > 3\varepsilon. \tag{1}$$

By Proposition 2, we may pick $k \in \omega$, $t \in G$ and $\mathbb{M}$ an infinite set such that

$$tz_m \in T_k^\varepsilon(\mathbf{x}) \text{ for } m \in \mathbb{M}.$$

So for $n > k$ and $m \in \mathbb{M}$ one has $||h(tz_m * x_n) - h(x_n)||_1 < \varepsilon$, and so in particular if $n > k$ and $n \in \mathbb{M}$ one has

$$||h(tz_n * x_n) - h(x_n)||_1 < \varepsilon. \tag{2}$$

By convergence at u and t, there is $N > k$, such that for all $n > N$ one has

$$||h(u * x_n) - h(x_n)||_1 < \varepsilon, \text{ and } ||h(t * x_n) - h(x_n)]||_1 < \varepsilon. \tag{3}$$

Combining, for $n > N$ with $n \in \mathbb{M}$, we have

$$\begin{aligned}
& ||h(u_n * x_n) - h(x_n)||_1 \\
\leq\ & ||h(u * z_n * x_n) - h(z_n * x_n)||_1 + ||h(z_n * x_n) - h(x_n)||_1 \\
=\ & ||h(u * z_n * x_n) - h(z_n * x_n)||_1 + ||h(tz_n * x_n) - h(x_n) - [h(tz_n * x_n) - h(z_n * x_n)]||_1 \\
\leq\ & ||h(u * z_n * x_n) - h(z_n * x_n)||_1 + ||h(tz_n * x_n) - h(x_n)||_1 + ||h(tz_n * x_n) - h(z_n * x_n)]||_1 \\
=\ & ||h(u * x_n) - h(x_n)||_1 + ||h(tz_n * x_n) - h(x_n)||_1 + ||h(t * x_n) - h(x_n)]||_1 \\
\leq\ & 3\varepsilon,
\end{aligned}$$

contradicting (1). □

7. From UCT to the Representation Theorem

The group-action approach in §6 opens a new perspective on the Representation Theorem of §2, permitting its extension from $\mathbb{R}$ to a locally compact metric group G (σ-compact in the representation theorem below, but this is not needed in the UCT below) equipped with a left-invariant metric d^G (cf. the Birkhoff-Kakutani Theorem [HewR, Th. 8.3]). The de Bruijn proof of BGT 1.3 remains the paradigm, but now needs to be based on an appropriate UCT. Such a UCT can be derived using the proof of the UCT of §6 by reinterpreting slow variation of $h : G \to \mathbb{R}$ at infinity now to mean that

$$|h(gx) - h(x)| \to 0 \text{ as } d^G(x, 1_G) \to \infty, \text{ for each } g \in G. \tag{SV-G}$$

This easily yields the following.

Theorem (UCT). *For G a locally compact metric group with left-invariant metric d^G, if $h : G \to \mathbb{R}$ is Baire and slowly varying, then the convergence in (SV-G) is uniform on compacts.*

The metric d^G is said to be *proper* if all the closed balls $\bar{B}_x(r) := \{y : d^G(x,y) \le r\}$ are compact, i.e. the metric has the *Heine-Borel* property: closed and bounded is equivalent to compact. (In geodesic geometry a proper metric space is called 'finitely compact', since an infinite bounded set has a point of accumulation – see [Bus], or [BridH] for a more recent text-book account of the extensive use of this concept.)

The UCT above leads in this context to the following generalization of the Representation Theorem for regularly varying functions, those for which, working additively,

$$h(gx) - h(x) \to k(g) \text{ as } d^G(x, 1_G) \to \infty, \text{ for each } g \in G.$$

See [BinO5, Part II] for a smooth extension to Euclidean spaces; in the current context one can only demand continuity.

Theorem (Karamata-de Bruijn Representation). *For G a σ-compact group with left Haar measure η and left-invariant proper metric d^G, if $h : G \to \mathbb{R}$ is Baire and regularly varying, then there are a homomorphism $\rho : G \to \mathbb{R}$, a continuous function $e : G \to \mathbb{R}$ vanishing at infinity, and a function $d : G \to \mathbb{R}$ convergent at infinity such that*

$$h(x) = \rho(x) + d(x) + \int_{\bar{B}_x} e(g)d\eta(g),$$

where as above $\bar{B}_x$ denotes the closed ball centered at 1_G of radius $d^G(x, 1_G)$.

