Almost completeness, shift-compactness, and continuity in groups

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Main Themes

A: Foundational Motivation

1. Motivation from $\mathbb{R}$: Measure-category duality, Cauchy dichotomy (Hamel pathology):

Classical examples: additivity (Ostrowski), subadditivity and convexity (Banach-Mehdi).

* Background: Measurability and Baire property (also Cantor and Baire Theorems).

2. Classical KBD and general shift theorems.

*. Kestelman versus Parthasarathy shift-compactness: what sort of compactness?

3. Measure-category duality via density topology.

4. Other fine topologies, associated generic behaviour, needing a Baire Theorem.

5. Category Embedding Theorem and Analytic Cantor and Baire Theorem

B: Semi-Polish Theorem

Theme: shift-compactness – that's what groups like best.

(Almost as good as local compactness.)

Statement and outline proof of theorem.

C: General Applications.

1. Effros Theorem.

2. Jones-Kominek Theorem.

3. Consecutive-form KBD.

4. van der Waerden combinatorics: the Ruziewicz theorem, Steinhaus Theorem from Ramsey.

5. Kingman's Theorem.

6. Reprise: Normed groups: KBD, Pettis, Baire homomorphism theorem

Background: almost a $\mathcal{G}_\delta$

Instant recall:

A is Lebesgue *measurable* – recall $A = H \backslash N$ with H a $\mathcal{G}_\delta$ and N of measure zero. It is *almost a* $\mathcal{G}_\delta$.

A is somewhere dense if cl(A) has a non-empty interior. If it is not somewhere dense, then it is *nowhere dense*.

A set is *meagre* if it is a countable union of nowhere dense sets. (First category).

A set A is *Baire* if $A = G \backslash M_1 \cup M_2$ with M_i meagre (quasi almost open)

so $A = H \cup M$ with H a $\mathcal{G}_\delta$ and M meagre. It is *almost* a $\mathcal{G}_\delta$.

Motivation: Kestelman-Borwein-Ditor Theorem

Theorem. *Let* $\{z_n\} \to 0$ *be a null sequence in* $\mathbb{R}$. *If* T *is a measurable/Baire subset of* $\mathbb{R}$, *then for generically all (= almost all/quasi all)* $t \in T$ *there is an infinite set* $\mathbb{M}_t$ *such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T. \tag{sub}$$

H. Kestelman, The convergent sequences belonging to a set, *J. London Math. Soc.* 22 (1947), 130-136.

H. Kestelman, On the functional equation $f(x + y) = f(x) + f(y)$, *Fund. Math.* 34 (1947), 144-147.

We call generalized versions of this result (null) **shift-compactness theorems**. Generalized versions have lots of applications: Steinhaus-Pettis Theorem ($A-A$ contains an interval around 0), Continuity of homomorphisms (automatic continuity), Uniform Boundedness Theorem, Uniform Convergence Theorem of Regular Variation.

Why compactness?

1. If $a_n \to a_0$ with $a_n \in T$, then $z_n := a_n - a_0 \to 0$.

For some $t \in T$, $t + z_n \in T$ for $n \in \mathbb{M}_t$ (infinite); take $s := t - a_0$

$s + a_n = (t - a_0) + a_n = t + z_n \in A$ converging through $\mathbb{M}_t$ to $s + u_0 = t \in T$.

Thus T is shift-compact.

2. Implies an (open) finite sub-covering theorem (covering after small shifts).

Why generically all?

Generic Dichotomy (Completeness Principle). *For* $F : \mathcal{B}a \to \mathcal{B}a$ *monotonic, if* $W \cap F(W) \neq \varnothing$ *for all non-meagre* $W \in \mathcal{G}_\delta$*, then, for each non-meagre* $T \in \mathcal{B}a$, $T \cap F(T)$ *is quasi almost all of* T.

That is, either

(i) *there is a non-meagre* $S \in \mathcal{G}_\delta$ *with* $S \cap F(S) = \varnothing$*, or,*

(ii) *for every non-meagre* $T \in \mathcal{B}a$, $T \cap F(T)$ *is quasi almost all of* T.

Easy Automatic Continuity

Theorem D (Darboux's Theorem). *If $f : \mathbb{R} \to \mathbb{R}$ is additive and locally bounded at some point, then f is linear.*

Proof. By additivity we may assume that f is locally bounded at the origin. So we may choose $\delta > 0$ and M such that, for all t with $|t| < \delta$, we have $|f(t)| < M$. For $\varepsilon > 0$ arbitrary, choose any integer N with $N > M/\varepsilon$. Now provided $|t| < \delta/N$, we have

$$N|f(t)| = |f(Nt)| < M, \text{ or } |f(t)| < M/N < \varepsilon,$$

giving continuity at 0. Linearity easily follows. $\square$

Strong Ostrowski Theorem. *If* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is additive and bounded (locally, above or below) on a null-shift-precompact set* S*, then* f *is locally bounded and hence linear.*

Proof. Suppose that f in not locally bounded in any neighbourhood of some point x. Then we may choose $z_n \longrightarrow 0$ such that $f(x+z_n) \geq n$, without loss of generality (otherwise replace f by $-f$). So $f(z_n) \geq n - f(x)$. Since S is null-shift-precompact, there are $t \in \mathbb{R}$ and an infinite $\mathbb{M}_t$ such that

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq S,$$

implying that f is unbounded on S locally at t (since $f(t+z_n) = f(t) + f(z_n)$), a contradiction. So f is locally bounded, and by Darboux's Theorem f is continuous and so linear. □

Corollary (Ostrowski Theorem, cf. Kes2). *If* $f:\mathbb{R}\to\mathbb{R}$ *is additive and bounded (above or below) on a set of positive measure* S*, then* f *is locally bounded and hence linear.*

Corollary (Banach-Mehdi Theorem,). *If* $f:\mathbb{R}\to\mathbb{R}$ *is additive and bounded (above or below) on a non-meagre Baire set* S*, then* f *is locally bounded and hence linear.*

