Dividend Policy Irrelevance: Ohlson's Uniqueness Principle in several variables

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Abstract

We extend to a multi-dimensional accounting variable x_t Ohlson's result that the equity valuation

$$P_0(R;d) = \sum_{t=1}^{\infty} R^{-t} d_t.$$

defined by the recurrence system

$$\left.\begin{array}{rcllll} x_{t+1} & = & \omega_{11}x_t & +\,\omega_{12}d_t & +\,\omega_{13}v_t, \\ d_{t+1} & = & \omega_{21}x_t & +\,\omega_{22}d_t & +\,\omega_{23}v_t, \\ v_{t+1} & = & & & +\,\omega_{33}v_t, \end{array}\right\}$$

and the given initial data vector (x_0, d_0, v_0), is independent of $\omega = (\omega_{21}, \omega_{22}, \omega_{23})$ iff $R = \omega_{11}$.

Subject to some mild regularity assumptions, an Identification Theorem describes for which recurrence systems Dividend Policy Irrelevance holds at R iff $R = \omega_{11}$. Up to similarity there are necessarily two extension to a two-dimensional accounting variable case. One is the system

$$\begin{array}{rllll} x_{t+1} = & \omega_{11}x_t & +\,\omega_{12}y_t & +\,\omega_{13}d_t & +\,\omega_{14}v_t, \\ y_{t+1} = & & \omega_{22}y_t, & & \\ d_{t+1} = & \omega_{31}x_t & +\,\omega_{32}y_t & +\,\omega_{33}d_t & +\,\omega_{34}v_t, \\ v_{t+1} = & & & & \omega_{44}v_t. \end{array}$$

with a regularity assumption which includes $\omega_{13} \cdot \omega_{31} \neq 0$ analogously to Ohlson's condition in the one variable case. But it is really the case of a three-dimensional accounting variable that exhibits the full story.

Subject to the regularity assumptions, it is shown that if dividend policy irrelevance holds at R then R is necessarily an eigenvalue. Subject to the regularity assumptions, a characterization of dividend policy irrelevance at R is deduced in the class of all systems having R as eigenvalue of the reduced system matrix (reduced by the exclusion of the hopefully irrelevant entries). Key to this is a factorization condition placed on the characteristic polynomial of the reduced matrix. A corollary to Theorem 4 in Section 1.3 offers the characterization in the $n = 2$ case.

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1 Introduction

1.1 Ohlson's Fundamental Uniqueness Principle

A fundamental insight in Accounting Theory since Preinreich (1936) is that valuation of a firm based on the discounted future stream of dividends d_t (for $t = 1, ...$) can be equivalently stated (after adjustment by the firm's current accounting book value B_0) in terms of the discounted future stream of an appropriately defined, earnings performance measure, known as 'residual income'. That is, there is an accounting identity connecting the value of the two income streams. At its simplest, the notion of residual income may be defined as earnings e_t in the year t less the notional opportunity cost, represented by the return which that firm's investments, recorded as end of previous year accounting book value B_{t-1}, would have earned had they been invested across the whole of its industrial sector; that is, residual income for the year t is $e_t - rB_{t-1}$where r is the average rate of return in the sector. In a stochastic environment the residual income is deemed to have expected value zero under a risk-neutral probability measure; in a deterministic analogue one might demand only that residual income tends to zero in the long run.

In fact this particular measure of earnings performance is by no means the only one to offer equivalence (= identity, after adjustment) with the dividend stream. An over-arching principle was recently uncovered by Ohlson characterizing a whole class of measures admitting such equivalence, as being those which are modelled in a particular, unique way, by means of the linear recurrence detailed below. To be specific the following theorem, announced in Ohlson (2003), was shown by Ohlson to lie at the heart of a great variety of accounting-theoretic valuation equations. See also Gietzmann (2004).

Define the equity valuation function generally by

$$P_0(R; d) = \sum_{t=1}^{\infty} R^{-t} d_t, \tag{1}$$

where the infinite vector of coefficients $d = (d_1, ..., d_t, ...)$ is generated by the linear system

$$\left.\begin{array}{rcllll} x_{t+1} & = & \omega_{11}x_t & + \omega_{12}d_t & + \omega_{13}v_t, \\ d_{t+1} & = & \omega_{21}x_t & + \omega_{22}d_t & + \omega_{23}v_t, \\ v_{t+1} & = & & & + \omega_{33}v_t, \end{array}\right\} \tag{2}$$

and by a given initial data vector (x_0, d_0, v_0).

It is interesting to ask whether the equity valuation, as modelled here, may depend solely on the 'current observations' (x_0, d_0, v_0) and on the knowledge of how the measures of performance x_t and v_t are defined, but without reference to the 'dividend policy', here interpreted as a particular choice of values for the coefficients $(\omega_{21}, \omega_{22}, \omega_{23})$.

Ohlson's Theorem asserts that if a 'non-spuriousness' condition holds, namely

$$\omega_{12} \cdot \omega_{21} \neq 0 \tag{3}$$

(requiring a mutual dependence of both variables x_t and d_t), then the 'equity' function $P_0(.,d)$, when evaluated at R, is independent of $\omega = (\omega_{21}, \omega_{22}, \omega_{23})$ iff $R = \omega_{11}$. That is, there is a unique value of R for which dividend policy is irrelevant to the value of equity.

Remarks. Note that in the third of the equations (2) it is usual to assume

$$0 \leq \omega_{33} < 1,$$

so that, in the long run, the variable v_t tends to zero, i.e.

$$\lim_{t \to \infty} v_t = 0, \tag{4}$$

which entirely places the focus on the relationship between x_t and d_t.

The significance of the equity valuation equation comes from interpreting ω_{11} as $R = 1 + r$, with r a constant rate of return, and d_t as future payments. The series of discounted future payments in (1) is then in turn interpreted as the current market value of a firm that makes these payments to its shareholders. The variable x_t of the first equation on the other hand may be given very wide-ranging interpretations in terms of accounting numbers, depending on the value attributed to ω_{12} and on the meaning attached to the variable v_t, always provided the long run condition (4) is satisfied. Once such an interpretation is selected for the first equation, the second and third equations describe a model for the evolution of future payments.

The motivation for a linear first-order dynamic comes from a desire to estimate year on year earnings relations from accounting data (and so to forecast them) by means of linear regression.

We note that if $R = \omega_{11}$, it follows that the equity valuation $P_0(R; d)$ is a linear combination of the initial data x_0, d_0, v_0 with coefficients independent of ω (see (34) in Section 6).

Actually, some further restrictions on the matrix $\Omega_2 = (\omega_{ij})_{ij \leq 2}$ are required; in particular one needs to restrict the growth rate of d_t (the dividend variable) so that the equity series is convergent. We refer to these as the **implicit assumptions** (see below in section 2.2); apart from expressing these in detail for the $n = 1$ case for clarification, we do not develop any explicit re-formulation of them here.

Examples. For an insight into the workings of Ohlson' Principle we consider three basic examples with $R = 1 + r = \omega_{11}$. We use the notation RIV_t for the residual income variable $e_t - rB_{t-1}$.

Consider first the very simplest of cases. Take $\omega_{12} = -1$ (resp. $\omega_{12} = 1$) and $v_0 = 0$ which obviously satisfies (4). The variable x_t may then be labelled the end of year '**savings account balance**' with d_t denoting end of year withdrawals (resp. deposits) and P_0 the present value of the total withdrawals (resp. savings) process. The second equation then describes a view of the planned intentions of the account-owner, but the theorem tells us that under the implicit assumptions the equity value is independent of the precise details of the plan (of the choice of values for $\omega_{21}, \omega_{22}, \omega_{23}$). Indeed we can see this here directly from the first equation which yields

$$\omega_{12} P = \omega_{12} \lim_{T\to\infty} \sum_{t=0}^{T} R^{-t} d_t = R \lim_{T\to\infty} \left(\frac{x_{T+1}}{R^{T+1}} \right) - \frac{x_0}{R} = -\frac{x_0}{R},$$

provided R exceeds in modulus the two eigenvalues κ_1 and κ_2 of $\Omega_2 = (\omega_{ij})_{ij \le 2}$.Indeed the assumed dynamics imply that for some constants L_1 and L_2

$$x_t = L_1 \kappa_1^t + L_2 \kappa_2^t.$$

Of course varying the details of the plan varies the eigenvalues. But provided the implicit assumption: $R > \max\{|k_1|, |\kappa_2|\}$ is observed P_0 remains independent of the choice of values for $\omega_{21}, \omega_{22}, \omega_{23}$.

As a second example, take $\omega_{12} = -R$ and $\omega_{13} = \omega_{33}$ with $v_t = RIV_t$, a choice justified on the grounds that RIV_t tends to zero in the long run. The variable x_t representing **pre-dividend book value**, i.e. $B_t + d_t$, evidently an earnings performance measure, satisfies the first equation of (2) identically. The third equation is then a standard deterministic model for the evolution of residual income. Here again the theorem asserts that P_0 is independent of the details of the dividend policy – of the values chosen for $\omega_{21}, \omega_{22}, \omega_{23}$.

For a final example take $\omega_{12} = -r$ and $\omega_{13} = \omega_{33}$ with $v_t = RIV_t - RIV_{t-1}$, the increment in Residual Income. The choice for v_t is again justified on the grounds that RIV_t tends to zero in the long run. Here $x_t = e_t$ and it is the **earnings** themselves that satisfy the first equation of (2) identically and again Dividend Policy Irrelevance holds.

Stochastic versions are readily devised by including noisy terms in (2).

