Infinite combinatorics and the foundations of regular variation
by

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Abstract The theory of regular variation is largely complete in one dimension, but is developed under regularity or smoothness assumptions. For functions of a real variable, Lebesgue measurability suffices, and so does having the property of Baire. We find here that the preceding two properties have common combinatorial generalizations, exemplified by 'containment up to translation of subsequences'. All of our combinatorial regularity properties are equivalent to the uniform convergence property.

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1 Introduction

The theory of regular variation, or of regularly varying functions, is a chapter in the classical theory of functions of a real variable, dating from the work of Karamata in 1930. It has found extensive use in probability theory, analysis (particularly Tauberian theory and complex analysis), number theory and other areas; see [8] for a monograph treatment, and [32] Chapter IV. Henceforth we identify our numerous references to [8] by BGT. The theory explores the consequences of a relationship of the form

$$f(\lambda x)/f(x) \to g(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0, \tag{RV}$$

for functions defined on $\mathbb{R}_+$. The limit function g must satisfy the Cauchy functional equation

$$g(\lambda\mu) = g(\lambda)g(\mu) \qquad \forall \lambda, \mu > 0. \tag{CFE}$$

Subject to a mild regularity condition, (CFE) forces g to be a power:

$$g(\lambda) = \lambda^{\rho} \qquad \forall \lambda > 0. \tag{ρ}$$

Then f is said to be *regularly varying* with *index* ρ, written $f \in R_{\rho}$.

The case $\rho = 0$ is basic. A function $f \in R_0$ is called *slowly varying*; slowly varying functions are often written ℓ (for *lente*, or *langsam*). The basic theorem of the subject is the Uniform Convergence Theorem (UCT), which states that if

$$\ell(\lambda x)/\ell(x) \to 1 \qquad (x \to \infty) \qquad \forall \lambda > 0, \tag{SV}$$

then the convergence is *uniform* on compact λ-sets in $(0, \infty)$.

The basic facts are:
(i) if ℓ is (Lebesgue) measurable, then the UCT holds;
(ii) if ℓ has the Baire property (for which see e.g. Kuratowski [35], Oxtoby [41]), then the UCT holds;
(iii) in general, the UCT need not hold.
Similarly, if f is measurable or has the Baire property, (CFE) implies (ρ), but not in general. See BGT §§1.1, 1.2; for background on the Cauchy functional equation, see [9] and [33], [1].

The UCT extends easily to regularly as well as slowly varying functions; see BGT Th. 1.5.2. The basic case is $\rho = 0$, so we lose nothing by restricting attention to it here.

The basic foundational question in the subject, which we address here, concerns the search for natural conditions for the above to hold, and in particular for a substantial common generalization of measurability and the Baire property. We find such a common generalization, which is actually both necessary and sufficient. See the Main Theorem in Section 3. The paper thus answers an old problem noted in BGT p. 11 Section 1.2.5.

While regular variation is usually used in the multiplicative formulation above, for proofs in the subject it is usually more convenient to use an additive formulation. Writing $h(x) := \log f(e^x)$ (or $\log \ell(e^x)$ as the case may be), $k(u) := \log g(e^u)$ and, following the letter convention of BGT, the relations above become

$$h(x+u) - h(x) \to k(u) \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}, \qquad (RV_+)$$

$$h(x+u) - h(x) \to 0 \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}, \qquad (SV_+)$$

$$k(u+v) = k(u) + k(v) \qquad \forall u, v \in \mathbb{R}. \qquad (CFE_+)$$

Here the functions are defined on $\mathbb{R}$, whereas in the multiplicative notation functions are defined on $\mathbb{R}_+$.

It is convenient to describe the context of the Uniform Convergence Theorem (UCT) by writing

$$\delta_x h(u) = h(u+x) - h(x)$$

and regarding $\delta_x h(u)$, with x as parameter, as though it were an 'approximately-additive' function of u (a term defined explicitly in [33] p. 424). Then, granted assumptions on the function h, (UCT) asserts that pointwise convergence of the family $\{\delta_x h\}_{x \in \mathbb{R}}$ implies uniform convergence over compact sets of u. In this context the following dual notation is thus natural:

$$k(u), \text{ or }, \qquad \partial h(u) := \lim_{x \to \infty} h(u+x) - h(x). \qquad (1)$$

2 Infinite combinatorics

The concepts we need for our analysis are embodied in the following definitions. They have been extracted from a close reading of the standard treatment of UCT in BGT, but whilst only implicit there, here they are now identified as quintessential.

Definitions - 1.

(i) The **ε-level set** (of $\delta_x h$) is defined to be the set

$$H^\varepsilon(x) := \{t : |\delta_x h(t)| < \varepsilon\} = \{t : |h(t+x) - h(x)| < \varepsilon\}.$$

(ii) For $\mathbf{x} = \{x_n : n \in \omega\}$ an arbitrary sequence tending to infinity, the **$\mathbf{x}$-stabilized ε-level set** (of h) is defined to be the set

$$T_k^\varepsilon(\mathbf{x}) = \bigcap\nolimits_{n=k}^{\infty} H^\varepsilon(x_n) \text{ for } k \in \omega.$$

Here ω denotes the set of natural numbers $0, 1, 2, ...$. Note that

$$T_0^\varepsilon(\mathbf{x}) \subseteq T_1^\varepsilon(\mathbf{x}) \subseteq T_2^\varepsilon(\mathbf{x}) \subseteq ... \text{ and } T_k^\varepsilon(\mathbf{x}) \subseteq T_k^\eta(\mathbf{x}) \text{ whenever } \varepsilon < \eta. \qquad (2)$$

If h is slowly varying, then $\mathbb{R} = \bigcup_{k\in\omega} T_k^\varepsilon(\mathbf{x})$.