An aside: averaging arguments

If T is null-shift precompact, then T is **averaging**, *for* $\{z_n\} \to 0$, *any* $u \in T$, *and with* $u_n := u + z_n$, *there are* $w \in \mathbb{R}^d$ *and* $\{v_n\} \subseteq T$ *such that, for infinitely many* $n \in \omega$, *we have:*

$$u_n = \frac{1}{2}w + \frac{1}{2}v_n.$$

Proof. In the averaging case, it is enough to show that $\frac{1}{2}T$ is subuniversal iff T is averaging. If $\frac{1}{2}T$ is subuniversal then, given $u_n \to u$, there are $w \in \mathbb{R}^d$ and some infinite $\mathbb{M}$ so that $\{-\frac{1}{2}w + u_n : n \in \mathbb{M}\} \subseteq \frac{1}{2}T$; hence, putting $v_n := 2u_n - w$, we have $\{v_n : n \in \mathbb{M}\} \subseteq T$. Conversely, if T is averaging and $\{z_n\} \to 0$, then for some x and some $\mathbb{M}$, $\{2x + 2z_n : n \in \mathbb{M}\} \subseteq T$, so $\{x + z_n : n \in \mathbb{M}\} \subseteq \frac{1}{2}T$ and hence $\frac{1}{2}T$ is subuniversal. Similar reasoning yields the reflecting case. $\square$

Why Analytic sets – because almost $\mathcal{G}_\delta$

In Polish spaces: Analytic = projection of a Borel set = continuous image of a Polish space.

$\mathcal{K}$-analytic = continuous image of a Lindelöf Čech-complete space.

* No Hamel basis is analytic. (Sierpiński, Jones)

*For T analytic, spanning $\mathbb{R}^d$ over $\mathbb{Q}$ (eg containing a Hamel basis) subadditive functions locally bounded on T are locally bounded above.

*For T analytic, spanning $\mathbb{R}^d$ over $\mathbb{Q}$ (eg containing a Hamel basis) convex functions locally bounded on T are continuous.

* Analytic sets are Baire sets, so almost $\mathcal{G}_\delta$ and in fact **directly** provide completeness arguments.

Two-in-one: weak category convergence

Definition . A sequence of autohomeomorphisms h_n of a topological space X satisfies the *weak category convergence* condition if:

For any non-empty open set U, there is an non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n \geq k} V \backslash h_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M_k in X such that, for $t \notin M_k$,

$$t \in V \Longrightarrow (\exists n \geq k)\ h_n(t) \in V. \tag{approx}$$

Category Embedding Theorem

Theorem (NHB/AO). *Let X be a topological space. Suppose given homeomorphisms $h_n : X \to X$ which satisfy the weak category convergence condition. Then, for any Baire subset T, for quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Satisfying the wcc

On the real line one proves that, for $z_n \to 0$, the shifts $h_n(t) = t + z_n$ satisfy wcc both

(i) in the usual (Euclidean norm) topology and

(ii) in the density topology.

Hence both versions of KBD follow from CET. The Baire version of KBD generalizes to metrizable topological groups that are topologically complete, and the measure version to locally compact topological groups.

$\mathcal{D}$ – the density topology on $\mathbb{R}$

Members of $\mathcal{D}$ are measurable sets s.t. all points of D are points of density 1. An arbitrary union of these is also measurable. So $\mathcal{E} \subseteq \mathcal{D}$ and so $\mathcal{D}$ is a refinement topology. One says it is a *fine* topology on $\mathbb{R}$.

A set A is Baire in $\mathcal{D}$ iff A is measurable

A is meagre in $\mathcal{D}$ iff $|A| = 0$.

So $\mathcal{D}$ is a Baire space.

Shifts are homeomorphisms: so $\mathcal{D}$ gives a semitopological group that is not paratopological.

Fine topologies in Analysis

One says $\mathcal{D}$ is a *fine* topology on $\mathbb{R}$. Fine topologies on R capture different notions of 'randomness', as in: Cohen reals (generic in the Euclidean topology), Solovay reals (generic in the density topology),

Gandy "reals" (Gandy-Sacks degrees), ... The Gandy-Harrington topology $\mathcal{GH}$ is used in Silver's Theorem.

The Ellentuck topology is used to study the Ramsey property (Mathias reals).

Fine topology is a standard tool in potential theory

[Nick's aphorism: $\pi = 3.1415...$ is one of the best fake random reals!].

From wcc to convergence

Theorem (Convergence to the identity). *Assume that the homeomorphisms* $h_n : X \to X$ *satisfy the weak category convergence condition* (wcc) *and that* X *is a Baire space. Suppose there is a countable family* $\mathcal{B}$ *of open subsets of* X *which generates a (coarser) Hausdorff topology on* X*. Then, for quasi-all (under the original topology)* t, *there is an infinite* $\mathbb{N}_t$ *such that under the coarser topology*

$$\mathrm{lim}_{m \in \mathbb{N}_t} h_m(t) = t.$$

Steinhaus-Picard Theorem

Fine Topology Interior Point Theorem. *Let $\mathbb{R}$ be given a shift-invariant topology τ under which it is a Baire space and suppose the homeomorphisms $h_n(x) = x + z_n$ satisfy* (1)*, whenever $\{z_n\} \to 0$ is a null sequence (in the Euclidean topology). For S Baire and non-meagre in τ, the difference set $S - S$ contains an interval around the origin.*

Proof. Suppose otherwise. Then for each positive integer n we may select

$$z_n \in (-1/n, +1/n) \backslash (S - S).$$

Since $\{z_n\} \to 0$ (in the Euclidean topology), the Category Embedding Theorem applies, and gives an $s \in S$ and an infinite $\mathbb{M}_s$ such that

$$\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S.$$

Then for any $m \in \mathbb{M}_s$,

$$s + z_m \in S \text{ , i.e. } z_m \in S - S,$$

a contradiction. $\square$

Baire Theorem: a fresh look

Analytic Baire Theorem – Category-heavy (van Mill: fat) **sets**.

In a Hausdorff space X, if A_n are $\mathcal{K}$-analytic, $\mathcal{M}$-heavy and dense in X, then $\bigcap_n A_n \neq \emptyset$.

Proof. Each A_n is a Baire set (Nikodym's Theorem) and co-null, so dense open modulo a meagre set. So $\bigcap_n A_n \neq \emptyset$. $\square$

$\mathcal{K}$-analytic spaces

Put $I = \mathbb{N}^{\mathbb{N}}$ (product topology with $\mathbb{N}$ discrete) and $i|n := (i_1, ..., i_n)$. $\mathcal{K} := \mathcal{K}(X)$ the compact subsets of X.