The above examples hardly do justice to the wide compass of Ohlson's Principle and the interested reader is referred to Ohlson (2003). The variable x_t should hereafter be regarded as a general accounting variable, which records some aspect of the current earning performance of the firm. Since the variable d_t is interpreted as a dividend payment, the second equation is seen as setting dividend policy. While x_{t+1} is capable of forecast (given the recorded x_t and given d_t), the forthcoming dividend d_{t+1} is being regarded as a hidden (latent) variable. On the other hand the quantity v_t is interpreted as describing a contribution to the future value x_{t+1} arising from some economic value unrecorded by the accounting variable x_t (perhaps thought of as arising from known assets of unproven or partially proven value, which are not currently enabled to generate income but will do so in the future). The third equation is therefore adjoined in order to model the up-dated guessed additional future value v_{t+1} as a patent variable. It is thus plausible for the variable v_t to be taken as decreasing, since the unproven asset may be regarded as gradually generating income that is recorded in x_t. The Principle thus asserts that provided the accounting (observable) variable x_t is appropriately defined, the equity (i) is independent of dividend policy, and (ii) depends only on the available initial data. Thus stated, the Uniqueness Principle is a theorem on the observability of dynamical systems and the reconstruction of the state vector, such as (x_t, d_t), from an 'observation vector' (here the scalar x_t) obtained by projecting out of the state $(x_t, d_t) \to x_t$. See for instance Russell [3].

In response to a question regarding a two-dimensional accounting variable setting, replacing x_t by $(x_t, y_t) \in R^2$, raised privately by Ohlson, this note extends the Ohlson theorem to a multi-dimensional accounting variable $z_t = (z_t^1, ..., z_t^n) \in R^n$.Actually the theorem due to Ohlson is capable of two interpretations according as (a) the rate R, or (b) dividend policy irrelevance, is placed centre stage. The two are of course inter-connected, but each needs a different theorem for the formulation of its generalization.

See Tippett and Warnock (1997) for some related literature.

1.2 A generalized Ohlson dynamic

To state our generalizations of Ohlson's Theorem we begin by defining a more general linear model than (2). For $a, b, w^T \in R^n$ and $\alpha, \beta, \gamma \in R$ consider the system

$$\begin{bmatrix} z_{t+1} \\ d_{t+1} \\ v_{t+1} \end{bmatrix} = \Omega \begin{bmatrix} z_t \\ d_t \\ v_t \end{bmatrix},$$

where Ω is an $(n+2) \times (n+2)$ real matrix and

$$\Omega_{n+2} = (\omega_{ij})_{i,j \le n+2} = \begin{bmatrix} A & b & a \\ w & \beta & \alpha \\ 0 & 0 & \gamma \end{bmatrix},$$

so that

$$\left.\begin{array}{rcllll} z_{t+1} & = & Az_t & + bd_t & + av_t, \\ d_{t+1} & = & wz_t & + \beta d_t & + \alpha v_t, \\ v_{t+1} & = & & & + \gamma v_t. \end{array}\right\} \qquad (\Omega)$$

Here $z_t \in R^n$ and will henceforth be called the **accounting state vector**, while d_t and v_t are reals defined for $t = 0, 1, 2, ...$. We will write $z_t = (z_t^1, ..., z_t^n)$, or when context permits more simply: $z_t = (x_t, ..., y_t)$. A is a real matrix of size $n \times n$, to be called the **reduced matrix** of the system (Ω). As before

$$P_0(R; d) = \sum_{t=1}^{\infty} R^{-t} d_t$$

is the equity series and we study the dependence of $P_0(R; d)$ on

$$\omega_{\text{div}} := (w, \beta, \alpha).$$

We assume the initial conditions as giving z_0 and d_0 so that z_1 is known.

Definition.

We will say that dividend policy irrelevance holds at R if the value of $P_0(R;d)$ does not depend on ω_{div} (i.e. $P_0(R;d)$ is unaltered by a change in ω_{div}).

It is useful to put

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A & b \\ w & \beta \end{bmatrix}, \qquad \bar{z}_t = (z_t, d_t)^T, \\ \bar{w} &= (w,\beta)^T, \qquad \bar{a} = (a,\alpha)^T, \qquad \bar{b} = (b,\beta)^T, \end{aligned}$$

and to call $\bar{A}$ the augmented matrix of size $n+1$ and $\bar{z}_t$ the **augmented state vector**. In this notation the original Ohlson dynamic has $A = \Omega_1 = (\omega_{11})$, $b = (\omega_{12})$, $z_t = x_t$ and

$$\bar{A} = \Omega_2 = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}, \qquad \bar{z}_t = (x_t, d_t)^T.$$

With this notation we may state our main results as follows.

1.3 Main Results: Uniqueness, Identification and Verification

Our first result is subject to some mild regularity assumptions. It assumes that ω_{11} is an eigenvalue of the reduced system and gives sufficient conditions that identify the circumstances under which the generalized system (Ω) obeys dividend policy irrelevance at R iff $R = \omega_{11}$ (uniquely). Theorem 2 below justifies the eigenvalue assumption and provides a converse result, namely that the sufficient conditions are also necessary. Our first result is this.

Theorem 1 (Generalized Uniqueness Principle: Sufficiency)
Assume the 'implicit assumptions' below and assume also that:

$$a_{11} \text{ is a positive eigenvalue of } A. \tag{5}$$

Suppose that $\xi = (\xi_2, ..., \xi_n)$ *is any solution to the in-homogeneous system of* $n-1$ *equations generated from the following identity in* κ*:*

$$g_A(\kappa,\xi) := \begin{vmatrix} 0 & a_{12} & ... & a_{1n} \\ a_{21}-\kappa\xi_2 & a'_{22}-\kappa & ... & a_{2n} \\ ... & & ... & \\ a_{n1}-\kappa\xi_n & a_{n2} & ... & a'_{nn}-\kappa \end{vmatrix} \equiv 0, \tag{6}$$

where $a'_{ii} = a_{ii} - a_{11}$ *and the constant term of the generating function* $g_A(\kappa,\xi)$ *is zero. Suppose further that for any* $b_1 \neq 0$ *we have*

$$b_i = b_1\xi_i, \text{for } i = 2, ...$$

Then dividend policy irrelevance holds at R*, i.e.* $P_0(R,d)$ *depends only on* A *and the initial data but not on* $\omega_{\text{div}} = (w,\beta,\alpha)$*, iff* $R = a_{11}$*.*

The above theorem is not intended to resolve which systems obey dividend policy irrelevance at some value of R. However, a minor modification to its proof provides necessary and sufficient conditions for identifying all systems Ω for which dividend policy irrelevance holds at some R provided R is an eigenvalue of the reduced matrix A. The starting point is the following result.

Theorem 2 (Identification Theorem: Necessity)

Assume the 'implicit assumptions' and that R satisfies the convergence constraint

$$R > \max\{|\kappa_i| : i = 1, ..., n+1\}, \tag{7}$$

where κ_i are the eigenvalues of $\bar{A}$, the augmented matrix of size $n+1$.

Suppose that dividend policy irrelevance holds at R, i.e. *$P_0(R, d)$ depends only on A and the initial data but not on $\omega_{\text{div}} = (w, \beta, \alpha)$. Then R is an eigenvalue of A. If, moreover, $R = a_{11} > 0$, then*

$$b_1 \neq 0,$$

and for some solution $\xi = (\xi_2, ..., \xi_n)$ to the in-homogeneous system of $n-1$ equations generated from the identity (6) it is the case that

$$b_i = b_1 \xi_i, \text{for } i = 2, ... \quad .$$

When interpreting R as a rate of return it would be usual to have this exceed any possible dividend rate, so the condition (7) assumed on R is acceptable.

Remark 1: An extension of Theorems 1 and 2. The 'implicit assumptions' may be weakened as follows to permit A and $\bar{A}$ to share eigenvalues other than a_{11}.

Theorem 3. *Suppose that the convergence constraint (7) holds for $R = a_{11}$. Suppose further that for any common eigenvalue λ of A and $\bar{A}$ it is the case that*

$$\lambda \neq a_{11},$$
$$\mathcal{N}((\lambda I - A)) \subseteq \{0\} \times R^{n-1}, \tag{8}$$

and

$$\lambda \text{ is a simple root of } |(\kappa I - A)_b|. \tag{9}$$

Then:

(i) Dividend policy irrelevance holds at R iff $R = a_{11}$ if b satisfies the hypothesis of Theorem 1.

(ii) If dividend policy irrelevance holds at $R = a_{11}$ then b satisfies the conclusion of Theorem 2.

Here M_b denotes the matrix M with its first column replaced by b. See Section 3.1.

We may deduce from Theorems 1, 2 and 3 that for the $n = 2$ case with $z_t = (x_t, y_t)$, subject to the implicit assumptions, there are only two systems (up to a non-singular transformation of the accounting variables) for which dividend policy independence holds iff $R = \omega_{11}$. The first is the accounting-canonical system:

$$\begin{array}{rllll} x_{t+1} = & \omega_{11}x_t & & +\omega_{13}d_t & +\omega_{14}v_t, \\ y_{t+1} = & \omega_{21}x_t & +\omega_{22}y_t & +\omega_{23}d_t & +\omega_{24}v_t, \\ d_{t+1} = & \omega_{31}x_t & +\omega_{32}y_t & +\omega_{33}d_t & +\omega_{34}v_t, \\ v_{t+1} = & & & & \omega_{44}v_t. \end{array}$$

Note that the regularity conditions include

$$0 \neq \chi_{\bar{A}}(\omega_{11}) = \omega_{13}\{\omega_{21}\omega_{32} - \omega_{31}(\omega_{22} - \omega_{11})\},$$

which in particular calls for $\omega_{13} \neq 0$.