Of course, in the locally compact setting above (needed to have a Haar measure), σ-compactness (σ-finiteness here) is equivalent to separability. In §8 below, local compactness gives us Haar measure, but as we shall see, even without this we still have a concept of Haar-null set, and this is crucially useful – see [Sol] and the end of §8 below.

8. Amenability

Our purpose here is to make connections via regular variation between

Steinhaus theory and amenability theory. Uniformity is built into regular variation, via the UCT, a new form of which we gave in §6 above. Uniformity is also built into amenability (a very important subject in the theory of topological groups: the term is meant to convey both its ordinary meaning, and 'mean-able' – existence of an invariant mean; for background, see the standard work by Paterson [Pat]). The Reiter and Følner conditions (below) are each equivalent to amenability; each holds uniformly on compacta when it holds pointwise. A second link comes via the Steinhaus Theorem (§4), on which the theory of regular variation is built. On a locally compact topological group one has a Haar measure, whose null sets can be thought of as the small or negligible sets. In fact, one may still be able to talk about Haar-null sets even when one has neither local compactness nor Haar measure. There is still a Steinhaus theorem in such contexts; this involves Solecki's concept of *amenability at* 1 [Sol] (below).

We shall show how, in a locally compact metric group G, the (uniform) Reiter condition of amenability theory may be deduced via UCT from its pointwise version; to be precise, for G again a locally compact metric group with left Haar measure η and $L_1(G,\eta)$ norm $||.||_1$, denote the non-negative densities by $P(G) := \{y \in L_1(G) : y \geq 0, \int y d\eta = 1\}$ and say that the (weak) *Reiter condition* ([Pat, Prop. 0.4]) holds for a net $\{x_\delta\}$ in $P(G)$ if

$$||g * x_\delta - x_\delta||_1 \to 0 \text{ for each } g \in G. \qquad (R)$$

In our context, this then holds uniformly on compact sets:

$$||g * x_\delta - x_\delta||_1 \to 0 \text{ on compact sets of } g \in G, \qquad (UR)$$

cf. [Pat] Prop. 4.4. The Reiter condition (R) is equivalent to amenability. (An invariant mean can be extracted as any weak* cluster-point of the net $\{\hat{x}_\nu\}$, where $\hat{x}$ represents x in the second dual $L_1(G)'' = L_\infty(G)'$.)

A more common condition equivalent to amenability is the *Følner condition*

$$\eta(gK_\delta \triangle K_\delta)/\eta(K_\delta) \to 0 \text{ for each } g \in G \qquad (F)$$

(here nets of non-null compacts K_δ may be replaced by sequences when the group G is σ-compact) – see [Pat] Ch. 4, esp. Prop. 4.10. This too holds uniformly on compact g-sets (condition UF):

$$\eta(gK_\delta \triangle K_\delta)/\eta(K_\delta) \to 0 \text{ uniformly for } g \text{ in compact sets.} \qquad (UF)$$

An alternative form of the Følner condition is: for each $\varepsilon > 0$ and compact subset C there is a non-null compact subset K such that

$$\eta(gK \triangle K)/\eta(K) < \varepsilon, \text{ for all } g \in C. \tag{F'}$$

Condition (R) reduces to (F) upon taking

$$x_\delta(t) = 1_{K_\delta}(t)/\eta(K_\delta).$$

For G locally compact metric, the deduction of (UR) from (R) is immediate on applying the UCT of §6 to the identity mapping $h(x) = x$, which is continuous.

Remark. This result allows a restatement of the Reiter condition for a locally compact group G in a measure-theoretic format in two ways. The first is the standard result ([Pat] Prop. 4.2) that G is amenable iff there is a net $\{x_\delta\}$ in $P(G)$ such that

$$||\mu * x_\delta - x_\delta||_1 \to 0 \text{ for every Borel probability measure } \mu \text{ on } G.$$

The second is the following Reiter-like condition ([Sol] §2 p. 699): for any Borel probability measure μ on G and $\varepsilon > 0$ there is a Borel probability measure ν such that for any compact subset K

$$|(\nu * \mu)(K) - \nu(K)| < \varepsilon. \tag{S}$$

This latter condition motivates a definition which is appropriate for a Polish group G which is not locally compact. Although Haar measure, i.e. a (left) translation-invariant, locally finite Borel measure, does not then exist on G, nevertheless, an analogue of the sets of Haar-measure zero can be developed. A subset N is (left) *Haar-null* if it is contained in a universally measurable set B and there is a Borel probability measure μ on G such that $\mu(gB) = 0$ for all $g \in G$. To study these Solecki introduced the following concept.