For X a Hausdorff space, a map $K : I \to \wp(X)$ is

(i) *compact-valued* if $K(i) \in \mathcal{K}$ for each $i \in I$, and

(ii) *singleton-valued* if each $K(i)$ is a singleton.

K is *upper semicontinuous* if, for each $i \in I$ and each open U in X with $K(i) \subseteq U$, there is n such that $K(i') \subseteq U$ for each i' with $i'|n = i|n$.

$A \subseteq X$ is a $\mathcal{K}$-*analytic* set if $A = K(I)$ for some compact-valued, upper-semicontinuous map $K : I \to \wp(X)$ (Frolík, 1962).

X is a $\mathcal{K}$-*analytic space* if X itself is a $\mathcal{K}$-analytic set.

By Jayne's thm (1976) this is equivalent to other defns: Choquet 1951, Sion 1960. Fremlin, 1980, defines a more general notion of *Čech-analyticity.*

Čech-analyticity

A in X is obtained from a family $\mathcal{H}$ of subsets of X by the Souslin operation, i.e. is *Souslin-*$\mathcal{H}$, if for each $i|n$ there are sets $H(i|n) \in \mathcal{H}$ such that

$$A = \bigcup_{i \in I} \bigcap_{n \in \omega} H(i|n).$$

Theorem Any $\mathcal{K}$-analytic set is Souslin-$\mathcal{F}$.

Proof. If K is upper semicontinuous and X is Hausdorff, then

$$K(I) = \bigcup_{i \in I} \bigcap_{n \in \omega} \mathrm{cl} K(i|n).$$

Indeed, as $K(i) \subseteq \bigcap_{n \in \omega} \mathrm{cl} K(i|n)$, the inclusion from left to right is clear; for the other direction, if $x \in (\bigcap_{n \in \omega} \mathrm{cl} K(i|n)) \setminus K(i)$ for some i, then there is U open with

$x \notin \mathrm{cl}U$ and $K(i) \subseteq U$, and so $K(i|n) \subseteq U$ for some n, yielding the contradiction that $x \notin \mathrm{cl}K(i|n) \subseteq \mathrm{cl}U$. □

Definition (Fremlin). For A completely regular, A is *Čech-analytic* if A is Souslin-$(\mathcal{F} \cup \mathcal{G})$ in some compactification. (See Hansell, Th. 5.3.; Fremlin's website.)

R. W. Hansell, Descriptive Topology, in Recent Progress in General Topology, 275-315, Elsevier, 1992.

The Analytic Cantor Theorem

Analytic Cantor Theorem . *Let* X *be a Hausdorff space, and let* $A = K(I)$ *be* $\mathcal{K}$*-analytic in* X, *where* K *is compact-valued and upper semicontinuous.*

Suppose that F_n *is a decreasing sequence of (non-empty) closed sets in* X *such that*

$$F_n \cap K(I(i_1, ..., i_n)) \neq \emptyset,$$

for some $i = (i_1, ...) \in I$ *and each* n. *Then*

$$K(i) \cap \bigcap_n F_n \neq \emptyset.$$

Equivalentl,y, if there are open sets V_n *in* I *with* $\mathrm{cl}V_{n+1} \subseteq V_n$ *and* $\mathit{diam}_I V_n \downarrow 0$ *such that* $F_n \cap K(V_n) \neq \emptyset$, *for each* n, *then*

(i) $\bigcap_n \mathrm{cl}V_n$ *is a singleton,* $\{i\}$ *say,*

(ii) $K(i) \cap \bigcap_n F_n \neq \emptyset$.

The restatement above is often more useful – $i \in I$ is usually constructed in a sequence of approximations V_n.

Corollary: The Gandy-Harrington Baire Theorem

Definitions. 1. Denote by $\mathcal{A}(\mathcal{T})$ the sets of $(X, \mathcal{T})$ that are $\mathcal{K}$-analytic

2. The representation $K : I \longrightarrow \mathcal{K}(X)$ is $\mathcal{T}$*-circumscribed* if there is an upper-semicontinuous determining system $\langle G(i|n) \rangle$ consisting of $\mathcal{T}$-open sets with $K(i) = \bigcap_{n\in\omega} G(i|n)$ and

$$K(I) = \bigcup_{i\in I} \bigcap_{n\in\omega} G(i|n),$$

Inspired by the Levi-van Mill Fat Sets Theorem:

Analytically heavy spaces

Fine Analytic Baire Theorem (Generalized Gandy-Harrington Theorem).

In a regular Hausdorff space, if $\mathcal{T}'$ is a refinement topology of $\mathcal{T}$, possessing a weak base (pseudo-base) $\mathcal{H} \subseteq \mathcal{A}(\mathcal{T}) \cap \mathcal{T}'$ whose elements are $\mathcal{T}'$-circumscribed, then $\mathcal{T}'$ is Baire.

In particular, this applies to a Polish space, the Gandy-Harrington $\mathcal{GH}$, the density $\mathcal{D}$ and the Ellentuck $\mathcal{E}l$ topologies.

Remark. *Analytically heavy*: any non-empty open $G \supseteq A \neq \emptyset$ with A analytic.

Application: Normed groups

Motivated by normed *vector* spaces where 'divergence' may be defined via $||x|| := d^X(x, e_X) \to \infty$, we have:

Definition. For X an *algebraic* group, say that $|| \cdot || : X \to \mathbb{R}_+$ is a *group-norm* if the following properties hold:

(i) Subadditivity (Triangle inequality): $||xy|| \leq ||x|| + ||y||$;

(ii) Positivity: $||x|| > 0$ for $x \neq e$;

(iii) Inversion (Symmetry): $||x^{-1}|| = ||x||$.

Induces two *Norm Topologies*: Norm yields a left and right invariant metric:

$$d_R^X(x,y) := ||xy^{-1}|| \text{ and } d_L^X(x,y) := ||x^{-1}y|| = d_R^X(x^{-1},y^{-1}).$$

1. $||x|| := d_R^X(x,e_X) = d_L^X(x,e_X)$

2. Under either norm topology, there is continuity of operations at e. At further distances the topology may be force the group operations to be increasingly 'less' continuous. See later.