The second is the following, but note that the implicit assumptions fail in regard to this system (since of course ω_{22} is an eigenvalue of both A and $\bar{A}$). Nevertheless it is also obtainable from solving the in-homogeneous system of Theorem 1 but requires Theorem 3 for justification.

$$\begin{array}{rllll} x_{t+1} = & \omega_{11}x_t & +\omega_{12}y_t & +\omega_{13}d_t & +\omega_{14}v_t, \\ y_{t+1} = & & \omega_{22}y_t, & & \\ d_{t+1} = & \omega_{31}x_t & +\omega_{32}y_t & +\omega_{33}d_t & +\omega_{34}v_t, \\ v_{t+1} = & & & & \omega_{44}v_t. \end{array}$$

Note that $\omega_{22} < \omega_{11}$ is required by the regularity assumptions; the eigenvectors of $A - \omega_{22}I$ are generated by $(0,1)^T$ if and only if $\omega_{12} = 0$, and so the system has dividend policy irrelevance at $R = \omega_{11}$ by Theorem 3 (since $|(\kappa I - A)_b|$ is linear and so condition (9) holds automatically). The condition that ω_{11} is not an eigenvalue of $\bar{A}$ asserts that

$$0 \neq \chi_{\bar{A}}(\omega_{11}) = -\omega_{13}\omega_{31}(\omega_{22} - \omega_{11}),$$

so that analogously to Ohlson's condition (3) for the case $n = 1$, here we have $\omega_{13}\omega_{31} \neq 0$. However, this system makes y_t as simple as v_t, so it offers scope for a different role, say for modelling in an uncoupled way some exceptional accounting items (as in the so-called 'dirty accounting' systems) – but note that $\omega_{22} < \omega_{11} = R$. In the Appendix we re-derive the case $n = 2$ by a more direct approach exhibiting both systems.

Remark 2. Dividend policy irrelevance is invariant under similarity transformations of A. Theorem 2 restricts our attention to the class of matrices A which have R as an eigenvalue and offers a necessary and sufficient test for dividend policy irrelevance. Any matrix with R as eigenvalue is similar to one with its first row having the entries $(R, 0, ..., 0)$. This means that an appropriate aggregation w_t of the accounting variables may be found which obeys an equation of the form

$$w_{t+1} = Rw_t + bd_t + av_t.$$

Let us call a system (Ω) accounting-canonical if its first row takes just such a form. (If we regard the similarity transformation as replacing the first accounting variable with the new variable w_t keeping the remaining variables, then the canonical system is 'spurious' in that the first equation makes use of only one of the accounting variables.) To check dividend policy irrelevance for a given matrix A with eigenvalue R it is thus enough to pass to an accounting-canonical similar matrix and apply the test there. By interpreting the test in terms of the original matrix A we may thus obtain not only (i) a characterization of all 'dividend-policy irrelevance' systems for which R is the only possible choice for a_{11}, (from the identification theorem, subject to regularity assumptions embodied in the implicit assumptions), but also (ii) a characterization of all systems equivalent under similarity to a canonical system.

The following theorem makes the direct connection to the similarity equivalent dynamics.

Theorem 4 (Verification Theorem)

Assume the 'implicit assumptions' below and suppose

$$R \text{ is an eigenvalue of } A. \tag{10}$$

Let u be an eigenvector of A to value R. Then dividend policy irrelevance holds at R, i.e. *$P_0(R;d)$ depends only on A and the initial data, but not on $\omega_{\text{div}} = (w,\beta,\alpha)$ iff one of the following cases i =1, ... , n arises.*

Case 1: *It is the case that $u_1 \neq 0$, and*

$$b_1 \neq 0 \text{ and } b_j = b_1(2u_1^{-1}u_j + \xi_j), \text{ for } j \neq 1,$$

for some $\xi = (\xi_2, ..., \xi_n)$ which is a solution to the in-homogeneous system of $n-1$ equations generated from the following identity in κ :

$$\bar{g}_A^1(\kappa,\xi) := \begin{vmatrix} \bar{a}_{11} & a_{12} & ... & a_{1n} \\ a_{21}-\kappa\xi_2 & \bar{a}_{22}-\kappa & ... & a_{2n} \\ ... & & ... & \\ a_{n1}-\kappa\xi_n & a_{n2} & ... & \bar{a}_{nn}-\kappa \end{vmatrix} \equiv 0. \tag{11}$$

Here $\bar{a}_{ii} = a_{ii} - R$ and the constant term of the generating function $\bar{g}_A^1(\kappa,\xi)$ is zero.

Case i :(i>1) *It is the case that $u_1 = u_2 = ... = u_{i-1} = 0$ and $u_i \neq 0$, and*

$$b_i \neq 0 \text{ and } b_j = b_i u_i^{-1}(u_j + \xi_j), \text{for } j \neq i,$$

for some $\xi = (\xi_1, ..., \xi_{i-1}, \xi_{i+1}, \xi_n)$ which is a solution to the in-homogeneous system of $n-1$ equations generated from the following identity in κ :

$$\bar{g}_A^i(\kappa,\xi) := \begin{vmatrix} \bar{a}_{11}-\kappa & a_{12} & ... & a_{1i}-\kappa\xi_1 & a_{1n} \\ ... & ... & ... & ... & ... \\ a_{i1} & a_{12} & ... & \bar{a}_{ii} & ... \\ & & & & \\ a_{n1} & a_{n2} & & a_{ni}-\kappa\xi_n & \bar{a}_{nn}-\kappa \end{vmatrix} \equiv 0. \tag{12}$$

Here again $\bar{a}_{ii} = a_{ii} - R$ and the constant term of the generating function $\bar{g}_A^i(\kappa,\xi)$ is zero.

Corollary For $n = 2$ the reduced matrices A identified by the in-homogeneous system take the form in case 1 of

$$A = \begin{pmatrix} R & 0 \\ f & g \end{pmatrix}, \text{ or } A = \begin{pmatrix} a & 0 \\ f & R \end{pmatrix}, \text{ with } \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix}, \text{ where } c \neq 0,$$

$$\text{or } A = \begin{pmatrix} R + b\xi & b \\ \xi(g - R) & g \end{pmatrix}, \text{ with } \mathbf{b} = c \begin{pmatrix} 1 \\ -\frac{a-R}{b} \end{pmatrix}, \text{ where } b \neq 0 \text{ and } c \neq 0.$$

subject to the 'implicit assumptions', and in case 2 of

$$A = \begin{pmatrix} a & b \\ 0 & R \end{pmatrix} \text{ or } \begin{pmatrix} R & b \\ 0 & g \end{pmatrix} \text{ with } \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix}, \text{where } h \neq 0,$$

$$\text{or } A = \begin{pmatrix} a & \frac{(a-R)(g-R)}{f} \\ f & g \end{pmatrix}, \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix}, \text{where } f \neq 0 \text{ and } h \neq 0,$$

again subject to the 'implicit assumptions'. See Section 11.3 in the Appendix for a calculation.

Remark 3. In the case $n = 2$ it is especially easy to check that the property 'R is an eigenvalue of A' is equivalent to 'A is similar to a matrix with first row $(R, 0)$'. See the appendix (section 11.1). However, being similar to a matrix with R in the 11 entry does not necessarily imply that R is an eigenvalue.

Remark 4. The leading term of the generating function $\bar{g}_A^1(\kappa, \xi)$ yields the following equation

$$a_{12}\xi_2 + ... + a_{1n}\xi_n = 0,$$

an orthogonality relation in $n - 1$ dimensional space. This implies for the case $n = 2$ a particularly simple structure, as the only equation to solve is $a_{12}\xi_2 = 0$. However, the case $n = 3$, where $b_2 a_{12} + b_3 a_{13} = 0$, exhibits the hall-marks of a general case, albeit there is only one direction for (b_2, b_3) to point, given (a_{12}, a_{13}). In the generic case (b_2, b_3) is uniquely determined. Otherwise, degeneracy occurs and there is a one-dimensional solution space for (b_2, b_3). Here, however, since evidently consistency issues also play a part, some further restrictions on the A matrix come into effect (see section 9 for details).

Remark 5. Given a matrix A, if we want $a_{11} = R$, where R is some required rate of return, for example $R = 1 + r$ with r the riskless interest rate, then (5) may be regarded as prescribing a value for any other one entry of A for which the corresponding co-factor is non-zero (in terms of all the other entries.)

1.4 Structure of the paper

We discuss in Section 2 the implicit assumptions. We then construct the proof of the Uniqueness and Identification Theorems initially under the blanket assumption (lasting till Section 6) that $v_0 = 0$. This is in the interests of clarity. We begin in Section 3 by finding a one-to-one transformation between the vector $d = (d_1, d_2, ...)$ and the vector $x = (x_1, x_2, ...)$. We use the transformation in Section 4 to characterize systems obeying dividend policy irrelevance in terms of a factorization formula for the characteristic polynomial of the (reduced) matrix A involving b and a_{11}, assuming that a_{11} is an eigenvalue of A. We also discover that systems obeying dividend policy irrelevance at R have R as an eigenvalue of A. In Section 5 we derive an equivalent formulation of the factorization condition in terms of the in-homogeneous system defined in the Identification Theorem and this enables us to deduce, for those b satisfying the in-homogeneous system, that $R = a_{11}$ iff dividend policy irrelevance holds at R. This completes the proof of the two theorems under the blanket assumption. Section 6 indicates what minor modifications are needed when we drop the assumption $v_0 = 0$. In Section 7 we deduce the form of dividend policy irrelevance systems for $n = 2$ when a_{11} is an eigenvalue of A. The Verification Theorem is proved in Section 8. We close the paper with a cursory discussion of the case $n = 3$ which has a much richer structure than the case $n = 2$. The Appendix contains a direct proof of the Identification Theorem for the case $n = 2$ and the explicit conclusions of the Verification Theorem.