Definition. A Polish (topological) group G is *amenable at* 1 if for any sequence of Borel probability measures μ_n all having 1_G in their support, there exist Borel probability measures ν_n and ν, with ν_n absolutely continuous w.r.t. μ_n, such that for all compact $K \subseteq G$

$$\lim_n(\nu * \nu_n)(K) = \nu(K).$$

That is, $\nu * \nu_n$ converges weakly to ν, for which see [Par, II.6].

Solecki [Sol, Prop. 1] proves that in a Polish group which is amenable at 1, the Haar-null sets form a proper σ-ideal; furthermore, a universally measurable subset A that is not Haar-null has the Steinhaus property, that is: 1_G is in the interior of $A^{-1}A$. The list of groups with this property [Sol, Prop 3.3] includes abelian Polish groups, locally compact second-countable groups, and countable products of the latter provided all but finitely many of them are amenable. See also §11.5.

Again there is here a connection with shift-compactness – we hope to return to this matter elsewhere.

9. The contextual question: beyond the reals

The classical theory of regular variation, as expounded in BGT, was described in the preface there as 'a chapter in classical real-variable theory'. But in BGT Appendix 1, several more general settings are briefly described. This raises the *contextual question*: what is the natural setting for regular variation?

In the probabilistic setting of extreme-value theory (EVT – see below), one studies maxima, say of flood heights. The basic setting is one-dimensional (height at a given coastal station, or price of a risky stock), but finite-dimensional settings are crucially important (heights at a number of coastal stations; prices of stocks in a portfolio of risky assets), and infinite-dimensional settings such as function spaces (heights along a threatened coastline). As we have seen above, topological groups and normed groups also provide settings in which the theory can be developed; see [BinO6] and the references cited there.

10. Discrete and continuous limits

The limits in (RV), (RV_+) are continuous limits. But the proofs of the UCT by contradiction are sequential, starting from a sequence witnessing to the contradiction and proceeding by finding a suitable subsequence. Furthermore, countability is built into the definition of a measure, and hence of measurability, and into the definition of Baire category.

It has long been known that one can build a sequential theory of regular variation; see BGT 1.9 for details and references.

The question of whether limits are discrete or continuous is linked to that of *weakening the quantifier* $\forall$ in (RV), (RV_+) ([BinG], I).

The setting of (RV) (Karamata theory) can be extended to that of

$$(f(tx) - f(x))/g(x) \to h(x) \qquad (x \to \infty) \qquad \forall\ t > 0 \qquad\qquad (deH)$$

(de Haan theory); see BGT Ch. 3. It had long been intriguing why the proof of the main result on weakening the quantifier for Karamata theory (BGT, Th. 1.4.3, p.19) should be so hard, and no easier than that of the corresponding theorem for de Haan theory (BGT Th. 3.2.5, p.141). The answer to this emerged in the recent study [BinO11]. See [BinO8] for details, of both category methods and links with infinite combinatorics – including such classic results as van der Waerden's theorem.

We note that one can develop a fruitful theory of regular variation in which limits need not exist (see BGT Ch. 2). Results of this type are harder, as although measurability is preserved under sequential limits, it is not preserved under upper or lower limits. A detailed study of the extent of the degradation that can result here was given in [BinO9], using the language of descriptive set theory.

11. Remarks

1. *Tauberian theorems.* Karamata [Kar1] created a sensation in 1930 by his short proof of the Hardy-Littlewood Tauberian theorem for Laplace transforms. His method of proof was based on polynomial approximation. Using his new theory of regular variation [Kar2], also of 1930, he was able to extend the result, to its modern form, the *Hardy-Littlewood-Karamata theorem* [Kar3]. For textbook accounts, see BGT Ch. 4, [Kor], IV (Karamata's heritage: regular variation). It was through Korevaar's interest in both Tauberian theorems and regular variation that de Bruijn entered this field in [KorvAEdB].

