A NEW LOOK AT REGULAR VARIATION

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See BOst1-11, CDAM website and Adam's home page

§1. An old look at regular variation

The theory of regular variation, or of regularly varying functions, is a chapter in the classical theory of functions of a real variable, dating from the work of Karamata in 1930. It has found extensive use in probability theory, analysis (particularly Tauberian theory and complex analysis), number theory and other areas; see [BGT] for a monograph treatment and [Kor] IV for Tauberian theorems. It explores the consequences of a relationship of the form

$$f(\lambda x)/f(x) \to g(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0. \tag{RV}$$

The limit function g must satisfy the Cauchy functional equation

$$g(\lambda\mu) = g(\lambda)g(\mu) \qquad \forall \lambda, \mu > 0. \tag{CFE}$$

Subject to a mild regularity condition, (CFE) forces g to be a power:

$$g(\lambda) = \lambda^{\rho} \qquad \forall \lambda > 0. \tag{ρ}$$

Then f is said to be *regularly varying* with *index* ρ, written $f \in R_\rho$. The case $\rho = 0$ is basic. A function $f \in R_0$ is called *slowly varying*; slowly varying functions are often written ℓ (for *lente*, or *langsam*). Recall:
f is (Lebesgue) measurable iff inverse images $f^{-1}(U)$ of open sets U are Lebesgue measurable,
f is Baire (has the Baire property) iff inverse images $f^{-1}(U)$ of open sets U have the Baire property, i.e. are a symmetric difference $G \Delta Q$ with G open and Q meagre (of first category – 'small') – i.e., 'nearly open'.
The basic theorem of the subject is the Uniform Convergence Theorem (UCT), which states that if

$$\ell(\lambda x)/\ell(x) \to 1 \qquad (x \to \infty) \qquad \forall \lambda > 0, \tag{SV}$$

then the convergence is *uniform* on compact λ-sets in $(0, \infty)$.

The basic facts are:

(i) if ℓ is (Lebesgue) measurable, then the UCT holds;

(ii) if ℓ is Baire, then the UCT holds;

(iii) in general, the UCT need not hold.

Similarly, if f is measurable or Baire, (CFE) implies (ρ), but not in general.

The basic foundational question in the subject, which we address here, concerns the search for natural conditions for the above to hold, and in particular for a substantial common generalization of measurability and the Baire property. We find two such common generalizations, both necessary and sufficient.

While regular variation is usually used in the multiplicative formulation above, for proofs in the subject it is usually more convenient to use an additive formulation. Writing $h(x) := \log f(e^x)$ (or $\log \ell(e^x)$ as the case may be), $k(u) := \log g(e^u)$, the relations above become

$$h(x+u) - h(x) \to k(u) \qquad (x \to \infty) \qquad \forall u \in \Re, \qquad (RV_+)$$

$$h(x+u) - h(x) \to 0 \qquad (x \to \infty) \qquad \forall u \in \Re, \qquad (SV_+)$$

$$k(u+v) = k(u) + k(v) \qquad \forall u, v \in \Re. \qquad (CFE_+)$$

Then the questions become:

(i) When does (SV_+) hold *uniformly* on compact u-sets?,

(ii) When is $k(u) = \rho u$ in (CFE_+)?

§2. *Why regular variation?*

Tauberian theory: BGT, Ch. 4.
For $\rho > 0$, $\int_0^\infty e^{-sx} d(x^\rho) = \rho \int_0^\infty e^{-sx} x^{\rho-1} dx = \rho\Gamma(\rho)/s^\rho = \Gamma(1+\rho)/s^\rho$: the Laplace-Stieltjes transform of a power x^ρ is $\Gamma(1+\rho)/s^\rho$. The Hardy-Littlewood-Karamata theorem (H&L 1914, Karamata 1931) extends this from powers to regularly varying functions, both ways.
Limit theorems in probability theory: BGT, Ch. 8. Recall two basic results:
The *Weak Law of Large Numbers* (WLLN): if $X_1, X_2, \ldots, X_n, \ldots$ are independent and identically distributed (iid) random variable, with partial sums $S_n := \sum_1^n X_k$ and mean μ,

$$S_n/n \to \mu \qquad (n \to \infty) \qquad \text{in probability.}$$

The *Central Limit Theorem* (CLT): if X_n are iid with mean μ and variance σ^2, $\Phi(x) := (1/\sqrt{2\pi}) \int_{-\infty}^{x} \exp\{-\frac{1}{2}y^2\} dy$,

$$(S_n - n\mu)/\sqrt{n}\sigma \to \Phi = N(0,1) \qquad (n \to \infty) \qquad \text{in distribution}$$

($\Phi = N(0,1)$ is the standard normal distribution function).

How far can we generalize, beyond existence of the mean and variance? If F is the distribution function of the X_n, both answers involve regular variation:

WLLN: The NASC for S_n/a_n to converge to a (non-zero) constant in probability is that the *truncated mean* $\int_{-x}^{x} y dF(y)$ be slowly varying;

CLT: The NASC for $(S_n - a_n)/b_n$ to converge in distribution to a (non-degenerate) limit is that the *truncated variance* $\int_{-x}^{x} y^2 dF(y)$ be slowly varying.

Complex Analysis: Levin-Pfluger theory, BGT Ch. 7.

Analytic Number Theory: BGT Ch. 6.