2 Implicit Assumptions

The main theorems make reference to the following assumptions. They are to be interpreted as restrictions on the reduced information carried by A and b.

(i)The eigenvalues of the two matrices A and $\bar{A}$ are all distinct from one another; in particular, if a_{11} is an eigenvalue of A, then

$$\chi_{\bar{A}}(\omega_{11}) \neq 0.$$

(ii) For each eigenvalue κ_i of $\bar{A}$ we have: κ_i is non-zero and

$$a_{11} > |\kappa_i|, \tag{13}$$

$$a_{11} > \gamma, \tag{14}$$

$$|(\kappa_i I - A)_b| \neq 0, \tag{15}$$

Evidently the condition (20) of the theorem requires that a_{11} is an eigenvalue of A, so the inequality (13) repeats an aspect of distinctness required in (i). In regard to (13) see Marden [1].

The assumption of distinct eigenvalues yields a more transparent proof. In any case minor perturbations will achieve distinctness.

Remark It transpires that we do not need to place any further restriction on the data: A, b, z_0, d_0, such as requiring invertibility between the initial segment of the dividend sequence $d = (d_1, ..., d_n)$ and ω_{div}.This invertibility is analyzed in the Appendix.

2.1 Implicit Assumptions for the case $n = 1$

For the case $n = 1$ (the original theorem of Ohlson) the implicit assumptions require that $\Omega_1 = A = (\omega_{11})$ and

$$\Omega_2 = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}$$

have distinct non-zero eigenvalues. Hence, referring to the characteristic polynomial of $\bar{A} = \Omega_2$, it is assumed that $0 \neq \chi_{\bar{A}}(\omega_{11}) =: \det(\omega_{11}I - \Omega) = -\omega_{12}\omega_{21}$. Now if $\omega_{12}\omega_{21} > 0$, then Ω_2 has two real eigenvalues (one positive, one negative) separated by ω_{11} thus failing inequality condition (13). Evidently the larger eigenvalue in this case renders $P_0(\omega_{11}, d)$ infinite, an unrealistic scenario, unless a special choice of initial conditions renders the function $P_0(R, d)$ dependent only on the smaller eigenvalue. Thus the inequality (13) requires $\omega_{12}\omega_{21} < 0$.

If the eigenvalues are to be real (13) further requires that $\omega_{12}\omega_{21} - 2\omega_{11}^2 < 2\omega_{11}\omega_{22}$.

Otherwise the eigenvalues of Ω_2 need to be complex conjugates. In this case referring again to the characteristic polynomial $\chi_{\bar{A}}$ this requires

$$(\omega_{11} + \omega_{22})^2 < 4(\omega_{11}\omega_{22} - \omega_{12}\omega_{21}),$$

so for given ω_{22} the value of $\omega_{12}\omega_{21}$ needs to be sufficiently negative, i.e.

$$-(\omega_{11} - \omega_{22})^2 > 4\omega_{12}\omega_{21}.$$

As for condition (15), since $A = (\omega_{11})$, this merely requires that $\omega_{12} \neq 0$, a condition already captured earlier. Of course (15) says $b_1 \neq 0$.

3 The reconstruction kernel - (i) when $v_0 = 0$

The aim of this section is to construct in the setting of equation (Ω) when $v_0 = 0$ a one-to-one transformation from the sequences x_t to the sequences d_t definable by reference to the pair A, b (subject to the implicit assumptions holding) and the eigenvalues of $\bar{A}$. Specifically (Ω) implies that

$$\begin{aligned} d_t &= \sum_{i=1}^{n+1} l_i \kappa_i^t, \\ x_t &= \sum_{i=1}^{n+1} L_i \kappa_i^t, \end{aligned}$$

where κ_i (for $i = 1, ..., n+1$) are the distinct eigenvalues of $\bar{A}$. We imagine temporarily that we will receive as information the sequence $\{x_t\}$ and ask how we might learn the values of d_t from the values x_t. This will be critical to our later context of more restrictive information.

We answer this question by showing that there is a function $\delta_b(\kappa)$ (depending also on A) such that for each i

$$l_i = L_i \delta_b(\kappa_i).$$

We will later find that if dividend policy irrelevance holds at R, and R is an eigenvalue of A which is not an eigenvalue of $\bar{A}$, then $\delta_b(\kappa)$ takes the simple form

$$\delta_b(\kappa) = \frac{\kappa - R}{b_1}.$$

From the standpoint of the reduced system A the variables d_t and v_t are on an equal footing (though they are latent and patent variables from an accounting perspective). For the sake of clarity we consider the notationally less cluttered situation arising when the informational variable v_t is absent (i.e. with $v_0 = 0$). Re-inclusion of the informational variable, and hence the general case, will follow automatically by a superposition argument (see Section 6).

The solution to the dynamic

$$\bar{z}_{t+1} = \bar{A}\bar{z}_t$$

implies the form of d_t to be

$$d_t = \sum_{i=1}^{n+1} l_i \kappa_i^t,$$

where κ_i are the distinct eigenvalues of the matrix $\bar{A}$ (and these do depend on ω_{div}). Writing $z_t = (x_t, ...)$ we regard x_t as implicitly defined from the remaining n variables of $\bar{z}_t$ by the equations of the reduced system

$$z_{t+1} = Az_t + bd_t.$$

It will be enough to solve instead (for x_t) the in-homogenous equation

$$z_{t+1} = Az_t + b\kappa^t, \tag{16}$$

where κ is a typical eigenvalue of $\bar{A}$ (since the superposition of each of the corresponding solutions for x_t gives the general solution).

Provided κ is not an eigenvalue of A the general solution of (16) takes the form for some constant matrix C

$$z_t = C\Lambda^t + z\kappa^t,$$

where

$$\Lambda = \text{diag}[\lambda_1, ...]$$

is the diagonal matrix of distinct eigenvalues of A, and z solves

$$\kappa z = Az + b, \tag{17}$$

and in particular

$$b \in \mathcal{R}((\kappa I - A)). \tag{18}$$

In the case considered we already know that the functions x_t and d_t are independent of the functions λ_i^t, so we have

$$x_t = x\kappa^t,$$

where $z = (x, ...)^T$ solves (17).

Remark. If κ is an eigenvalue of A, then although the general solution of (16) is of the form

$$z_t = C\Lambda^t + ct\kappa^t,$$

we have $c = 0$, since z_t is independent of $t\kappa^t$. So it is still the case here that for some z

$$z_t = z\kappa^t,$$

hence $z = (x, ...)^T$ still solves (17).

We may now find x by using Cramer's Rule. With this aim define the reconstruction kernel by

$$\delta_b(\kappa) =: \frac{|(\kappa I - A)|}{|(\kappa I - A)_b|}, \tag{19}$$

where M_b denotes the matrix M with its first column replaced by b. We thus find that x may be represented in the form

$$x = \frac{|(\kappa I - A)_b|}{|\kappa I - A|} = \delta_b(\kappa)^{-1},$$

provided κ is not an eigenvalue of A. Evidently the form of (19) requires us to assume in addition that

$$|(\kappa I - A)_b| \neq 0,$$

which is a condition on b.

We may now superpose the representations x as κ ranges over κ_i with b held fixed. We note for future reference that the highest power of κ in $|(\kappa I - A)_b|$ is $n-1$ and the coefficient at the leading power is b_1.

It follows from our calculations that the sequence x_t reconstructs the sequence d_t in the following sense. Suppose that

$$d_t = \sum_{i=1}^{n+1} l_i \kappa_i^t$$

where the l_i could be obtained from observing a segment of the sequence d_t though this is not permitted. By (19) we obtain from superposition of the solutions of (16) a representation of x_t in the form

$$x_t = \sum_{i=1}^{n+1} \delta_b(\kappa_i)^{-1} l_i \kappa_i^t,$$

provided $\delta_b(\kappa_i)$ is non-zero. But the representation of x_t

$$x_t = \sum_{i=1}^{n+1} L_i \kappa_i^t,$$

which may be obtained from the observed sequence x_t, yields, by comparing with the alternative representation, that

$$l_i = L_i \delta_b(\kappa_i). \tag{20}$$

3.1 An extension

Observe now that if for a given κ a solution z exists for the equation

$$(\kappa I - A)z = b, \tag{21}$$

then partitioning $(\kappa I - A) = [m_1|...]$ we have, as in Cramer's Rule, that

$$|(\kappa I - A)_b| \ = |(\kappa I - A)_{(x \cdot m_1 + ...)}| \ = x \cdot |\kappa I - A|. \tag{22}$$

By (18) a solution z exists when $\kappa = \kappa_i$ is an eigenvalue of A, and consequently, the left-hand side vanishes, i.e. $|(\kappa_i I - A)_b| = 0$. Thus the polynomial $|(\kappa I - A)_b|$ has a factor $(\kappa - \kappa_i)$ just as $|\kappa I - A|$ has.

Now we may always take limits in equation (21) through a sequence of values κ approaching κ_i (but omitting eigenvalues of A), and we claim that the corresponding unique solution $z(\kappa)$ of the equation converges to a solution for $\bar{z} = (\bar{x}, ...)$ of (21) with $\kappa = \kappa_i$. Furthermore, given this claim we have by (22) that

$$\bar{x} = \lim_{\kappa \to \kappa_i} \frac{|(\kappa I - A)_b|}{|\kappa I - A|}.$$

Evidently this limit operation is equivalent to evaluation of the ratio

$$\frac{|(\kappa I - A)_b|}{|\kappa I - A|}$$

at κ_i after removal (cancellation) of the common factor $(\kappa - \kappa_i)$ in the numerator and denominator since $\kappa = \kappa_i$ is a simple root of $|\kappa I - A|$.