(iii) We say that a set S is **universal** (resp. **subuniversal**) if for any null sequence $z_n \to 0$, there are $s \in \mathbb{R}$ and a co-finite (resp. infinite) set $\mathbb{M}_s$ such that

$$\{s + z_m : m \in \mathbb{M}_s\} \subseteq S.$$

We shall also say that a universal set S *includes by translation* the null sequences. (Omission of 'by translation' is not to be taken as implying translation.) We say that a subuniversal set *traps* null sequences, to abbreviate 'includes by translation a subsequence of'. Subuniversality, a property possessed by various 'large' sets (see below), is linked both to compactness and additivity through 'shift-compactness': see [12] for a topological analysis.

Clearly an open interval is universal and hence also subuniversal. Indeed, suppose z is in the interior of S, and suppose u_m converges to u; then with $y = z - u$ we see that

$$y + u_m = z - (u - u_m)$$

is ultimately in S. A subuniversal set is necessarily uncountable: see [9].

We shall later be concerned with bounded and or convergent sequences $\{u_n\}$. Of course, for S subuniversal, if $\{u_n\}$ is a bounded sequence, we may pass to a convergent subequence with limit u, for which the corresponding subsequence $z_n := u_n - u$ is null, and so there are $t \in \mathbb{R}$ and an infinite set $\mathbb{M}_t$ such that

$$\{t + u_m : m \in \mathbb{M}_t\} \subseteq S.$$

The reason that the above definition is phrased in terms of null sequences is that we may wish to have $s \in S$, as in the next theorem. The following result is due in this form in the measure case to Borwein and Ditor [16], but was already known much earlier albeit in somewhat weaker form by Kestelman ([30] Th. 3), and rediscovered by Trautner [45] (see [8] p. xix and footnote p. 10).

Theorem (Kestelman-Borwein-Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null (resp. non-meagre), then, for almost all (resp. for quasi-all) $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

For the proof see [9]. In the next definition we use bounded sequences.

Definition - 2. The basic **No Trumps** combinatorial principle (there are several), denoted $\mathbf{NT}(\{T_k : k \in \omega\})$, refers to a family of subsets of reals $\{T_k : k \in \omega\}$ and means the following.

For every bounded sequence of reals $\{u_m : m \in \omega\}$ there are $k \in \omega$, $t \in \mathbb{R}$ and an infinite set $\mathbb{M} \subseteq \omega$ such that

$$u_m + t \in T_k \text{ for all } m \text{ in } \mathbb{M}.$$

In words: the translate of some *subsequence* of $\{u_m\}$ is contained in some T_k. As with universality (resp. subuniversality), we will also say that the family $\{T_k : k \in \omega\}$ *includes by translation* (resp. *traps*) the *bounded sequences.* (See Section 5 for the background on this terminology.)

If for some k the set T_k is subuniversal then $\mathbf{NT}(\{T_k : k \in \omega\})$ holds; thus the latter is less restrictive, especially if, as it may happen in applications, the family $\{T_k : k \in \omega\}$ is increasing, as e.g. in (2).

Here again we note that if $\{T_n\}$ is a family of sets such that for some n the set T_n contains an interval, then the family traps sequences. This observation ties in with the standard textbook approach to UCT, where a number of proofs arrange to use measurability and Steinhaus's Theorem (see BGT Theorem 1.1.1 p. 2) to manufacture an interval that traps a convergent sequence. One can also relate the sequence trapping property directly to the notion of 'automatic continuity'. Here the natural point of departure from the present perspective is the limit function of (1) which, assuming it exists, is additive. We study in [9] the present combinatorial insights, as they impinge on the Ostrowski and Steinhaus Theorems; there is also the expected connection with the natural classes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ associated with automatic continuity, as defined by Ger and Kuczma (see [33] p. 206 or [24], and also [10], [11]).

The existing literature is on universality and has mostly concentrated not on inclusion but on exclusion, even of images of *entire* convergent sequences (affine images, including translates); see for example [31] in regard to sets of positive measure avoiding translates of a given convergent sequence (see [36] for additional references). Our rather different approach is motivated by the relationship which we demonstrate between UCT and 'positive' rather than 'negative' combinatorics.

To clarify the status of the weaker concept of subuniversality in its present context of measure and category we refer to the notions of Lusin set (or, to use the modern transliteration, Luzin set), Sierpiński set, Hamel basis, and automatic continuity. We recall that a *Luzin* set is one which meets any nowhere dense set in at most a countable set. Similarly a *Sierpiński* set is one which meets any set of measure zero in at most a countable set. See [34], [37] p. 32 (where there is a historical attribution to Mahlo, and the two concepts are described as $\mathcal{I}$-Luzin sets for the appropriate σ-ideal $\mathcal{I}$), or [38] for a survey of 'special' subsets of the real line. An altogether more fruitful viewpoint on the similarity comes from giving $\mathbb{R}$ the density topology; in the first place we may interpret a Sierpiński set then as a Luzin set in the density topology, secondly, and more thematically, the two forms of the Kestelman-Borwein-Ditor Theorem become unified, as two corollaries of one more general theorem, the Category Embedding Theorem (for which see [9]), as do for the same reasons the classical category and measure versions of the UCT (see [13] for an approach to the UCT via measure-category duality).

A Luzin set is measurable and is of measure zero; furthermore, it is of second category, but fails to have the Baire property. See e.g. [33], p. 63 for proofs. Similarly every Sierpiński set is strongly meagre, see [42].

Proposition 1. *Assume the Continuum Hypothesis (CH). There exists a Luzin set (resp. Sierpiński set) which contains a Hamel basis and contains all sequences up to translation. Its difference set has empty interior.*

See the end of the paper for a remark on the set-theoretic character of such a set under *Gödel's Axiom of Constructibility* ($V = L$).

3 The UCT and its equivalents

We begin by noting the following strong property of the stabilized ε-level sets.