3. Proofs of UCT.

BGT Ch. 1 contains *six*:

First proof (Delange, 1955): Direct, uses quantitative measure theory.

Second proof (Charles Goldie, for BGT): Direct, uses qualitative measure theory (whether measure is zero or positive is all that matters), translates to Baire (changing 'null set' to 'meagre set').

Third proof (Matuszewska 1965): Indirect (by contradiction), qualitative, measure/Baire.

Fourth proof (Csiszár & Erdös, 1964): Ditto.

Fifth proof (Elliott, 1979-80): Indirect, quantitative measure theory, Egorov's theorem.

Sixth proof (Trautner, 1987: BGT 2nd ed., 1989): Indirect, qualitative, 'covering principle'.

Seventh proof (BOst1): Indirect, qualitative, measure/Baire. Uses an infinite combinatorial principle, No Trumps (NT – named by analogy with Jensen's Diamond $\diamond$ and Ostaszewski's Club $\clubsuit$).

Eighth proof (BOst1): Indirect, qualitative, measure/Baire. Uses Bounded Equivalence Principle: the following are equivalent:

1. UCT, i.e. $h(x+u) - h(x) \to 0$ $(x \to \infty)$ uniformly on compact u-sets;
2. $h(x_n + u_n) - h(x_n) \to 0$ $(n \to \infty)$ for all sequences $x_n \to \infty$ and bounded sequences u_n;
3. h slowly varying and satisfies NT (for which h measurable or h Baire suffices).

4. Density Topology

A point x is called a *density point* of a set A if $|A \cap (x - \delta, x + \delta)|/2\delta \to 1$ as $\delta \to 0$ ($|.|$ is Lebesgue measure). Recall the *Lebesgue Density Theorem*: almost all points of a measurable set A are density points of A. Call a set U *open* if all its points are density points. These open sets do define a topology, the *density topology* or d-topology (Goffman, 1950). That it links measurability and the Baire property is shown by the following result (Kechris, 1995): A is Lebesgue measurable iff it has the Baire property under the density topology.

5. Kestelman-Borwein-Ditor Theorem (KBD Theorem)

THEOREM (KBD: Kestelman 1947, Borwein & Ditor 1978, Trautner 1987). If $z_n \to 0$, T is measurable and non-null/Baire and non-meagre, then for all $t \in T$ off a null/meagre set, there is an infinite set M_t such that

$$\{t + z_m : m \in M_t\} \subset T.$$

This result is an infinite combinatorial principle, playing a role analogous to that of NT.

Ninth proof of UCT (BOst11): Topological KBD Theorem and density topology/Euclidean topology – indirect, qualitative, measure/Baire. Asserts primacy of the Baire approach over the measure approach.

6. BOst so far

BOst1: Foundations of regular variation

BOst2: Very slowly varying functions II

BOst3: Beyond the theorems of Steinhaus and Ostrowski: combinatorial versions

BOst4: Beyond Lebesgue and Baire: generic regular variation

BOst5: Generic subadditive functions

BOst6: New automatic properties: subadditivity, convexity, uniformity

BOst7: Analytic automaticity: the theorems of Jones and Kominek

BOst8: The converse Ostrowski theorem

BOst9: Genericity and the Kestelman-Borwein-Ditor theorem

BOst10: Homotopy and the Kestelman-Borwein-Ditor theorem

BOst11: Duality and the Kestelman-Borwein-Ditor theorem

BOst12: Topological regular variation and group actions.

7. Overview

BGT stands the test of time (20 years now) fairly well, except for:
(a) foundations – what are the *right* conditions (we know now, from BOst),
(b) higher dimensions – important in probability, e.g. for portfolio analysis in mathematical finance.
Re (a): the BOst papers need integrating, into a BOst book.
Re (b): We have made a start! Quite a lot of this extends to d-dimensions. But there is much still to do!

8 (if time allows): Descriptive set theory

Call a subset A of a Polish space X *analytic* if it is of the form $f(Y)$ for some continuous $f : Y \to X$ with Y Polish. Call A *co-analytic* if its complement is analytic. The classes of analytic and co-analytic sets, and their intersection, are written $\mathbf{\Sigma}_1^1$, $\mathbf{\Pi}_1^1$, $\mathbf{\Delta}_1^1$ ($\mathbf{\Delta}_1^1$ is the class B of Borel sets, by Suslin's theorem). Then $\mathbf{\Sigma}_{n+1}^1$ is the class of projections (onto the first coordinate) of a two-variable $\mathbf{\Sigma}_n^1$, $\mathbf{\Pi}_{n+1}^1$ their complements, $\mathbf{\Delta}_{n+1}^1$ the intersection (i.e., the ambiguous class of level n), etc., thus defining the *projective hierarchy*. In BOst1, we advocate $\mathbf{\Delta}_2^1$ – the ambiguous class of second level – as the appropriate setting for regular variation.

Studying the foundations of regular variation forces one to do things that analysts generally avoid, e.g.:
(i) examining one's axioms (ZFC – the usual Zermelo-Fraenkel set theory plus the Axiom of Choice – is only one possibility);
(ii) delving into the structure of the real number system – e.g., by regarding it as an infinite-dimensional vector space over the rationals, taking a Hamel basis (using the Axiom of Choice! – or Zorn's Lemma), and looking at its structure via descriptive set theory.
NHB, 1.11.07