Note the converse: if d^X is a one-sidedly invariant metric, then $||x|| := d^X(x,e_X)$ is a norm.

3. **Birkhoff-Kakutani Theorem**: a metrizable *topological* group has a right-invariant metric. In this case:

i) if X is a Baire space under the norm topology, then the wcc holds under the norm topology,

ii) if, additionally, X is locally compact, X has a Haar measure and the wcc can be verified for the *Haar-density topology*.

In infinite dimensional spaces: category beats measure (lack of compactness) making the Baire case primary.

Normed groups: their history

1. **Birkhoff-Kakutani Theorem reviewed**. Really a normability theorem:

Birkhoff-Kakutani Normability Theorem. *A first-countable right topological group X is a normed group iff inversion and multiplication are continuous at the identity.*

2. Early use by A. D. Michal and his collaborators was in providing a canonical setting for differential calculus; example:noteworthy generalization of the implicit function theorem by Bartle.

3. In name the group-norm makes an explicit appearance in 1950 in Pettis in the course of his classic closed-graph theorem (in connection with Banach's closed-graph theorem and the Banach-Kuratowski category dichotomy for groups).

4. Reappears in the group context in 1963 under the name ‘length function’, motivated by word length, in the work of R. C. Lyndon on Nielsen’s Subgroup Theorem.

5. Gromov theory has a normed group context.

Canonical Example

For a metric space (X, d^X) consider $Auth(X)$ the *algebraic* group of homeomorphisms $h : X \to X$ (under composition). Identity is $e_X(x) = x$. The *supremum metric,* if finite, is a candidate metric

$$\hat{d}(h, h') := \mathsf{sup}_x\, d^X(h(x), h'(x)).$$

OK – so restrict attention to $\mathcal{H}(X)$ those $h(x)$ such that $\mathsf{sup}_x\, d^X(h(x), e_X(x)) < \infty$ (bounded elements).

Properties

1. $\hat{d}$ is right invariant, so denote it by $d_R^{\mathcal{H}}$

$$\hat{d}(hg, h'g) = \mathsf{sup}_x\, d^X(h(g(x)), h'(g(x))) = \mathsf{sup}_y\, d^X(h(y), h'(y)) = \hat{d}(h, h').$$

2. So

$$||h|| := \hat{d}(h, e_X) \text{ defines a norm on } \mathcal{H}(X),$$

$$d_R^{\mathcal{H}}(g,h) = ||g^{-1}h|| \text{ and } d_L^{\mathcal{H}}(g,h) = ||gh^{-1}||.$$

3. Complete? Topologically complete – yes, if X compact.

4. Complete under the symmetrized topology: $d_S^{\mathcal{H}}(g,h,) = \max\{d_R^{\mathcal{H}}, d_L^{\mathcal{H}}\} = \max\{\hat{d}(g,h), \hat{d}(g^{-1},h^{-1})\}$, if X, d^X complete

5. Bitopology at work: $d_R^{\mathcal{H}} \leq d_S^{\mathcal{H}}$ so a finer topology, but latter need not be invariant.

6. Baire iff non-meagre.

7. The norm enables continuity of action:

$\mathcal{H}(X)$ acts on X via $(h, x) \to h(x)$. Action is continuous as a map from $(\mathcal{H}(X), \hat{d}) \times (X, d^X) \to (X, d^X)$, so can use to develop topological dynamics.

Lemma *Under $\hat{d}$ on $\mathcal{H}(X)$ and d^X on X, the evaluation map $(h, t) \to h(t)$ from $\mathcal{H} \times X$ to X is continuous.*

Proof. Fix h_0 and x_0. The result follows from continuity of h_0 at x_0 via

$$\begin{aligned} d^X(h_0(x_0), h(x)) &\le d^X(h_0(x_0), h_0(x)) + d^X(h_0(x), h(x)) \\ &\le d^X(h_0(t_0), h_0(t)) + d^{\mathcal{H}}(h, h_0). \qquad \square \end{aligned}$$

Normed vis-a-vis compact-open

Instead, one may consider the compact-open topology, but recall ...

Salient features of the *compact-open*.

∘ For composition to be continuous *local compactness* is essential (Dugundji Ch. XII.2, van Mill Ch.1),

∘ For T compact this topology is *admissible* (= makes $Auth(X)$ a topological group),

∘ Admissibility in the X non-compact situation not currently fully understood,

◦ In the locally compact case there exist counter-examples with non-continuous inversion,

◦ Additional properties, e.g. local connectedness usually invoked – see Dijkstra for the strongest results.

◦ Arens: if T is separable metric, and the compact-open topology on $\mathcal{C}\ (T, \mathbb{R})$ is metrizable, then T is necessarily locally compact and σ-compact, and conversely (see e.g. Engelking, p.165 and 266).

Saving grace: a complete subgroup

Definition. Say that h is *bi-uniformly continuous* if both h and i h^{-1} are uniformly continuous wrt d^X. Write

$$\mathcal{H}_u(X) = \{h \in \mathcal{H}_{unif}(X) : h^{-1} \in \mathcal{H}_{unif}\} \subseteq \mathcal{H}(X).$$

Theorem. *For X complete under d^X $\mathcal{H}_u(X)$ is complete.*

$\mathcal{H}_u(X)$ is a topological dual of X.

Almost completeness

Characterization Theorem for Almost completeness. *In a separable normed group X under d_R^X, the following are equivalent:*

(i) X *is a non-meagre absolute* $\mathcal{G}_\delta$ *modulo a meagre set (i.e. is almost complete);*

(ii) X *contains a non-meagre analytic subset;*

(iii) X *is non-meagre analytic modulo a meagre set.*

Idea goes back to Frolík, studied in particular by Aarts & Lutzer, E. Michael.