Proposition 2 (Sequence inclusion). *Suppose the UCT holds for a function h. Let $\mathbf{u}$ be any bounded sequence, and let $\varepsilon > 0$. Then, for every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level set $T_k^\varepsilon(\mathbf{x})$ for some k includes the sequence $\mathbf{u}$. In particular, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ trap bounded sequences.*

For a proof see Section 4.1. Our main result is the following 'converse' (see Section 5 for the terminology 'No Trumps').

Theorem 1 (Main Theorem, or UCT). *For h slowly varying, the following are equivalent.*
(i) *The UCT holds for h.*
(ii) *The principle $\mathbf{1\text{-}NT}_h$ holds: for every $\varepsilon > 0$ and every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ of h trap bounded sequences by translation. That is:*
$(\forall\varepsilon > 0)(\forall\mathbf{x})\ \mathbf{NT}(\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\})$.
(iii) *For every $\varepsilon > 0$ and for every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ of h include all the bounded sequences.*

That this is indeed the sought-for generalization of the UCT in BGT is shown by the special case of the following general result. We term the latter the No Trumps Theorem, as it justifies the combinatorial framework of No Trumps.

Theorem 2 (No Trumps Theorem). *Let T be an interval. Suppose that $T = \bigcup_{k\in\omega} T_k$, where the sets T_k are measurable/Baire. Then the sets $\{T_k : k \in \omega\}$ include bounded sequences by translation, i.e. $\mathbf{NT}(\{T_k : k \in \omega\})$.*

The idea behind the next theorem comes from a re-interpretation of what is referred to as the 'fourth proof of UCT' in BGT, p. 9, which proof is a reworking of one due to Csiszár and Erdős, see [18].

Theorem 3 (Trapping Families Theorem, after Csiszár and Erdős**).** *Suppose the slowly varying function h is measurable, or has the property of Baire. Let $\mathbf{x} = \{x_n\}$ be any sequence tending to infinity. Then, the stabilized*

ε*-level sets* $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ *include bounded sequences by translation, i.e.* $\mathbf{NT}(\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\})$.

As the proof of this theorem is only implicitly given in BGT, p. 9, being bound up with a different context, we repeat the short proof, in isolated form, for convenience in Section 4.3. In fact, much more is true (see [10]); we restrict attention here to the simplest case, which suffices for our present purposes. Theorem 3 combined with the Main Theorem yields as immediate the following corollary.

Corollary (Classical UCT). *Suppose the slowly varying function* h *is measurable, or has the property of Baire. Then*

$$h(x+u) - h(x) \to 0, \text{ as } x \to \infty,$$

uniformly for u *in a compact set.*

We have already seen in the discussion of subuniversality the equivalence of trapping null sequences and bounded sequences. This simple equivalence is reflected in a more powerful result which is at the heart of a whole chain of equivalent formulations of the UCT.

Theorem 4 (The Bounded Equivalence Principle). *For* h *a slowly varying function the following are equivalent.*
(i) *The UCT holds for* h.
(ii) *Whenever* $\{u_n\}$ *is a bounded sequence, and* $\{x_n\}$ *tends to infinity*

$$\lim_{n\to\infty}(h(u_n + x_n) - h(x_n)) = 0.$$

(iii) *Whenever* $\{z_n\}$ *is a null sequence, and* $\{x_n\}$ *tends to infinity*

$$\lim_{n\to\infty}(h(z_n + x_n) - h(x_n)) = 0.$$

In BGT p.7 condition (iii) is derived when the slowly varying h is measurable or Baire as a first step in a direct proof of the UCT. The broader picture is formulated in the next theorem and in the diagram below it.

Theorem 5 (Equivalence Theorem). *For* h *a slowly varying function the following are equivalent.*

(i) *The principle* $\mathbf{1\text{-}NT}_h$ *holds: the family* $\{T_n^\varepsilon(\mathbf{x}) : n \in \omega\}$ *traps bounded sequences for any* ***real*** *sequence* $\mathbf{x}$ *tending to infinity, and any positive* ε*. That is:*

$(\forall\varepsilon > 0)(\forall \text{ real } \mathbf{x})\ \mathbf{NT}(\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\})$.

(ii) *Whenever* $\{u_n\}$ *is a bounded sequence, and* $\{x_n\}$ *tends to infinity*

$$\lim_{n\to\infty}(h(u_n + x_n) - h(x_n)) = 0. \tag{3}$$

(ii)* *For any sequence* $\mathbf{x}$ *tending to infinity, and any positive* ε*, the family* $\{T_n^\varepsilon(\mathbf{x}) : n \in \omega\}$ ***ultimately contains*** *almost all of any bounded sequence* $\mathbf{u}$*. That is, for any bounded sequence* $\mathbf{u} = \{u_n\}$*, there is* k *such that*

$$\{u_m : m > k\} \subseteq T_n^\varepsilon(\mathbf{x}) \text{ for all } n > k. \tag{4}$$

(iii) *Whenever* $\{u_n\}$ *is a bounded sequence, and* $\mathbf{m} = \{m_n\}$ *is an* ***integer*** *sequence tending to infinity*

$$\lim_{n\to\infty}(h(u_n + m_n) - h(m_n)) = 0. \tag{5}$$

(iv) $\mathbf{2\text{-}NT}_h$ *holds: the family* $\{T_n^\varepsilon(\mathbf{m}) : n \in \omega\}$ *traps bounded sequences for any* ***integer*** *sequence* $\mathbf{m}$ *tending to infinity, and any positive* ε*. That is:*

$(\forall\varepsilon > 0)(\forall \text{ integer } \mathbf{m})\ \mathbf{NT}(\{T_k^\varepsilon(\mathbf{m}) : k \in \omega\})$.

(v) $\mathbf{3\text{-}NT}_h$ *holds: for all* $\varepsilon > 0$*, the family* $\{T_n^\varepsilon(\mathbf{m}) : n \in \omega\}$ *traps bounded sequences with* $\mathbf{m}$ *restricted to just the one sequence* $\mathbf{id}$ *defined by* $m_n = n$*. That is:*

$(\forall\varepsilon > 0)\ \mathbf{NT}(\{T_k^\varepsilon(\mathbf{id}) : k \in \omega\})$.