2. *Zorn's lemma.* The Hamel pathology of §1 depends on Hamel bases – bases for the real numbers (as vectors) over the rationals (as ground field). As is well known, that all vector spaces have a basis is equivalent to the Axiom of Choice (AC), itself equivalent to Zorn's lemma. Recall that, while in general a subset of the reals is non-measurable, exhibiting one requires some form of AC, as in the classic example by Vitali. That the UCT is false without some regularity condition such as measurability was shown in the original paper by Korevaar, van Aardenne-Ehrenfest and de Bruijn (see e.g. BGT 1.2.4 for a simple example, involving a Hamel basis). De Bruijn's ongoing interest in Tauberian theory is seen in his paper with van der Meiden [dBvM] on Gelfand theory (this paper also illustrates his interest in where one needs

Zorn's lemma). Tauberian theory was transformed in 1932 by Wiener, who used general kernels. One of the first triumphs of modern functional as distinct from classical analysis was Gelfand's use of his theory to both extend and simplify the Wiener Tauberian theory.
3. *Extreme-value theory.* While regular variation entered probability theory in the context of addition of random variables (BGT 8.3-4), it is also crucially useful for maxima of random variables – the context of extreme-value theory (EVT – BGT 8.13). Early work here was by Gnedenko in 1943 (see [Bin4] for details and references). The subject became of pressing concern to the entire Netherlands, in particular to its mathematical community, following the disastrous floods of 31 January – 1 February 1953[3]. As a result, a number of mathematicians from the Netherlands have worked on EVT, including Balkema and de Haan already cited. Applications to EVT, particularly flood heights and finance [BalE], have been important stimuli to the development of regular variation in many dimensions.
4. *Analytic sets.* That analytic sets are relevant to the category and measure aspects of regular variation is suggested by Nikodym's theorem ([Rog, 1.2.9]). This concerns preservation of the Baire property and measurability under the Souslin operation. It would take us too far afield to discuss this important matter more fully here; for details and references, we refer to [BinO10], [BinO11], [Ost3], [Ost4].
5. *Steinhaus' Theorem and paradoxical decompositions.* We noted in §8 Solecki's results on the Steinhaus property in groups G that are amenable at 1. The theory of amenability may be re-cast in the language of paradoxical decompositions of Banach-Tarski type: non-amenability is paradoxicality. The crux is group-theoretic, and concerns the presence or absence of a free subgroup in G on two generators ([Pat, 0.6]). See [W] for background and Solecki [Sol] for detailed statements (Th. 1, Cor. 2 for positive results, Th. 3 for negative results).
6. *Automatic continuity.* The Characterization Theorem of regular variation (BGT, 1.4.1) mentioned in §1 identifies the limit in (RV_+), which is a homomorphism, by means of its continuity. Relevant here are *automatic continuity* theorems, for which see e.g. [BinO10], Hoffmann-Jørgensen's article [Rog, 3.2], and [Ros].

[3] The first author was a schoolboy of seven, and remembers vividly the shock caused by the heavy loss of life. The floods in the UK were worst in Essex; a full account is given in [Gri]. They were much worse in the Netherlands.

7. *Further results.*
(i) *Beurling slow and regular variation.* There is an interesting complement to the theory of regular variation (Karamata and de Haan theory, in the terminology of BGT), introduced by Beurling (in unpublished lectures), in connection with Tauberian theory. See BGT §2.11 and 3.10 for the theory up to 1989, [BinO11] and [BinO12] for recent results. The approach adopted there is in terms of 'asymptotic group actions', and 'asymptotic cocycles', cf. [Ost2].
(ii) *Boundedness.* There are versions of many results in regular variation ('O-versions') involving boundedness rather than convergence; see e.g. BGT Ch. 2. In more recent work, the context of regular variation has been greatly generalized (beyond its original setting of real analysis), e.g. [Ost2]. For background and detail here, see e.g. [BinO6] and references there.
(iii) *Regular variation of measures.* One of the motivations for extending regular variation beyond one dimension comes from extreme-value theory, as above. For background and detail here, see e.g. [BinO3] and references there.
8. *Historical background.* For reasons of space, we must refer elsewhere to the extensive and interesting history of regular variation. For a bibliography up to 1989, see BGT; for historical remarks see e.g. [Bin1], [Bin2] and references there.

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Postscript

The first author was fortunate to hear a course of lectures by N. G. de Bruijn at the St. Andrews Colloquium in 1968, when he was a second-year research student. Both authors have been influenced by his work, and take pleasure in dedicating this paper to his memory.

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