Analytic Shift Theorem

Analytic Shift Theorem. *In a normed group under the topology d_R^X, for $z_n \to e_X$ null, A a $\mathcal{K}$-analytic and non-meagre subset: for a non-meagre set of $t \in A$ with co-meagre Baire envelope, there is an infinite set $\mathbb{M}_t$ and points $a_n \in A$ converging to t such that*

$$\{ta_m^{-1}z_m a_m : m \in \mathbb{M}_t\} \subseteq A.$$

In particular, if the normed group is topological, for quasi all $t \in A$, there is an infinite set $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq A.$$

Analytic Squared Pettis Theorem. *For X a normed group, if A is analytic and non-meagre under d_R^X, then e_X is an interior point of $(AA^{-1})^2$.*

Normed versus topological: Equivalence Theorem

Theorem. *A normed group is topological iff the* d_R^X *topology is equivalent to the* d_L^X *topology. Furthermore, either of the following is equivalent to this condition:*

(i) each conjugacy $\gamma_t(x) := txt^{-1}$ *is continuous at* e *in norm,*

(ii) inversion is continuous in either d_R^X *or* d_L^X.

Corollary: *An abelian group equipped with a group norm is topological under the norm topology.*

Theorem. *For X a normed group which is separable, topologically complete, if each $\gamma_g(x) = gxg^{-1}$ is Baire, then X is topological.*

Theorem (Borel/analytic inversion) *For X a normed group which is separable, topologically complete, if inversion $x \to x^{-1}$ from (X, d_R) to (X, d_R) is Borel (or has analytic graph) then X is topological.*

Oscillation of a group and Cauchy dichotomy

Definition. Recall $\gamma_t(x) := txt^{-1}$ denotes conjugacy. Write

$$\omega(t) := \lim_{\delta \searrow 0} \omega_\delta(t), \text{ where } \omega_\delta(t) := \sup_{||z|| \leq \delta} ||\gamma_t(z)||,$$

and call the function $\omega(\cdot)$ the *oscillation of the norm.* Conjugacy γ_t is continuous for those t for which $\omega(t) = 0$. Write

$$\Omega(\varepsilon) := \{t : \omega(t) < \varepsilon\}.$$

As γ_t is a homomorphism, a normed group inherits the Cauchy dichotomy: just like homomorphisms they are pathological or given a bit of regularity topological

Theorem (Uniform continuity of oscillation). *For X a normed group*

$$\omega(t) - 2||s|| \leq \omega(st) \leq \omega(t) + 2||s||, \text{ for all } s, t \in X.$$

Hence

$$0 \leq \omega(s) \leq 2||s||, \text{ for all } s \in X,$$

implying uniform continuity in the d_R^X topology and norm-boundedness of the oscillation.

Definition. A point z lies in the *topological centre* $Z_\Gamma(X)$ of the normed group X, if $\gamma_z(x)$ is continuous (at e in norm).

Theorem. *In a topologically complete, separable, connected normed group X, if the topological centre is non-meagre, then X is a topological group.*

Dense Oscillation Theorem. *In a normed group X*

$$\bigcap_{n\in\mathbb{N}} \mathrm{cl}\,[\Omega(1/n)] = \bigcap_{n\in\mathbb{N}} \Omega(1/n) = Z_{\Gamma}.$$

Hence, if for each $\varepsilon > 0$ the ε-shifting points are dense, equivalently $\Omega(\varepsilon) = \{t : \omega(t) < \varepsilon\}$ is dense for each $\varepsilon > 0$, then the normed group is topological.

More generally, if for some open W and all $\varepsilon > 0$ the set $\Omega(\varepsilon) \cap W$ is dense in W, then $\omega = 0$ on W; n particular,

(i) *if $e_X \in W$ and X is connected and Baire under its norm topology, then X is a topological group,*

(ii) *if X is separable, connected and topologically complete in its norm topology, then X is a topological group.*

Semi-Polish Theorem

The Semi-Polish Theorem. *For a normed group* X *under* d_R^X*, if the space* X *is non-meagre and semi-Polish (i.e. is Polish under* d_S*), then it is a Polish topological group.*

Ingredients

Baire Homomorphism Theorem. *Let X and Y be normed groups analytic in the right norm-topology with X non-meagre. If $f : X \to Y$ is a Baire homomorphism, then f is continuous.*

Lemma 1 (Levi's Open Mapping Theorem). *Let X be a regular classically analytic space. Then X is Baire iff $X = f(P)$ for some f continuous and defined on some Polish space P with the property that there exists a set X' which is a dense metrizable $\mathcal{G}_\delta$ in X such that the* **restriction map** *$f|P' : P' \to X'$ where $P' = f^{-1}(X')$* **is open***.*

Lemma 2. *For X a normed group, if (X, d_S) is Polish and (X, d_R) non-meagre, then there is a subset Y of X which is a dense absolute-$\mathcal{G}_\delta$ in (X, d_R), and on which the d_S and d_R topologies agree.*

Proof. The embedding $j : (X, d_S) \to (X, d_R)$ with $j(x) = x$ makes (X, d_R) is analytic, and being non-meagre is Baire, by Theorem I. Apply Levi's Theorem to $f = j$ to obtain a set Y that is a dense $\mathcal{G}_\delta$ in (X, d_R), s.t. every open set in (Y, d_S) is open in (Y, d_R). Every open set in (Y, d_R) is open in (Y, d_S), since d_S is a refinement of d_R. Thus the two topologies agree on the $\mathcal{G}_\delta$ subset Y. As Y is a $\mathcal{G}_\delta$ subset of (X, d_R), it is also a $\mathcal{G}_\delta$ subset in the complete space (X, d_S), and so (Y, d_S) is topologically complete. So too is (Y, d_R), being homeomorphic to (Y, d_S). Working in Y, we have $y_n \to_R y$ iff $y_n \to_F y$ iff $y_n \to_L y$. $\square$

Lemma 3. *If in the setting of Lemma 2 the three topologies* d_R, d_L, d_S *agree on a dense absolutely-*$\mathcal{G}_\delta$ *set* Y *of* (X, d_R), *then for any* $\tau \in Y$ *the conjugacy* $\gamma_\tau(x) := \tau x \tau^{-1}$ *is continuous.*

Proof. We work in (X, d_R). Let $\tau \in Y$. Fix $\varepsilon > 0$; then $T := Y \cap B_\varepsilon(\tau)$ is analytic and non-meagre, since X is Baire. By the Analytic Shift Theorem there is $t \in T$ and t_n in T with t_n converging to t (in d_R) and an infinite $\mathbb{M}_t$ such that $\{tt_m^{-1}z_m t_m : m \in \mathbb{M}_t\} \subseteq T$. Since the three topologies agree on Y and as the subsequence $tt_m^{-1}z_m t_m$ converges to t in Y under d_R, it also converges to t under d_L.