(vi) *The UCT holds for* h*.*

In particular, for h *slowly varying, the three combinatorial principles* $\mathbf{1\text{-}NT}_h$*,* $\mathbf{2\text{-}NT}_h$*,* $\mathbf{3\text{-}NT}_h$ *involving sequence trapping are all equivalent.*

The assertion (ii)* , which is actually a transcription of (ii), clearly alludes to some further variations on the $\mathbf{i\text{-}NT}_h$ theme. The sequence $\{T_k^\varepsilon(\mathbf{y}) : k \in \omega\}$ may have one of three 'inclusion properties' in relation to a bounded sequence $\mathbf{u}$. For some k, $T_k^\varepsilon(\mathbf{y})$ could:

(F) include all of $\mathbf{u}$, i.e. *fully* include $\mathbf{u}$, or,

(A) include *almost* all terms of $\mathbf{u}$, or,

(ST) include a *subsequence* of $\mathbf{u}$ *by translation*, i.e. precisely $\mathbf{NT}$ itself.

We refer to these various strengthenings of trapping as $F/A/ST$ analogues of trapping. Furthermore the inclusion property might be applied to:

($\mathbf{x}$) $\mathbf{y}$ ranging over real sequences $\mathbf{x}$,
($\mathbf{m}$) $\mathbf{y}$ ranging over integer sequences $\mathbf{m} = \{m_n\}$,
($\mathbf{id}$) $\mathbf{y}$ restricted to just the one integer sequence $\mathbf{id}$ defined by $m_n = n$.

The implications can be summarized in a 'contingency table', shown below in the style of the *Cichoń diagram*, for which see [23]. The minimal one is thus $\mathbf{NT}_h := \mathbf{3\text{-}NT}_h$ (referring to the sequence $\mathbf{id}$).

When restricted to a slowly varying function h all these properties are equivalent.

$$\begin{array}{ccccccc} & ST(\mathbf{x}) & \Longrightarrow & ST(\mathbf{m}) & \Longrightarrow & ST(\mathbf{id}) & \Longrightarrow UCT_h \\ & \Uparrow & & \Uparrow & & \Uparrow & \\ & A(\mathbf{x}) & \Longrightarrow & A(\mathbf{m}) & \Longrightarrow & A(\mathbf{id}) & \\ & \Uparrow & & \Uparrow & & \Uparrow & \\ UCT_h \Longrightarrow & F(\mathbf{x}) & \Longrightarrow & F(\mathbf{m}) & \Longrightarrow & F(\mathbf{id}) & \end{array}$$

Here

$$P(\cdot) = F/A/ST \text{ analogue of the property } \forall\varepsilon\forall(\cdot)\mathbf{NT}(\{T_k^\varepsilon(\cdot) : k \in \omega\}),$$

$$\begin{aligned} F &= \text{Full inclusion,} \\ A &= \text{Almost inclusion,} \\ ST &= \text{Subsequence inclusion by translation.} \end{aligned}$$

Of course in combination with the Trapping families theorem, the equivalence theorem contributes a 'sixth' proof of UCT complementing the five given in BGT, Chapter 1.

As a consequence of the equivalence principle, in the general setting of a regularly varying function h, one may relax the definition of the associated limit function in (1), that is, the limit may be taken there sequentially rather than continuously. Other variations are possible: see the remarks at the end.

4 Proofs

4.1 Proof that UCT implies sequence inclusion

Suppose given two sequences $\mathbf{x} = \{x_n\}$ and $\mathbf{u} = \{u_n\}$ with $x_n \to \infty$ and u_n bounded. If the sequence $\{u_m\}$ lies in the compact interval $[a, b]$ then, for any $\varepsilon > 0$, there is k so large that, for any u in $[a, b]$ and any $n \geq k$, we have

$$|h(u + x_n) - h(x_n)| < \varepsilon.$$

This means that any such u is in $T_k^\varepsilon(\mathbf{x})$, so in particular $\{u_m : m \in \omega\} \subset T_k^\varepsilon(\mathbf{x})$. □

4.2 Proof of the Main theorem (UCT)

From the last Proposition we already know that (i) implies (iii) and (iii) implies (ii). It remains to prove that (ii) implies (i).

So suppose that UCT fails for some function h.

Suppose that for the two sequences $\mathbf{x} = \{x_n\}$ and $\mathbf{u} = \{u_n\}$ with $x_n \to \infty$ and u_n bounded there is an $\varepsilon > 0$ such that for $n = 1, 2, ..$ we have

$$|h(x_n + u_n) - h(x_n)| \geq 2\varepsilon. \tag{6}$$

Note that if $y \in T_k^\varepsilon(\mathbf{x})$ then we have, for $n = k, k + 1, ...$, that

$$|h(x_n + u_n) - h(x_n + y)| \geq \varepsilon. \tag{7}$$

Indeed, otherwise we would have

$$|h(x_n + u_n) - h(x_n + y)| < \varepsilon$$

and

$$|h(x_n + y) - h(x_n)| < \varepsilon,$$

contradicting (6).

Now, by the trapping assumption, for infinitely many m in, say $\mathbb{M}$, we have

$$y_m = u_m + z \in T_k^\varepsilon(\mathbf{x}) \text{ for } m \in \mathbb{M}.$$

Now, for any such $m \in \mathbb{M}$ with $m > k$, by (7) with $y = y_m$, we have that for $n = m$:

$$|h(x_m + u_m) - h(x_m + u_m + z)| \geq \varepsilon.$$

Putting $v_m = x_m + u_m$ this yields that

$$|h(z + v_m) - h(v_m)| \geq \varepsilon,$$

which contradicts that h is slowly varying. Hence the assumption (6) is untenable, and thus after all UCT holds. □

4.3 The No Trumps and the Trapping Families Theorem

The No Trumps Theorem follows immediately from the Kestelman-Borwein-Ditor Theorem; indeed if the interval T is the union of the measurable/Baire sets T_k, then for some k the set T_k is non-null/non-meagre. (Compare the remark after the definition of the **NT** combinatorial principle.) This follows the exposition of the infinite combinatorics of subuniversality followed in [9]. As to the Trapping Families Theorem, one way to derive it is from the No Trumps Theorem by taking $T_k := T_k^{\varepsilon}$ and noting that these are measurable/Baire if the slowly varying function h is measurable/Baire.