In fact we require $\bar{x} \neq 0$, and this is equivalent to requiring that $|(\kappa I - A)_b|$ has a simple root at $\kappa = \kappa_i$, as required by (9). Subject to this condition, it is possible also to interpret $\delta_b(\kappa_i)$ as

$$\bar{\delta}_b(\kappa_i) = \lim_{\kappa \to \kappa_i} \delta_b(\kappa),$$

i.e. as being the same as $\delta_b(\kappa)$ evaluated at κ_i after reduction to 'lowest terms' (i.e. cancellation of the common factor $(\kappa - \kappa_i)$ in the numerator and denominator).

To verify the claim, let $\{v_1, ..., v_n\}$ be the independent eigenvectors of A with $Av_j = \lambda_j v_j$. Since $(I - A)v_j = (\kappa - \lambda_j)v_j$ we observe that if $\kappa = \lambda_1$ the linear span $Lin\{v_1\}$ equals $\mathcal{N}((\lambda_1 I - A))$, the null space of $(\lambda_1 I - A)$, and $Lin\{v_2, ...v_n\}$ equals $\mathcal{R}((\lambda_1 I - A))$, the range of $(\lambda_1 I - A)$. But by (18) we have $b \in \mathcal{R}((\lambda_1 I - A))$, so that $b = \beta_2 v_2 + \beta_3 v_3 + ...$. Now for any κ put

$$z(\kappa) = \beta_2(\kappa - \lambda_2)^{-1} v_2 + ...,$$

then $z = z(\kappa)$ is a solution to $(\kappa I - A)z = b$, since

$$(\kappa I - A)\{\beta_2(\kappa - \lambda_2)^{-1} v_2 + ...\} = \beta_2 v_2 + ... = b.$$

Of course $z(\kappa)$ is the unique solution of $(\kappa I - A)z = b$ whenever κ is not an eigenvalue of A. Now letting $\kappa \to \lambda_1$ we obtain the limiting result that, as claimed,

$$\bar{z} = \lim_{\kappa \to \lambda_1} z(\kappa) = \beta_2(\lambda_1 - \lambda_2)^{-1} v_2 + ...$$

Now note that $\bar{\delta}_b(\kappa_i)^{-1}$ will be the only solution for x iff $\mathcal{N}(\kappa_i I - A) \subseteq \{0\} \times R^{n-1}$. Indeed, the solution set of (21) for $\kappa = \kappa_i$ has the form

$$\bar{z} + \mathcal{N}((\kappa_i I - A)).$$

It is instructive to observe that in the case $n = 1$ we have $b = b_1$, so that

$$\delta_b(\kappa) = \frac{\kappa - a_{11}}{b_1}.$$

Evidently when A is of size 2×2 the polynomial $|(\kappa I - A)_b|$ is linear and so automatically $|(\kappa I - A)_b|$ has a simple root at $\kappa = \kappa_i$ if, as discussed above, an eigenvalue κ_i of $\bar{A}$ is an eigenvalue of A. Under these circumstances

$$|(\kappa I - A)_b| = b_1(\kappa - \kappa_i)$$

and

$$|\kappa I - A| = (\kappa - a_{11})(\kappa - \kappa_i),$$

so that again

$$\delta_b(\kappa) = \frac{\kappa - a_{11}}{b_1}.$$

We will later find that for vectors b defined as in Theorem 1, this formula remains valid for general n. (See formula (33) in Section 5.) Thus $\bar{\delta}_b(\kappa) = \delta_b(\kappa) \neq 0$ for $\kappa \neq a_{11}$.

4 Equity and the reconstruction kernel: when $v_0 = 0$

In this section we use the transformation from $d = (d_t)$ to $x = (x_t)$ to identify the circumstances when equity depends only on A and on the initial data.

Recalling that

$$P_0(R; d) = \sum_{t=1}^{\infty} R^{-t} d_t,$$

and assuming that R satisfies the convergence constraint (7) we have

$$P_0(R; d) = \sum_{t=1}^{\infty} R^{-t} d_t = \sum_{t=1}^{\infty} \sum_{i=1}^{n+1} l_i \kappa_i^t R^{-t}.$$

Changing the order of summation we obtain

$$P_0(R; d) = \sum_{i=1}^{n+1} l_i \frac{\kappa_i}{R - \kappa_i}. \tag{23}$$

See Ashton (1995) for a discussion of this formula. The earliest form of this equation is due to Ohlson (1989).

We assume that for each eigenvalue κ_i of $\bar{A}$ we have

$$a_{11} > |\kappa_i|.$$

Thus referring to (14) we have from (23) via (20) that

$$P_0(R; d) = -\sum_{i=1}^{n+1} \frac{|\kappa_i I - A|}{(\kappa_i - R)|(\kappa_i - A)_b|} L_i \kappa_i. \tag{24}$$

In order to understand this equation, suppose first that for some constant μ we have for $i = 1, ..., n+1$

$$\frac{|\kappa_i I - A|}{(\kappa_i - R)|(\kappa_i - A)_b|} = \mu. \tag{25}$$

Then

$$P_0(R; d) = -\mu \sum L_i \kappa_i = -\mu x_1. \tag{26}$$

So provided μ depends on A and b the equity $P_0(R; d)$ will depend only on A and the initial data, since

$$\begin{aligned} x_1 &= z_1^1 = a_{11} z_0^1 + a_{12} z_0^2 + ... + a_{1n} z_0^n + b_1 d_0 + a_1 v_0, \\ &= a_{11} x_0 + ... + a_{1n} y_0 + b_1 d_0 + a_1 v_0. \end{aligned} \tag{27}$$

Note that in the case $n = 1$ we have of course

$$P_0(a_{11}; d) = -\frac{\kappa_1 - a_{11}}{(\kappa_1 - a_{11}) b_1} L_1 \kappa_1 = -\frac{L_1 \kappa_1}{b_1} = -\frac{x_1}{b_1}.$$

We will see in a moment that, quite generally, under the assumption of dividend policy irrelevance, $\mu = b_1^{-1}$.

Returning to our argument. Equation (24) implies that equity is expressible by a functional of the form

$$P(u_1, u_2, ...) = c_1 u_1 + ... + c_{n+1} u_{n+1}$$

for some coefficients $c_1, ..., c_{n+1}$ independent of $L_1, .., L_{n+1}$, when for $u = (u_1, ..., u_{n+1})$ the substitution $(L_1 \kappa_1, ..., L_{n+1} \kappa_{n+1})$ is made. It is the case however that

$$u_1 + ... + u_{n+1} = \sum L_i \kappa_i = k, \tag{28}$$

where $k = a_{11} x_0 + ... + a_{1n} y_0 + b_1 d_0 + a_1 v_0$ and this value remains constant as ω_{div} changes. Intuition suggest that we have

$$c_1 = c_2 = ... = c_n = c_{n+1}.$$

We prove this assertion now by taking $w = 0$ and $\beta = \kappa$,an arbitrary value restricted only by $|\kappa| < R$ and avoiding the eigenvalues of A. For this choice of w

and β the eigenvalues of $\bar{A}$ are seen to be κ together with those of A. Moreover by (2)

$$d_{t+1} = \kappa d_t$$

so in this case we have

$$d_t = d_0 \kappa^t$$

i.e. $l_1 = .. = l_n = 0$ and $l_{n+1} = d_0$. It follows from (20) that $L_1 = .. = L_n = 0$ and $L_{n+1}\kappa = x_1$. Hence we have

$$P_0(R; d) = -\sum_{i=1}^{n+1} \frac{|\kappa_i I - A|}{(\kappa_i - R)|(\kappa_i I - A)_b|} L_i \kappa_i = -x_1 \frac{|\kappa I - A|}{(\kappa - R)|(\kappa I - A)_b|}.$$

By (27) the term x_1 is independent of κ and so if equity is to be independent of κ we necessarily have for some constant μ that

$$\frac{|\kappa I - A|}{(\kappa - R)|(\kappa - A)_b|} = \mu,$$

for all κ with $|\kappa| < R$ omitting the eigenvalues of A. We may now regard numerator and denominator as polynomials in κ and the equality

$$|\kappa I - A| = \mu(\kappa I - R)|(\kappa I - A)_b|, \tag{29}$$

holds for a continuum of values. Note that the coefficient at the leading power of κ in $|\kappa I - A|$ is unity and that in $|(\kappa I - A)_b|$ is b_1. The two polynomials are identical by the Interpolation Theorem, so we may equate coefficients at κ^n to obtain

$$\mu b_1 = 1,$$

and therefore in particular

$$b_1 \neq 0 \text{ and } \mu = b_1^{-1}.$$

In conclusion we have

$$|\kappa I - A| = b_1^{-1}(\kappa - R)|(\kappa - A)_b|.$$

Conversely, if the above factorization holds then and if κ_i are the eigenvalues of $\bar{A}$ arising when w and β take arbitrary values, we have

$$\frac{|\kappa_i I - A|}{(\kappa_i - R)|(\kappa_i I - A)_b|} = b_1^{-1},$$

i.e. (25) holds. Hence by (26) we have

$$P_0(R; d) = -b_1^{-1} \sum L_i \kappa_i = -b_1^{-1} x_1,$$

so that the equity is independent of w and β.