Using the identity $d_L(tt_m^{-1}z_m t_m, t) = d_L(t_m^{-1}z_m t_m, e) = d_L(z_m t_m, t_m)$, we note that

$$\begin{aligned}||t^{-1}z_m t|| &= d_L(t, z_m t) \le d_L(t, t_m) + d_L(t_m, z_m t_m) + d_L(z_m t_m, z_m t)\\ &\le d_L(t, t_m) + d_L(tt_m^{-1}z_m t_m, t) + d_L(t_m, t) \to 0,\end{aligned}$$

as $m \to \infty$ through $\mathbb{M}_t$. So $d_L(t, z_m t) < \varepsilon$ for large enough $m \in \mathbb{M}_t$. In particular, for any integer N, there is $m = m(\varepsilon) > N$ with $d_R(t, tz_m) < \varepsilon$. Then, as $d_L(\tau, t) = d_R(\tau, t)$,

$$\begin{aligned} ||\tau^{-1} z_m \tau|| &= d_L(z_m\tau, \tau) \leq d_L(z_m\tau, z_m t) + d_L(z_m t, t) + d_L(t, \tau) \\ &\leq d_L(\tau, t) + d_L(tz_m, t) + d_L(t, \tau) \leq 3\varepsilon. \end{aligned}$$

Inductively, taking successively for $\varepsilon = 1/n$ and $k(n) := m(\varepsilon) > k(n-1)$, one has $||\tau^{-1} z_{k(n)} \tau|| \to 0$. By the weak continuity criterion (Lemma 3.5 of [Bost-N], p. 37),

$$\gamma(x) := \tau^{-1} x \tau$$

is continuous.

Since (X, d_R^X) is analytic and metric, each open set U is analytic, so $\gamma_\tau^{-1}(U) = \gamma(U)$ is analytic, so has the Baire property by Nikodym's Theorem. So $\gamma_\tau(x) =$

$\tau x \tau^{-1} = \gamma^{-1}(x)$ is a Baire homomorphism, and so is continouous – by the Baire Homomorphism Theorem. $\square$

Proof of Semi-Polish Theorem. Under d_R, the set $Z_\Gamma := \{x : \gamma_x$ is continuous$\}$ is a *close*d (subsemigroup) of X ([Bost-N], Prop. 3.43). So $X =$cl$_R Y \subseteq Z_\Gamma$, i.e. γ_x is continuous for all x, and so (X, d_R^X) is a topological group. So $x_n \to_R x$ iff $x_n^{-1} \to_R x^{-1}$ iff $x_n \to_L x$ iff $x_n \to_S x$. So (X, d_R^X) is a Polish topological group. $\square$

General applications

1. The Effros Theorem

Definition. A group $G \subset \mathcal{H}(X)$ acts *weakly* on a space X if $(g, x) \to g(x)$ is continuous *separately* in g and in x.

A group $G \subset \mathcal{H}(X)$ acts *transitively* on a space X if for each x, y in X there is g in X such that $g(x) = y$.

The group acts *micro-transitively* on X if for U open in G and $x \in X$ the set $\{h(x) : h \in U\}$ is a neighbourhood of x.

In a metric space a set is *analytic* if it is a continuous image of a complete separable metric space.

The Effros Open Mapping Principle. *Let G be a Polish topological group acting transitively on a separable metrizable space X. The following are equivalent.*

(i) G *acts micro-transitively on* X,

(ii) X *is Polish,*

(iii) X *is of second category*

More generally, for G an analytic **normed group** *acting transitively on a separable metrizable space X:*

$$\text{(iii)} \implies \text{(i)},$$

i.e., if X is of second category, then G acts micro-transitively on X.

Remark. Jan van Mill gave the stronger result that for G an *analytic* topological group, but actually his proof only assumes in effect a **normed** group structure.

Crimping Theorem

Theorem (Crimping Theorem). *Let T be a Polish space with a complete metric d. Suppose that a closed subgroup $\mathcal{G}$ of $\mathcal{H}_u(T)$ acts on T transitively, i.e. for any s, t in T there is h in $\mathcal{G}$ such that $h(t) = s$. Then for each $\varepsilon > 0$ and $t \in T$, there is $\delta > 0$ such that for any s with $d^T(s, t) < \delta$, there exists h in $\mathcal{G}$ with $||h||_{\mathcal{H}} < \varepsilon$ such that $h(t) = s$.*

Consequently:

(i) *if* y, z *are in* $B_\delta(t)$, *then there exists* h *in* $\mathcal{G}$ *with* $||h||_{\mathcal{H}} < 2\varepsilon$ *such that* $h(y) = z$,

(ii) *Moreover, for each* $z_n \to t$ *there are* h_n *in* $\mathcal{G}$ *converging to the identity such that* $h_n(t) = z_n$.

The Crimping Theorem implies the following classical result.

Ungar's Theorem *Let* $\mathcal{G}$ *be a subgroup of* $\mathcal{H}(X)$. *Let* X *be a compact metric space on which* $\mathcal{G}$ *acts transitively. For each* $\varepsilon > 0$, *there is* $\delta > 0$ *such that for* x, y *with* $d(x, y) < \delta$ *there is* $h \in \mathcal{G}$ *such that* $h(x) = y$ *and* $||h|| < \varepsilon$.

PS. KBD background to the Effros Open Mapping Principle

Let $\psi : G \times X \to X$ *be a continuous action that is transitive on* X. Banach's shift-theorem:

Theorem (Banach, cf. Hoffmann-Jørgensen, in proof of Th. 2.2.12 p. 349). *For* z_n *in* X *define maps* $\psi_n : G \to X$ *by setting*

$$\psi_n(g) = g(z_n).$$

For X *non-meagre and* C *Baire and co-meagre in* X, *for some* $h \in G$

$$\{\psi_n(h) : n \in \omega\} \subseteq C.$$

This needs a result based on the Kuratowski-Ulam Theorem (Category version of Fubini).