Alternatively, one may use the argument below extracted from the Csiszár-Erdős proof [18] of the UCT. Without loss of generality we take $T = [-1, 1]$. Now let $\mathbf{u} = \{u_n\}$ be a bounded sequence, which we may as well assume is convergent to some u_0. We assume that $|u_n - u_0| \leq 1$. We are to show that for some z, some K, and some infinite $\mathbb{M} \subset \omega$, we have $z + u_m \in T_K$.

By assumption, each T_k is measurable [Baire], so there is K such that T_K has positive measure [is non-meagre]. Let

$$Z_K = \mathbf{u}(T_K) := \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} (T_K - u_n) .$$

We now quote almost verbatim from BGT p. 9. 'In the measurable case all the $Z_{n,K}$ have measure $|T_K|$, and as they are subsets of the fixed bounded interval $[u_0 - 2, u_0 + 2]$, Z_K is a subset of the same interval having measure

$$|Z_K| = \lim_{j \to \infty} \left| \bigcup_{n=j}^{\infty} (T_K - u_n) \right| \geq |T_K| > 0.$$

So Z_K is non-empty.

In the Baire case T_K contains some set $I \backslash M$, where I is an open interval of length $\delta > 0$, and M is meagre. So each set $T_K - u_n$ contains $I^n \backslash M^n$, where $I^n = I - u_n$ is an open interval of length δ and $M^n := M_n - u_n$ is meagre. Choosing J so large that $|u_i - u_j| < \delta$ for all $i, j \geq J$, the intervals $I^J, I^{J+1}, ...$ all overlap each other, and so $\bigcup_{n=j}^{\infty} I^n$, for $j = J, J+1, ...$, is a decreasing sequence of intervals, all of length $\geq \delta$ and all contained in the interval $[u_0 - 2, u_0 + 2]$; hence $I^0 = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} I^n$ is an interval of length $\geq \delta$. Since Z_K contains $I^0 \backslash \bigcup_{n=j}^{\infty} M^n$, it follows that Z_K is non-meagre, so non-empty.' Thus in either case, there is a point $z \in Z_K$.

This means that $z \in T_K - u_n$ for infinitely many n. Say that

$$z \in T_K - u_m \text{ for } m \in \mathbb{M}.$$

Without loss of generality, $m \in \mathbb{M}$ implies $m > K$.

Consider $m \in \mathbb{M}$. By definition, for some $y = y_m$, we have $z = y_m - u_m$ with $y_m \in T_K$. But this says that

$$z + u_m \in T_K \text{ for } m \in \mathbb{M},$$

as required. □

Corollary. *The Trapping Families Theorem holds.*

Proof. Let h be measurable or Baire slowly varying. Let $\mathbf{x} = \{x_n\}$ be a fixed sequence tending to infinity and let $\varepsilon > 0$ be fixed.

By assumption of slow variation, we have

$$[-1,1] = \bigcup_k I_k, \text{ where } I_k = [-1,1] \cap \bigcup_k T_k^\varepsilon(\mathbf{x})$$

and

$$T_k^\varepsilon(\mathbf{x}) = \bigcap_{n=k}^{\infty} \{y : |h(y + x_n) - h(x_n)| < \varepsilon\}.$$

The corollary is now immediate, as the sets $T_k := T_k^\varepsilon(\mathbf{x})$ are, by assumption, measurable [Baire]. □

Comment. A forcing argument due to A. Miller (quoted in Section 5) shows why there is duality present here between measure and category; his proof tells us that the amount by which the subsequence needs to be translated is 'generic' in nature.

4.4 Proof of the Bounded Equivalence Principle

First we note that (i) implies (ii). Suppose otherwise. Then for some $\varepsilon > 0$, some $x_n \to \infty$, and some bounded $\{u_n\}$ we have

$$|h(x_n + u_n) - h(x_n)| \geq \varepsilon.$$

Passing to a subsequence we may now assume that u_n is convergent with limit u. But now the inequality contradicts the assertion of uniform boundedness over the compact set $\{u_n : n = 0, 1, 2, ..\}$.

Clearly (iii) is a special case of (ii).

Finally, we must show that (iii) implies the UCT.

Suppose otherwise. Then, for the slowly varying function h, there are $\varepsilon > 0$, some convergent u_n with limit u and some $y_n \to \infty$ such that

$$|h(y_n + u_n) - h(y_n)| \geq \varepsilon. \tag{8}$$

Write $z_n = u_n - u$. Now $h(y_n + u) - h(y_n) \to 0$ (convergence at u); setting $x_n := u + y_n$ (so that $x_n \to \infty$) we have $x_n + z_n = u_n + y_n$ and thus we may apply (iii) to the sequences $x_n$and z_n to deduce that

$$|h(y_n + u_n) - h(y_n)| = |h(x_n + z_n) - h(x_n)| + |h(y_n + u) - h(y_n)| \to 0,$$

contradicting (8).

4.5 Proof of the Equivalence Theorem

In what follows if we assert that a combinatorial principle holds, then it is to be understood implicitly that it holds for all $\varepsilon > 0$.

(a) The equivalence of (i) and (vi) is the substance of our Main Theorem UCT.

(b) We prove that (i) implies (ii). This is the hardest part of the proof. All the other steps are either simple, or in just one case a nearly verbatim repetition of the current step with $\mathbf{x}$ replaced by $\mathbf{m}$.