We now note that as a consequence of dividend policy irrelevance holding for $R = a_{11}$ we have

$$|\kappa I - A| = b_1^{-1}(\kappa - a_{11})|(\kappa I - A)_b|. \tag{30}$$

Thus we obtain dividend policy irrelevance at $R = a_{11}$ iff the characteristic polynomial may be factored by $(\kappa - a_{11})$ in this particular way.

5 Equivalence of factoring and the in-homogeneous system

In the last section we showed that dividend policy irrelevance holds at $R = a_{11}$ iff the following factorization occurs

$$|\kappa I - A| = b_1^{-1}(\kappa - a_{11})|(\kappa I - A)_b|. \tag{31}$$

In this section we shall derive an equivalent assertion characterizing b as a solution of a system of equations derived from A as stated in the Identification Theorem.

This has two consequences. First, we have shown dividend policy irrelevance for $R = a_{11}$ for any such b. Second, if for such a b we have dividend policy irrelevance at some R, then by (29) we have for some non-zero constant μ the factorization

$$|\kappa I - A| = \mu(\kappa - R)|(\kappa I - A)_b|,$$

as well as the factorization (31). Hence

$$R = a_{11},$$

and so we have established the uniqueness assertion in Part (i) of the Identification Theorem.

We now observe that the factorization (31) is equivalent to the identity

$$\begin{aligned} & (\kappa - a_{11})c_{11}(\kappa I - A) + \sum_{i=2}^{n}(-1)^{1+i}a_i c_{i1}(\kappa I - A) \\ = \; & b_1^{-1}(\kappa - a_{11})\sum_{i=1}^{n}(-1)^{1+i}b_i c_{i1}(\kappa I - A) \\ = \; & (\kappa - a_{11})c_{11}(\kappa I - A) + \sum_{i=2}^{n}(-1)^{1+i}b_1^{-1}b_i(\kappa - a_{11})c_{i1}(\kappa I - A), \end{aligned}$$

or

$$\sum_{i=2}^{n}(-1)^{1+i}[a_i - b_1^{-1}b_i(\kappa - a_{11})]c_{i1}(\kappa I - A) = 0.$$

Write

$$\tau = \kappa - a_{11}, \qquad \xi_i = b_1^{-1}b_i, \qquad A' = A - a_{11}I$$

to obtain

$$\sum_{i=2}^{n}(-1)^{1+i}[a_i - \xi_i\tau]c_{i1}(\tau I - A') = 0.$$

Thus $\xi = (\xi_2, ..., \xi_n)$ solves the identity in τ given by

$$g_A(\tau, \xi) \equiv 0, \tag{32}$$

where

$$g_A(\tau,\xi) = \begin{vmatrix} 0 & a_{12} & \dots & a_{1n} \\ a_{21} - \xi_2\tau & a_{22} - a_{11} - \tau & & \\ \dots & & & \\ a_{n1} - \xi_n\tau & a_{n2} & \dots & a_{nn} - a_{11} - \tau \end{vmatrix}.$$

Observe that $g_A(0) = 0$ since a_{11} is an eigenvalue of A. Now $g_A(\tau)$ is a polynomial of degree $n-1$ with a zero constant term. The identity (32) may thus be read as generating $n-1$ equations to be satisfied by the $n-1$ variables $\xi_2, ..., \xi_n$ where the equations arise from setting the coefficients of $g_A(\tau)$ to zero.

Remark. If b is selected so that $b_i = b_1\xi_i$ where ξ solves the identity $g_A(\tau,\xi) \equiv 0$, then evidently by (31) we have, just as in the case $n = 1$, that

$$\delta(\kappa) = \frac{\kappa - a_{11}}{b_1}, \tag{33}$$

which is meaningful even if κ assumes the value of any eigenvalue common to A and $\bar{A}$.

6 Reconstruction kernel - (ii) when $v_0 \neq 0$

In this section we show how to amend the proof of the Identification Theorem when $v_0 \neq 0$.

The solution to the dynamic

$$\bar{z}_{t+1} = \bar{A}\bar{z}_t + \bar{a}v_0\gamma^t$$

has the form

$$\bar{z}_t = \sum l_i\kappa_i^t + fv_0\gamma^t,$$

where κ_i are the distinct eigenvalues of the matrix $\bar{A}$ and f defines a particular solution, i.e. solves

$$(\gamma I - A)f = \bar{a}.$$

This implies d_t to be of the form

$$d_t = \sum_{i=1}^{n} l_i\kappa_i^t + l_0\gamma^t = \sum_{i=0}^{n} l_i\kappa_i^t,$$

where $\kappa_0 = \gamma$ and l_0 depends linearly on v_0, i.e. $l_0 = v_0 l_0'$ (and l_0' does not depend on v_0).

Given this form of d_t we again wish to solve for x_t when x_t is implicitly defined by the equations

$$z_{t+1} = Az_t + bd_t + av_0\gamma^t.$$

By linearity it will be enough to solve separately two equations, namely

$$z_{t+1} = Az_t + b\sum_{i=0}^{n} l_i\kappa_i^t,$$

and

$$z_{t+1} = Az_t + av_0\gamma^t.$$

Thus the first equation gives an x solution $\sum_{i=0}^{n} L_i\kappa_i^t$ where

$$l_i = L_i\delta_b(\kappa_i),$$

and the second gives an x solution $L\gamma^t$ with

$$v_0 = L\delta_a(\gamma),$$

where

$$\delta_a(\kappa) =: \frac{|(\kappa I - A)|}{|(\kappa I - A)_a|},$$

so that L does not depend on $\omega = (w, \beta, \alpha)$.

Now one may read off x_t from the two equations by super-position as taking the form

$$x_t = \sum_{i=0}^{n} L_i\kappa_i^t + L\gamma^t.$$

In particular

$$x_1 - L\gamma = \sum_{i=0}^{n} L_i\kappa_i.$$

Now as before assuming that

$$R > |\kappa_i|$$

for $i = 0, 1, ..., n+1$ (note the new condition for $i = 0$) we have

$$\begin{aligned} P_0(R; d) &= \sum_{t=1}^{\infty} R^{-t} d_t \\ &= \sum_{t=1}^{\infty} R^{-t} \sum_{i=0}^{n} l_i\kappa_i^t \\ &= \sum_{i=0}^{n} l_i \frac{\kappa_i}{R - \kappa_i}. \end{aligned}$$

Hence

$$P_0(a_{11}; d) = -\sum_{i=0}^{n} L_i\kappa_i \frac{\delta_b(\kappa_i)}{\kappa_i - a_{11}} = -\mu\sum_{i=0}^{n} L_i\kappa_i = -\mu(x_1 - L\gamma), \tag{34}$$

where, as earlier,

$$x_1 = a_{11}x_0 + ... + a_{1n}y_0 + b_1v_0,$$

and so $P_0(a_{11}, d)$ does not depend on $\bar{w}$.

7 Example: the $n = 2$ case

In this section we use the Identification Theorem to find dividend irrelevant systems for $n = 2$ with a_{11} an eigenvalue of A. For clarity put

$$A = \begin{pmatrix} a & b \\ f & g \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix}.$$

In this section we ask when is Dividend Irrelevance at R equivalent to the statement $a = R$. Here we have

$$\begin{vmatrix} 0 & b \\ f - \tau\xi_2 & g - a - \tau \end{vmatrix} = 0,$$

yielding the identity $b(f - \tau\xi_2) \equiv 0$. The constant term is zero since $bf = 0$ asserts that a is an eigenvalue of A. So we obtain the one equation

$$b\xi_2 = 0.$$

Now either $b \neq 0$ yielding the solution $f = 0$ and $h = 0$ (as $0 = \xi_2 = b_1^{-1}b_2 = c^{-1}h$), or $b = 0$ yielding the solution f and ξ_2 arbitrary (i.e. $b_1^{-1}b_2 = c^{-1}h$ arbitrary). The two systems are thus

$$\begin{aligned} x_{t+1} &= ax_t + by_t + cd_t, \\ y_{t+1} &= gy_t. \end{aligned}$$

and

$$\begin{aligned} x_{t+1} &= ax_t + cd_t, \\ y_{t+1} &= fx_t + gy_t + hd_t. \end{aligned}$$

8 Proof of the Verification Theorem

In this section we deduce the Verification Theorem from the Identification Theorem when the matrix A has R as an eigenvalue. This is done in a series of four steps.

Let $u = (u_1, ..., u_n)^T$ be an eigenvector of A to eigenvalue R. The first step is to choose a non-singular matrix $P = [u, ...]$ so that $B = P^{-1}AP$ has $b_{11} = R$ so that the Identification Theorem may be applied to B. We delay the details of P until the last step which is where they are needed. The details depend on the first index i such that $u_i \neq 0$ and follow a familiar algorithm (reduction to lower triangular form).

The second step is to use the matrix P to make a change of basis (i.e. a change of accounting variable) to obtain the equivalent system in which B is the reduced matrix, so the aim here is to identify the resulting coefficient columns $\tilde{b}$ and $\tilde{a}$ for the variables d_t and v_t. Specifically this second step calls for a change variables from z_t to Z_t by putting $z_t = PZ_t$. We rewrite the reduced system in the form

$$PZ_{t+1} = APZ_t + bd_t + av_t,$$

which yields the equivalent reduced system as

$$Z_{t+1} = BZ_t + \tilde{b}\, d_t + \tilde{a} v_t,$$

with

$$\tilde{b} = P^{-1} b \text{ and } \tilde{a} = P^{-1} a.$$

The third step identifies the relation between $\tilde{b}$ and B which is necessary and sufficient for the system to satisfy dividend policy irrelevance. The fourth and final step is to characterize the equivalent relation between b and A. This involves setting up some helpful notation and then some tedious calculations will follow involving the explicit form of P^{-1} and P.