Lemma (Becker, cf. Hoffmann-Jørgensen, Prop. 2.2.1 p. 340). *For $z \in X$ fixed, and $\psi_z(g) = g(z)$, if C Baire and co-meagre in X, then $\psi_z^{-1}(C)$ is co-meagre.*

Proof of Theorem. By Lemma 1, for $n = 0, 1, 2,$... each set $\psi_n^{-1}(C)$ is co-meagre in G. Let $h \in \bigcap_{n\in\omega} \psi_n^{-1}(C)$. Then $\psi_n(h) \in C$ for each $n \in \omega$. $\square$

Remark. If $z_n \to z_0$, then $\psi_n \to \psi_0$ pointwise, because, since the action of G on X is continuous, $g(z_n) \to g(z_0)$. (More specificlly: g is a homeomorphism.) In particular, for $X = G$ with G a topological group and $z_0 = e_G$, we have $\psi_n(g) = gz_n \to g$ and ψ_n converges to the identity.

2. Analytic thinning: Jones, Kominek Theorems

Theorems of Jones and Kominek. *Let f be additive and either have (Jones) a continuous restriction, or (Kominek) a bounded restriction, $f|T$, where T is some analytic set spanning $\mathbb{R}$. Then f is continuous.*

Theorem (Compact Spanning Approximation). *For T analytic, if the linear span of T is non-null or is non-meagre, then there exists a compact subset of T which spans all the reals. If T is symmetric about the origin, then the compact spanning subset may be taken symmetric.*

Proof. If T is non-null or non-meagre, then T spans all the reals (by the Analytic Dichotomy Lemma); then for some $\varepsilon_i \in \{\pm 1\}$, $\varepsilon_1 T + ... + \varepsilon_d T$ has positive measure/

is non-meagre. Hence for some K compact $\varepsilon_1 K + ... + \varepsilon_d K$ has positive measure/ is non-meagre. Hence K spans some and hence all reals.

Let T be symmetric. If T spans the reals, then so does $T_+ = T \cap \mathbb{R}_+$. Choose a compact $K_+ \subseteq T_+$ to span the reals. Then $K := K_+ \cup (-K_+) \subseteq T$ is compact, symmetric and spans the reals. $\square$

As a corollary, we deduce the relation between the theorems of Jones and Kominek.

Kominek $\Longrightarrow$ Jones

Theorem. *Kominek's Theorem implies Jones's Theorem.*

Proof. If T is an analytic spanning set, then it contains a compact spanning set K. If f is continuous on T, then f is bounded on the compact set K. By Kominek's Theorem, as f is additive and bounded on a compact spanning set, f is continuous. $\square$

3. Combinatorics: Consecutive Embedding

(Category Embedding Theorem - Consecutive form.) *Let X be a Baire space. Suppose the homeomorphisms $h_n : X \to X$ satisfy the weak category convergence condition conjunctively. Then, for any Baire set T, for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t), h_{m+1}(t) : m \in \mathbb{M}_t\} \subseteq T.$$

4a. van der Waerden-type Combinatorics

Ruziewicz's Theorem (cf. Kemperman after Lemma 2.1 for the measure case)**.** *Given p positive real numbers $k_1, ..., k_p$ and any Baire non-meagre/measurable non-null set T, there exist d and points $x_0 < x_1 < ... < x_p$ in T such that*

$$x_i - x_{i-1} = k_i d, \qquad i = 1, ..., p.$$

Proof. Given $k_1, ..., k_p$, define a null sequence by the condition $z_{pm+i} = (k_1 + ... + k_i)2^{-m}$ $(i = 1, ..., p)$. Then there is $t \in T$ and m such that

$$\{t + z_{mp+1}, ..., t + z_{mp+p}\} \subseteq T.$$

Taking $d = 2^{-m}$, $x_0 = t$ and for $i = 1, ..., p$

$$x_i = t + z_{mp+i} = t + (k_1 + ... + k_i)d,$$

we have $x_0 < x_1 < ... < x_p$ and

$$x_{i+1} - x_i = k_i d. \qquad \square$$

Remarks. 1. If each $k_i = 1$ above, then the sequence $x_0, ..., x_p$ is an arithmetic progression of arbitrarily small step d (which we can take as 2^{-m} with m arbitrarily large) and arbitrarily large length p. So if $\mathbb{R}$ is partitioned into a finite number of Baire/measurable cells, one cell T is necessarily non-meagre/measurable, and contains arbitrarily long arithmetic progressions of arbitrarily short step. This is similar to the van der Waerden theorem.

4b. After Ruziewicz: Ramsey Property

Definition. Say that a set S has the *strong (weak) Ramsey distance property* if for any convergent sequence $\{u_n\}$ there is an infinite set (a set with two members) $\mathbb{M}$ such that

$$\{u_n - u_m : m, n \in \mathbb{M} \text{ with } m \neq n\} \subseteq S.$$

Thinking of the points of S as those having a particular colour, S has the strong Ramsey distance property if any convergent sequence has a subsequence all of whose pairwise distances have this colour.

Combinatorial Steinhaus Theorem. *For an additive subgroup S of $\mathbb{R}$, the following are equivalent:*

(i) $S = \mathbb{R}$,

(ii) S *is shift-compact,*

(iii) S *is null-shift-compact,*

(iv) S *has the strong Ramsey distance property,*

(v) S *has the weak Ramsey distance property,*

(vi) S *has the finite covering property,*

(vii) S *has finite index in* $\mathbb{R}$*.*

5. Simultaneous Embeddings

Kingman's Theorem (for Category). *If* $\{S_k : k = 1, 2, ...\}$ *are Baire and essentially unbounded in the category sense, then for quasi all* η *and each* $k \in \mathbb{N}$ *there exists an unbounded subset* $\mathbb{J}_\eta^k$ *of* $\mathbb{N}$ *with*

$$\{n\eta : n \in \mathbb{J}_\eta^k\} \subset S_k.$$

In particular this is so if the sets S_k *are open and unbounded.*

Kingman's Theorem (for Measure). *If* $\{S_k : k = 1, 2, ...\}$ *are measurable and essentially unbounded in the measure sense, then for almost all* η *and each* $k \in \mathbb{N}$ *there exists an unbounded subset* $\mathbb{J}_\eta^k$ *of* $\mathbb{Q}_+$ *with*

$$\{q\eta : q \in \mathbb{J}_\eta^k\} \subset S_k.$$

6. Regular variation: what are regularly varying functions?

The theory of regular variation, or of regularly varying functions, explores the consequences of a relationship of the form

$$f(\lambda x)/f(x) \to g(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0, \tag{RV}$$

for functions defined on $\mathbb{R}_+$. The limit function g must satisfy the Cauchy functional equation

$$g(\lambda\mu) = g(\lambda)g(\mu) \qquad \forall \lambda, \mu > 0. \tag{CFE}$$

Subject to a mild regularity condition, (CFE) forces g to be a power:

$$g(\lambda) = \lambda^\rho \qquad \forall \lambda > 0. \tag{ρ}$$

Then f is said to be *regularly varying* with *index* ρ, written $f \in R_\rho$.