Suppose that (3) fails. Then for some $\eta > 0$

$$|h(u_n + x_n) - h(x_n)| \geq \eta, \tag{9}$$

for a subsequence $\mathbb{M}^0 \subset \mathbb{N}$ of $n's$. As $u = \{u_n\}$ is a bounded sequence, by passing to a subsequence $\mathbb{M} \subset \mathbb{M}^0$, we may suppose that $\{u_m\}$ converges for $m \in \mathbb{M}$, to u say.

We begin by establishing that, for the subsequence of $\{u_m\}$ convergent to u, we have

$$\lim_{m \in \mathbb{M}} (h(u + x_m) - h(u_m + x_m)) = 0,$$

where the limit is taken down the subsequence $\mathbb{M}$. More precisely, we show that, with $\varepsilon = \eta/3 > 0$, there is $N = N(u)$ such that if $n > N$ and $n \in \mathbb{M}$, then

$$|h(u + x_n) - h(u_n + x_n)| < 2\varepsilon.$$

Define

$$y_n = u + x_n,$$

which tends to infinity. By the sequence trapping hypothesis, there are t, n and $\mathbb{M}_1 \subset \mathbb{M}$ such that

$$u_m - u + t \in T_n^\varepsilon(\mathbf{y}),$$

provided $m \in \mathbb{M}_1$. Let $M_1 = \min \mathbb{M}_1$. Since h is slowly varying, we have

$$\lim_{n\to\infty} |h(t + y_n) - h(y_n)| = 0.$$

That is, transcribing the result, there is M_2 such that, for $n \geq M_2$, we have

$$|h(t + u_m + x_n) - h(u + x_n)| < \varepsilon. \tag{10}$$

Finally, since h is slowly varying, we also have

$$\lim_{n\to\infty} |h(u + x_n) - h(x_n)| = 0,$$

so there is M_3 such that, for $n \geq M_3$, we have

$$|h(u + x_n) - h(x_n)| < \varepsilon. \tag{11}$$

Consider now any $k > N(u) = \max\{M_1, M_2, M_3, n\}$ with $k \in \mathbb{M}_1$. We have, since $k > n$, that

$$u_k - u + t \in T_n^\varepsilon(\mathbf{y}) \subseteq H_k^\varepsilon(\mathbf{y}).$$

Put $v = u_k - u + t$. Then

$$|h(v + y_k) - h(y_k)| < \varepsilon.$$

Substituting in this last inequality for v and for y_k, we obtain

$$|h((u_k - u + t) + (u + x_k)) - h(u + x_k)| < \varepsilon,$$

i.e.

$$|h(t + u_k + x_k) - h(u + x_k)| < \varepsilon. \tag{12}$$

Combining (10) and (12) we obtain

$$\begin{aligned}|h(u+x_k)-h(u_k+x_k)| &\leq |h(t+u_k+x_k)-h(u_k+x_k)|+|h(t+u_k+x_k)-h(u+x_k)| \\ &< 2\varepsilon.\end{aligned}$$

Finally, referring to (11), we obtain

$$\begin{aligned}|h(x_k)-h(u_k+x_k)| &\leq |h(u+x_k)-h(u_k+x_k)|+|h(u+x_k)-h(x_k)| \\ &< 2\varepsilon+\varepsilon=3\varepsilon.\end{aligned}$$

This contradicts (9). □ (b)

(c) The assertion (ii)* is a restatement of (ii). Indeed, (3) implies that, for every $\varepsilon > 0$, there is k such that $u_n \in H(x_n)$, for every $n > k$; hence $\{u_m : m > k\} \subseteq T_k^\varepsilon(\mathbf{x})$ from the definition of $T_k^\varepsilon(\mathbf{x})$. So (4) follows from (2). For the reverse direction note that (4) implies that $u_n \in H(x_n)$, for every $n > k$. □ (c)

(d) Since (ii)* asserts that $\mathbf{u}$ is trapped without any need for translation, we have a fortiori (i). □ (d)

(e) We show that (ii) and (iii) are equivalent. Clearly (ii) implies (iii). To see that (iii) implies (ii) write $x_n = m_n + v_n$, where $m_n \in \omega$ and $0 < v_n < 1$ and $w_n = u_n + v_n$, then we have

$$\begin{aligned}h(x_n+u_n)-h(x_n) &= [h(m_n+u_n+v_n)-h(m_n)]-[h(m_n+v_n)-h(m_n)] \\ &= [h(m_n+w_n)-h(m_n)]-[h(m_n+v_n)-h(m_n)] \\ &\to 0-0=0,\end{aligned}$$

in view of (iii). □ (e)

(f) We now proceed by analogy and prove that (iii) is equivalent to (iv). Indeed (b) with $\mathbf{x}$ replaced by $\mathbf{m}$ proves that (iv) implies (iii). Now (iii) is equivalent to the following (just as (ii) and (ii)* were):

(iii)* *For any integer sequence* $\mathbf{m}$ *tending to infinity, and any positive* ε*, the family* $\{T_n^\varepsilon(\mathbf{m}) : n \in \omega\}$ ***ultimately contains*** *almost all of any bounded sequence* $\{u_n\}$*.*

That is, for any bounded sequence $\{u_n\}$, there is k such that

$$\{u_m : m > k\} \subseteq T_n^\varepsilon(\mathbf{m}), \text{ for all } n > k,$$

so a fortiori $\mathbf{2}\text{-}\mathbf{NT}_h(\{T_k^\varepsilon(\mathbf{m}) : k \in \omega\})$ holds for all $\mathbf{m}$. $\square$ (f)

(g) Clearly if $\mathbf{2}\text{-}\mathbf{NT}_h(\{T_k^\varepsilon(\mathbf{m}) : k \in \omega\})$ holds for all $\mathbf{m}$, then in particular $\mathbf{3}\text{-}\mathbf{NT}_h(\{T_k^\varepsilon(\mathbf{id}) : k \in \omega\})$ holds. Noting that

$$\bigcap_{n=m_k}^{\infty} H^\varepsilon(n) \subseteq \bigcap_{n=k}^{\infty} H^\varepsilon(m_n),$$

we see that if $\mathbf{3}\text{-}\mathbf{NT}_h(\{T_k^\varepsilon(\mathbf{id}) : k \in \omega\})$ holds, then $\mathbf{2}\text{-}\mathbf{NT}_h(\{T_k^\varepsilon(\mathbf{m}) : k \in \omega\})$ holds for all $\mathbf{m}$. $\square$ (g)

Comment. If (3) holds for $\{u_n\}$ any bounded sequence, and $\{x_n\}$ any real sequence tending to infinity, then one can prove directly that UCT holds for h by repeating the proof step given in BGT p. 8. Clearly the property (3) follows from UCT.