The third step in fact applies the Characterization Theorem to the pair $B, \tilde{b}$. We deduce that $\tilde{b}_1 \neq 0$ and that there is some ξ, solving the identity $g_B(\kappa, \xi) \equiv 0$, for which we have

$$\tilde{b} = \tilde{b}_1 \xi.$$

The final step is to characterize ξ in terms of A. For this we need to compute $g_B(\kappa; \xi)$ in terms of A. To do so, it is helpful first to write $g_B(\kappa; v)$ in the form

$$g_B(\kappa; v) = |B - RI - \kappa(I^- + V^-)|.$$

Here I^- denotes the $n-1$ identity matrix bordered by zeros thus:

$$I^- = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & & \ldots & \\ 0 & 0 & \ldots & 1 \end{bmatrix},$$

and $V^- = [v|\mathbf{0}]$ where $v_1 = 0$, so that in fact

$$V^- = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ v_2 & 0 & \ldots & 0 \\ \ldots & & \ldots & \\ v_n & 0 & \ldots & 0 \end{bmatrix}.$$

Thus the identity $g_B(\kappa; v) \equiv 0$ may be written in terms of A

$$\begin{aligned} g_B(\kappa; v) &= |P^{-1}AP - RI - \kappa(I^- + V^-)| \\ &= |A - RI - \kappa P(I^- + V^-)P^{-1}|. \end{aligned}$$

Now, finally, we must compute $P(I^- + V^-)P^{-1}$ which can only be done when P has been selected. So we now argue by cases, selecting P and computing $P(I^- + V^-)P^{-1}$, according to the first index i with $u_i \neq 0$.

Case 1. Write $u = (\mu, \bar{u}^T)$ and suppose $\mu \neq 0$. We take P in the partitioned form

$$P = \begin{bmatrix} \mu & 0 \\ \bar{u} & I_{n-1} \end{bmatrix},$$

i.e., $P = [u, e_2, e_3, ..]$. We see that

$$P^{-1} = \begin{bmatrix} \mu^{-1} & 0 \\ -\mu^{-1}\bar{u} & I_{n-1} \end{bmatrix}$$

since indeed

$$\begin{bmatrix} \mu^{-1} & 0 \\ -\mu^{-1}\bar{u} & I_{n-1} \end{bmatrix} \begin{bmatrix} \mu & 0 \\ \bar{u} & I_{n-1} \end{bmatrix} = \begin{matrix} 1 & 0 \\ 0 & I_{n-1} \end{matrix} = I.$$

We compute that

$$\begin{aligned} PI^{-}P^{-1} &= \begin{bmatrix} \mu & 0 \\ \bar{u} & I_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} \mu^{-1} & 0 \\ -\mu^{-1}\bar{u} & I_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \mu & 0 \\ \bar{u} & I_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\mu^{-1}\bar{u} & I_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ -\mu^{-1}\bar{u} & I_{n-1} \end{bmatrix} \end{aligned}$$

Moreover

$$\begin{aligned} PV^{-}P^{-1} &= \begin{bmatrix} \mu & 0 \\ \bar{u} & I_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix} \begin{bmatrix} \mu^{-1} & 0 \\ -\mu^{-1}\bar{u} & I_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \mu & 0 \\ \bar{u} & I_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mu^{-1}v & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ \mu^{-1}v & 0 \end{bmatrix}. \end{aligned}$$

Thus with $\lambda_i = \mu^{-1}(\xi_i - u_i)$ for $i = 2, 3, ...,$ we have obtained the identity

$$\begin{aligned} g_B(\kappa, \xi) &= |A - RI - \kappa P(I^{-} + V^{-})P^{-1}| \\ &= \bar{g}_A^1(\kappa, \lambda) := \begin{vmatrix} \bar{a}_{11} & a_{12} & ... & a_{1n} \\ a_{21} - \kappa\lambda_2 & \bar{a}_{22} - \kappa & ... & a_{2n} \\ ... & & ... & \\ a_{n1} - \kappa\lambda_n & a_{n2} & ... & \bar{a}_{nn} - \kappa \end{vmatrix} \equiv 0, \end{aligned}$$

where $\bar{a}_{ii} = a_{ii} - R$. As before $\bar{g}_A^1(0) = 0$, since R is an eigenvalue of A and so of B (indeed we have $g_B(0) = |A - RI| = 0$).

Recall that $\tilde{b}_1 \neq 0$ so that with $\xi_1 = 1$, we have

$$\tilde{b} = \tilde{b}_1 \xi,$$

where ξ solves $g_B(\kappa; \xi) \equiv 0$. But

$$\tilde{b} = P^{-1}b = \begin{bmatrix} \mu^{-1} & 0 \\ -\mu^{-1}\bar{u} & I_{n-1} \end{bmatrix} \begin{bmatrix} b_1 \\ \bar{b} \end{bmatrix} = \begin{bmatrix} \mu^{-1}b_1 \\ -\mu^{-1}b_1\bar{u} + \bar{b} \end{bmatrix}$$

so $b_1 \neq 0$ and we have

$$b = \tilde{b}_1 P\xi = \mu^{-1} b_1 \begin{bmatrix} \mu & 0 \\ \bar{u} & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \xi \end{bmatrix} = \mu^{-1} b_1 \begin{bmatrix} \mu \\ \bar{u} + \xi \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ \mu^{-1}\bar{u} + \mu^{-1}\xi \end{bmatrix}$$

But $\lambda = \mu^{-1}(\xi - \bar{u})$ hence

$$b = b_1 \begin{bmatrix} 1 \\ 2\mu^{-1}\bar{u} + \lambda \end{bmatrix},$$

where $(\mu, \bar{u})$ is an eigenvalue of A to value R and λ solves the system $\bar{g}_A^1(\kappa, \lambda) \equiv 0$.

Case i. Here without loss of generality we suppose that the eigenvector u has $u_1 = 0$ and $u_2 = \mu \neq 0$.This time write $\bar{u} = (u_3, ..., u_n)^T$ and take

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \mu & 0 & 0 \\ \bar{u} & 0 & I_{n-2} \end{bmatrix}.$$

Arguing from elementary row operations we see that

$$P^{-1} = \begin{bmatrix} 0 & \mu^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & -\mu^{-1}\bar{u} & I \end{bmatrix}.$$

Alternatively observe that

$$\begin{bmatrix} 0 & \mu^{-1} & \mathbf{0}^T \\ 1 & 0 & \mathbf{0}^T \\ 0 & -\mu^{-1}\bar{u} & \mathbf{I} \end{bmatrix} \begin{bmatrix} 0 & 1 & \mathbf{0}^T \\ \mu & 0 & \mathbf{0}^T \\ \bar{u} & \mathbf{0} & \mathbf{I}_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}.$$

Hence we compute PI^-P^{-1} to be

$$\begin{aligned} PI^-P^{-1} &= \begin{bmatrix} 0 & 1 & 0 \\ \mu & 0 & 0 \\ \bar{u} & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & \mu^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & -\mu^{-1}\bar{u} & I \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ \mu & 0 & 0 \\ \bar{u} & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\mu^{-1}\bar{u} & I \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\mu^{-1}\bar{u} & I \end{bmatrix} \end{aligned}$$

and we compute that

$$
\begin{aligned}
PV^-P^{-1} &= \begin{bmatrix} 0 & 1 & 0 \\ \mu & 0 & 0 \\ \bar{u} & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ v_2 & 0 & 0 \\ \bar{v} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \mu^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & -\mu^{-1}\bar{u} & I \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ \mu & 0 & 0 \\ \bar{u} & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu^{-1}v_2 & 0 \\ 0 & \mu^{-1}\bar{v} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \mu^{-1}v_2 & 0 \\ 0 & 0 & 0 \\ 0 & \mu^{-1}\bar{v} & 0 \end{bmatrix}.
\end{aligned}
$$

Hence

$$
P(V^- + I^-)P^{-1} = \begin{bmatrix} 1 & \mu^{-1}v_2 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mu^{-1}(\bar{v} - \bar{u}) & \mathbf{I} \end{bmatrix}
$$

Putting $\lambda_1 = \mu^{-1}v_2$ and $\lambda_j = \mu^{-1}(v_j - u_j)$ for $j = 3, ...$ we obtain in this case that

$$
g_B(\kappa; v) = \bar{g}_A^2(\kappa; \lambda) = \begin{vmatrix} \bar{a}_{22} - \kappa & a_{12} - \kappa\lambda_1 & \cdots & a_{1n} \\ a_{21} & \bar{a}_{22} & \cdots & a_{2n} \\ \cdots & & \cdots & \\ a_{n1} & a_{n2} - \kappa\lambda_n & \cdots & \bar{a}_{nn} - \kappa \end{vmatrix}.
$$

Now we observe that

$$
\tilde{b} = P^{-1}b = \begin{bmatrix} 0 & \mu^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & -\mu^{-1}\bar{u} & I \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \bar{b} \end{bmatrix} = \begin{bmatrix} \mu^{-1}b_2 \\ b_1 \\ -\mu^{-1}b_2\bar{u} + \bar{b} \end{bmatrix}
$$

so that $\tilde{b}_1 = \mu^{-1}b_2$. Thus $b_2 \neq 0$ and we have putting $\xi_1 = 1$ as before that

$$
b = P\tilde{b} = \tilde{b}_1 P\xi = \mu^{-1}b_2 \begin{bmatrix} 0 & 1 & 0 \\ \mu & 0 & 0 \\ \bar{u} & 0 & I \end{bmatrix} \begin{bmatrix} 1 \\ \mu^{-1}v_2 \\ \mu^{-1}(\bar{v} - \bar{u}) \end{bmatrix} = \mu^{-1}b_2 \begin{bmatrix} \mu^{-1}v_2 \\ \mu \\ \bar{u} + \mu^{-1}(\bar{v} - \bar{u}) \end{bmatrix}
$$

so that

$$
b = b_2 \begin{bmatrix} \mu^{-1}\lambda_1 \\ 1 \\ \mu^{-1}\bar{u} + \mu^{-1}\bar{\lambda} \end{bmatrix},
$$

where $(0, \mu, \bar{u})$ is an eigenvalue of A to value R, and $\lambda = (\lambda_1, \bar{\lambda})$ solves $\bar{g}_A^2(\kappa; \lambda) \equiv 0$.