Uniform Convergence Theorem: *For f Baire/measurable the convergence above is uniform on compact sets of λ.*

Background Information

An algebraic group with a topology under which $(x, y) \to xy$ is **se***parately* continuous is called a **se***mitopological* group.

An algebraic group with a topology under which $(x, y) \to xy$ is *jointly* continuous is called a *paratopological* group.

A topological space is Čech-complete if it embeds as a $\mathcal{G}_\delta$ subset in some/any compactification.

So in a compact group H, if a subgroup G is not a $\mathcal{G}_\delta$, then G isn't complete.

Examples [cf. Charatonik et al.]: open mappings – an $\mathcal{F}_{\sigma\delta}$; ditto: montone open, light open.

Saving grace: if H is Borel (more generally: **analytic**), then, we're still 'almost' in a complete situation.

Bouziad's Theorems (1996)

B1. A Čech-analytic (in particular a Čech-complete) Baire semitopological group is a topological group.

B2. A pointwise countably complete (in particular, p-space), Baire, left topological group with separately continuous action on a p-space has continuous action.

History:

1936 Montgomery: A completely metrizable semitopological group is paratopological.

1957 Ellis: A locally compact semitopological group is paratopological.

1957 Ellis: A locally compact paratopological group is topological.

1960 Żelazko: A completely metrizable paratopological group is topological.

1982 Brand: A Čech-complete paratopological group is topological.

1993 Bouziad: For a Čech-complete semitopological group, the group is topological iff paracompact.

In fact it is paracompact, because every Baire p-space is paratopological.

1994 Reznichenko: (no proof; true also by B2) A Čech-complete Baire semitopological group is a topological group.

Abelian Normability

A norm is *abelian* if $||xy| = ||yx||$. So a norm on an abelian group is an abelian norm.

Lemma. If the norm is abelian, the normed group is topological.

Abelian normability of $\mathcal{H}(X)$. *Suppose that*

$$||f||_\infty := \text{sup}\{||f||_g : g \in X\}, \text{ where } ||f||_g := ||gfg^{-1}||_X,$$

is finite for f in $\mathcal{H}(X)$ – eg if the metric d_R^X is bounded, or in particular if X is compact.

Then:

(i) $||f||_\infty$ *is abelian and* $\mathcal{H}(X)$ *under* $||f||_\infty$ *is topological.*

(ii) *The norm* $||f||_\infty$ *is equivalent to the supremum norm* $||f||_{\mathcal{H}}$ *iff* (n-adm) *holds, i.e. for* $||f_n||_{\mathcal{H}} \to 0$ *and for arbitrary* g_n *in* $\mathcal{H}(X)$,

$$||g_n f_n g_n^{-1}||_{\mathcal{H}} \to 0.$$

Equivalently, for $||z_n||_{\mathcal{H}} \to 0$ *(i.e..* z_n *converging to* e*), arbitrary* g_n *in* $\mathcal{H}(X)$, *and arbitrary* $y_n \in X$,

$$||g_n(z_n(y_n))g_n(y_n)^{-1}||_X \to 0.$$

(iii) *In particular, if* X *is compact,* $\mathcal{H}(X) = \mathcal{H}_u(X)$ *is a topological group under the supremum norm* $||f||_{\mathcal{H}}$.

The Pettis Theorem

Context: a normed group induces two metrics $d_R^X(x,y) := ||xy^{-1}||$ (right invariant) and $d_L^X(x,y) := ||x^{-1}y||$ (left invariant). Here preference is given to the right-invariant version in the theorems below. KBD Theorem holds but in a more involved format, but does leads to the following elegant result:

Theorem (Squared Pettis Theorem). *Let X be a topologically complete normed group and A Baire non-meagre under the right norm. Then e_X is an interior point of $(AA^{-1})^2$.*

Squaring and higher powers of AA^{-1} were studied by Henstock (1963) and E. Følner (Math. Scand. 2(1954),5-18; cf. Bogoliouboff).

When X is normed locally compact there exists an invariant Haarmeasure4 on X and KBD holds as well as a measure version of the Pettis Theorem (without squaring).

Baire Homomorphism Theorem. *Let X and Y be normed groups with Y K-analytic and X topologically complete. If $f : X \to Y$ is a Baire homomorphism, then f is continuous.*

KBD – normed group Haar version

Say that the group-norm $||.||$ on X has the *Heine-Borel property* if under either of the norm topologies a set is compact iff it is closed and norm-bounded, or equivalently if norm-bounded sets are *precompact.*

In such circumstances closed balls are compact and the norm topology is locally compact. A norm can always be replaced with an equivalent one that has the Heine-Borel property.

Kestelman-Borwein-Ditor Theorem – normed Haar version. *In a normed group X whose right norm topology has the Heine-Borel property, if $\{z_n\} \to e_X$ (a null sequence converging to the identity) and T is (right) Haar-measurable, then for almost all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{z_m t : m \in \mathbb{M}_t\} \subseteq T.$$

Open Problems

1. **Countable productivity of shift-comapctness?**

For a metrizable, separable, topological group G and subsets thereof A_i $(i \in \omega)$ that have covering shift-compactness, i.e. open covers have shifted-subcovers that are finite, for arbitrarily small shifts. Is the product $\prod_{i\in\omega} A_i$ (with Tychonoff topology) covering shift-compact? (Yes if the A_i are subgroups.)

2.The covering property above is implied by sequential shift-compactness. Are the two equivalent?

3. Non-specific question: examine the pathology of normed groups, i.e broaden our understanding of when normed is topological?