4.6 Proof of the Luzin set proposition

In the Luzin [resp. Sierpiński] case, let $\{N_\alpha : \alpha < \omega_1\}$ list all closed nowhere dense sets in $\mathbb{R}$ [all the $\mathcal{G}_\delta$-sets of measure zero] and let $\{\{u_\alpha^n\} : \alpha < \omega_1\}$ list all sequences. We construct, by transfinite induction, points t_α for $\alpha < \omega_1$ so that the sets $T_\alpha = \{t_\beta : \beta \leq \alpha\}$ avoid certain forbidden sets. The forbidden sets will have union a first category set [be a set of measure zero] and so it will be possible to select the next point in the transfinite induction.

For more clarity we give the construction in two parts.

First part. Here we neglect the Hamel basis property; we modify the construction to accommodate this in the second part.

To secure the Luzin [Sierpiński] property, we aim to have

$$T_{\omega_1} \cap \bigcup_{\delta<\beta} N_\delta \subset T_\beta,$$

for $\beta < \omega_1$, as then $T = T_{\omega_1}$ meets any N_δ in at most a countable set. This can be arranged in the induction by ensuring that for $\alpha < \omega_1$we have for all $\beta < \alpha$ that

$$T_\alpha \cap \bigcup_{\delta<\beta} N_\delta \subset T_\beta. \tag{13}$$

We also require that the difference set of each T_α avoids $\mathbb{Q}$. Thus $T = T_{\omega_1}$ is the required Luzin set and $T - T$ avoids $\mathbb{Q}$, which implies that $T - T$ has empty interior.

Actually, it is more convenient to carry out the induction over limit ordinals. Suppose that T_α has been defined with α a limit ordinal, so that (13) holds, and

$$T_\alpha - T_\alpha \cap \mathbb{Q} = \emptyset.$$

We intend to select t so that the translates $t + u_\alpha^n$ shall all be included in $T_{\alpha+\omega}$, that is, so that $T_{\alpha+\omega} = T_\alpha \cup \{t + u_\alpha^n : n \in \omega\}$.

Consider our requirements. For the Luzin [Sierpiński] property at $\alpha + \omega$ in place of α in (13), we require:

$$t + u_\alpha^n \notin \bigcup_{\delta<\alpha} N_\delta \text{ i.e. } t \notin \bigcup_{\delta<\alpha} (N_\delta - u_\alpha^n).$$

For the forbidden differences to occur we require that for $\beta < \alpha$ we have

$$\pm(t + u_\alpha^n - t_\beta) \notin \mathbb{Q} \text{ i.e. } t \notin (\mathbb{Q} + t_\beta - u_\alpha^n).$$

Thus t must be selected to avoid the first category set [the measure zero set]

$$C = \bigcup_{\beta<\alpha} \bigcup_{n\in\omega} \left[\bigcup_{\delta<\alpha} (N_\delta - u_\alpha^n) \cup (\mathbb{Q} + t_\beta - u_\alpha^n) \right].$$

Note that it is not possible to arrange that the vectors in $T_\alpha \cup \{ t + u_\alpha^n : n \in \omega\}$ do not introduce linear dependencies over $\mathbb{Q}$. For instance if the sequence $\mathbf{u}_\alpha = \{u^n\}$ is such that

$$u^{n+1} \in \text{conv}_{\mathbb{Q}}\{u^1, .., u^n\},$$

then for any t we have

$$t + u^{n+1} \in \text{conv}_{\mathbb{Q}}\{t + u^1, .., t + u^n\}$$

and we introduce linear dependencies (over $\mathbb{Q}$). The best that we can achieve is to include a Hamel basis in our Luzin [Sierpiński] set.

Second part. Here we show how to modify the construction in the first part so as to ensure that the set T contains a Hamel basis. We mimic an idea due to Erdős (see [33] p. 267). Let $\{x_\alpha : \alpha < \omega_1\}$ list all real numbers. We assume, as before, that T_α has been defined inductively with the properties identified before and in addition the property that: for $\delta < \alpha$ the points x_δ are represented as rational convex combinations of members of T_α.

We suppose at stage α that x_α is not a rational convex combination of members of T_α. We need to include in the construction of $T_{\alpha+\omega}\backslash T_\alpha$ two real numbers u, v such that x_α will be represented as

$$x_\alpha = u + v.$$

We thus require that

$$\begin{aligned}
\{u, v\} &\notin \bigcup_{\delta<\alpha} N_\delta, \text{ i.e. } u \notin \bigcup_{\delta<\alpha} N_\delta \text{ and } u \notin \bigcup_{\delta<\alpha} x_\alpha - N_\delta, \\
\pm(u - v) &\notin \mathbb{Q}, \text{ i.e. } 2u \notin \mathbb{Q} + x_\alpha, \text{ and also } 2u \notin \mathbb{Q} - x_\alpha, \\
\pm(u - t_\beta) &\notin \mathbb{Q}, \text{ i.e. } u \notin \mathbb{Q} + t_\beta, \text{ and also } u \notin \mathbb{Q} - t_\beta, \\
\pm(v - t_\beta) &\notin \mathbb{Q}, \text{ i.e. } u \notin x_\alpha - t_\beta + \mathbb{Q}, \text{ and also } u \notin \mathbb{Q} + t_\beta - x_\alpha.
\end{aligned}$$

Again such a choice of u is clearly possible. We put $t_\alpha = u, t_{\alpha+1} = x_\alpha - u, t_{\alpha+n+2} = t + u_\alpha^n$ with t selected as earlier but with $T_{\alpha+2}$ replacing T_α. Evidently, this ensures that x_α is represented, that $T - T$ contains no intervals, and T meets every nowhere dense set in at most a countable set. □

Comment. In the absence of the assumption of (CH) the argument may be modified to give a set of reals of power continuum such that the set
(i) contains no non-empty perfect subset (so has inner measure zero),
(ii) has difference set with empty interior,
(iii) contains all sequences up to translation, and
(iv) contains a Hamel basis.

5 Complements

This section is devoted to some open problems, thoughts on directions of generalization, and comments to the main material which would have been out of place elsewhere.

Beyond the real line. The theory as presented here is, to quote the preface of BGT, 'essentially a chapter in real variable theory'. We mention here the availability of a well-developed theory going beyond the real line, for which see [19]. We raise the possibility of extending the theory of regular variation in this direction.

No Trumps. The term No Trumps in Definition 2, a combinatorial principle, is used in close analogy with earlier combinatorial principles, in particular Jensen's Diamond $\diamondsuit$ [27] and Ostaszewski's Club $\clubsuit$ [40] and its weakening in another direction: 'Stick' in [25]. The argument in the proof of the No Trumps Theorem is implicit in [18] and explicit in [6], p.482 and [8], p.9. The intuition behind our formulation may be gleaned from forcing arguments in [36], [37], [38].

Effective versions of the trapping property. Are there 'effective' versions (see [39] Chapter 3) of the Existence Theorem (for trapping families, cf. Section 1)? Here we refer to the light-face versions of the bold-face projective classes introduced in Section 2, so that the hyper-arithmetic sets are effective versions of the Borel sets. For example, what may be said about a Σ^1_1 set trapping by translation a hyperarithmetic sequence?

The $\mathbf{NT}_{\mathbf{\Gamma}}$ *property.* Let $\mathbf{NT}_{\mathbf{\Gamma}}$ be the statement that **3-NT**$_h$ holds, i.e. $(\forall\varepsilon > 0)\mathbf{NT}(\{T^{\varepsilon}_k(\mathbf{id}) : k \in \omega\})$, for all functions h of a class $\mathbf{\Gamma}$. The statement holds in the models of Solovay [44] and of Shelah [43] for any $\mathbf{\Gamma}$. One natural candidate is the ambiguous class of the second level in the projective hierarchy, the class $\mathbf{\Delta}^1_2$ (see [29] for a definition in terms of universal and existential quantifiers of type 1). This, as we argue in the companion paper [15], is a natural class for analysts to work in, whenever the lim sup operation is in use. We know that the class of models of (PD) with $\mathbf{\Gamma} = \mathbf{\Delta}^1_2$ satisfies $\mathbf{NT}_\Gamma$. What other classes of models of (ZF) and classes $\mathbf{\Gamma}$ have this property?

Similar sequences: generic arguments. One can see that a non-meagre set A with the Baire property traps sequences by an amendment of a forcing argument given by Miller in [36]. Let $\{u_n\}$ be a convergent sequence with limit u. Specifically, suppose that A is co-meagre in the interval (a, b). Choose $\varepsilon > 0$ and a rational q so that $a + \varepsilon < q < b - \varepsilon$. Thus for some N we have that $a + \varepsilon < q + (u_n - u) < b - \varepsilon$ for all $n > N$. Let $x \in (-\varepsilon, \varepsilon)$ be a Cohen real. Then for every $n \in \omega$, the number $q + (u_n - u) + x$ is a Cohen real. Since $a < q + (u_n - u) + x < b$ we deduce that for $n > N$ we have $q + x - u + u_n \in A$. Thus a translate of almost all of the sequence $\{u_n\}$ is in A. A similar argument may be given replacing 'Cohen real' by (Solovay) 'random real' to show that a translate of almost all of any sequence $\{u_n\}$

is contained in a measurable set A of positive measure. This pin-points the 'generic' nature of the arguments in Section 4.3.

Non-duality between measure and category. We have been lucky in the Existence Theorem (for trapping families) in that the measure/category analogy holds. See [21], [2], [3] for its limitations.

Continuum Hypothesis. In elucidating the sequence trapping property we restricted ourselves to the simplest context, that of assuming CH. We draw the reader's attention to two alternative hypotheses: Martin's Axiom (see [22]) and the Covering Property Axiom CPA (see [17]). We note also that the example of Section 4.6, derived under the continuum hypothesis, may be derived to be in the class $\mathbf{\Delta}_2^1$ (see above) when making the stronger assumption of Gödel's Axiom $V = L$, cf. [20].

Multi-dimensional regular variation. As mentioned earlier, the theory in BGT deals with regular variation in one dimension. In recent years, much effort has been devoted to extensions of this theory to many dimensions, including infinitely many dimensions. Since the motivation is mainly probabilistic, we give the probabilistic formulation:

$$nP(\mathbb{X}/a_n \in \cdot) \to \mu(\cdot),$$

where $\mathbb{X}$ is a random vector (possibly infinite dimensional), a_n is a sequence and μ is a measure. For background here, see e.g. [26]. See e.g. [14] for a development along these lines.

Postscript.

This paper is, for the first author, a return to the foundational first sections of BGT with the benefit of twenty-one years' worth of hindsight – or, in the case of [6], [7], twenty-six. It may be regarded as 'the missing zeroth chapter' of BGT. For a similar return to the motivating last chapter of BGT, on probability theory, see [5].

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