9 The more general case $n = 3$

In this section we analyse the solutions to the identity

$$
g_A(\kappa, \xi) = \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} - \kappa\xi_2 & a'_{22} - \kappa & a_{23} \\ a_{31} - \kappa\xi_3 & a_{32} & a'_{33} - \kappa \end{vmatrix} \equiv 0,
$$

where we put $a'_{22} = a_{22} - a_{11}$ etc. and $A' = A - a_{11}I$.

Using co-factor notation we have

$$g_A(\kappa) = -\kappa^2[\xi_2 a_{12} + \xi_3 a_{13}] + \kappa[\xi_2 c_{21}(A') - \xi_3 c_{31}(A') - c_{22}(A') - c_{33}(A')] + |A'|$$

and we are to solve

$$\begin{aligned} \xi_2 a_{12} + \xi_3 a_{13} &= 0, \\ -\xi_2 c_{21}(A') + \xi_3 c_{31}(A') &= c_{22}(A') + c_{33}(A'), \end{aligned}$$

where

$$c_{22}(A') = -a_{13}a_{31}, c_{33}(A') = -a_{12}a_{21}.$$

This system of equations has determinant

$$D = a_{12}c_{31}(A') + a_{13}c_{21}(A').$$

9.1 The case $D \neq 0$

Here we obtain uniquely

$$\left.\begin{aligned} \xi_2 = &\quad -a_{13}[c_{22}(A') - c_{33}(A')]D^{-1}, \\ \xi_3 = &\quad a_{12}[c_{22}(A') - c_{33}(A')]D^{-1}. \end{aligned}\right\}$$

9.2 The degenerate case $D = 0$

We require for consistency that

$$c_{22}(A') + c_{33}(A') = 0,$$

whereupon the one-dimensional solution space for (ξ_2, ξ_3) is in the span of $(a_{13}, -a_{12})$. Thus for this to be the solution space we need

$$\begin{aligned} a_{12}a_{21} + a_{13}a_{31} &= 0, \\ a_{12}c_{31}(A') + a_{13}c_{21}(A') &= 0. \end{aligned}$$

Regarded as a system of equations for (a_{12}, a_{13}) this has determinant equal to

$$a_{21}c_{21}(A') - a_{31}c_{31}(A') = |A'|,$$

which is zero. Thus degeneracy occurs and (a_{12}, a_{13}) lies in the span of $(a_{31}, -a_{21})$. This means

$$(\xi_2, \xi_3) = \mu(a_{13}, -a_{12}), (a_{12}, a_{13}) = \lambda(a_{31}, -a_{21})$$

so

$$(\xi_2, \xi_3) = -\lambda\mu(a_{21}, a_{31})$$

and that the dynamic system in the degenerate case has coefficient matrix (with $\kappa = -\lambda\mu$)

$$\Omega = \begin{bmatrix} a_{11} & \lambda a_{31} & -\lambda a_{21} & b_1 & a_1 \\ a_{21} & a_{22} & a_{23} & \kappa b_1 a_{21} & a_2 \\ a_{31} & a_{32} & a_{33} & \kappa b_1 a_{31} & a_3 \\ w_1 & w_2 & w_3 & \beta & \alpha \\ & & & & \gamma \end{bmatrix}.$$

10 References

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11 Appendix 1: A direct approach for $n = 2$

This appendix is devoted to studying the case $n = 2$ explicitly, which in principle entails re-working the sequence of sections 3,4,5. However, we will take for granted the arguments of Section 4 leading to (30), which are centred on the Interpolation Theorem, as there is nothing to be gained from rewriting these with $n = 2$. We thus begin by showing how the reconstruction kernel of section 3, more precisely the equations (19) and (20), may be obtained directly for the case $n = 2$. In the next subsection we analyse explicitly the factorization identity (30) of section 5. In the last subsection we offer a direct proof that the property that R is an eigenvalue of A is equivalent to A being similar to a matrix with R in its top-left corner (11-entry) and zero in the top right corner (the 12 entry). The proof is of curiosity value as it relies on finding the appropriate change of variable without direct reference to the eigenvector to value R.

11.1 Reconstruction kernel when $n = 2$

Put

$$A = \begin{pmatrix} a & b \\ f & g \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix}.$$

We may suppose that b or $g \neq 0$ so for definiteness say that $b \neq 0$. One can now use the first row/equation to define y_t thus

$$by_t = x_{t+1} - ax_t - cd_t,$$

in order to substitute into the second equation in the form

$$by_{t+1} = bfx_t + bgy_t + bhd_t$$

to obtain

$$x_{t+2} - ax_{t+1} - cd_{t+1} = g(x_{t+1} - ax_t - cd_t) + bfx_t + bhd_t$$

or

$$x_{t+2} - (a+g)x_{t+1} + (ag - bf)x_t = cd_{t+1} - (gc - bh)d_t.$$

The substitution of $x_t = \sum L_i \kappa_i^t$ and $d_t = \sum l_i \kappa_i^t$ followed by comparison of the coefficients of L_i for each i gives, as in (20), that

$$L_i[\kappa_i^2 - (a+g)\kappa_i + (ag - bf)] = l_i[c\kappa_i - (gc - bh)].$$

If $b = 0$ and $g \neq 0$ the same argument applies but now substitute from the second equation into the first.

Note that in the case when $b = 0$ we have of course

$$cd_t = x_{t+1} - ax_t,$$

so if also $c \neq 0$ we obtain on substituting $x_t = \sum L_i \kappa_i^t$ and $d_t = \sum l_i \kappa_i^t$ the same reconstruction as when $n = 1$ namely

$$cl_i = L_i(\kappa_i - a).$$

11.2 Analysis of the factorization condition for $n = 2$

In the notation of section 11.1, we will now show that the factorization condition (30) of section 5, requires that $\mu = c^{-1}$ and

$$bf = 0, \qquad bh = 0.$$

The first condition merely guarantees that a is an eigenvalue of A. Note that $P_0(a;d) = -x_1/c$. There are two solutions one with A having $(a, 0)$ as a row, the other with $(a, 0)$ as a column, conditions equivalent to a being an eigenvalue.

Solution (i) $b = 0$. We call this the accounting canonical case, as

$$A = \begin{pmatrix} a & 0 \\ f & g \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix},$$

i.e.

$$\begin{array}{rlll} x_{t+1} = & ax_t & & +cd_t, \\ y_{t+1} = & fx_t & +gy_t & +hd_t. \end{array}$$

Solution (ii) $b \neq 0$. Here

$$A = \begin{pmatrix} a & b \\ 0 & g \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

i.e.

$$\begin{array}{rlll} x_{t+1} = & ax_t & +by_t & +cd_t, \\ y_{t+1} = & & gy_t. & \end{array}$$

Proof. We compute that in this case

$$\begin{aligned} |\kappa - A| &= \begin{vmatrix} \kappa - a & b \\ f & \kappa - g \end{vmatrix} \\ &= (\kappa - a)(\kappa - g) - bf \\ &= \kappa^2 - (a + g)\kappa + (ag - bf), \\ |(\kappa I - A)_{\mathbf{b}}| &= \det \begin{vmatrix} c & -b \\ h & \kappa - g \end{vmatrix} = c\kappa - (cg - bh). \end{aligned}$$

We require a to be an eigenvalue of A.This implies that

$$bf = 0.$$

The factorization requires that

$$(\kappa - a)(\kappa - g) = c\mu(\kappa - a)\left(\kappa - \frac{cg - bh}{c}\right)$$

so that $c\mu = 1$ and

$$g = \frac{cg - bh}{c}$$

or

$$cg = cg - bh$$

i.e.

$$bh = 0.$$

11.3 Analysis of the Verification Theorem for $n = 2$

In this section we consider the in-homogeneous system of Theorem 4 and so analyse systems for which Dividend Policy Irrelevance holds at R. As in section 11.1 take

$$A = \begin{pmatrix} a & b \\ f & g \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix},$$

and now assume that R is an eigenvalue of A. Thus

$$(a - R)(g - R) - bf = 0. \tag{35}$$

We must solve in case 1

$$\begin{aligned} \bar{g}^1_A(\kappa, \xi) &= \begin{vmatrix} a - R & b \\ f - \kappa\xi & g - R - \kappa \end{vmatrix} \\ &= (a - R)(g - R - \kappa) - b(f - \kappa\xi) \\ &= [(a - R)(g - R) - bf] - \kappa[a - R - b\xi] \\ &= -\kappa[a - R - b\xi]. \end{aligned}$$

Hence for some ξ

$$a = R + b\xi. \tag{36}$$

Recall that R is an eigenvalue of A. This implies that (35) is equivalent to

$$b\xi(g - R) - bf = 0,$$

so either $b = 0$ (i.e. $a = R$ or $g = R$) or else

$$f = \xi(g - R).$$

Thus

$$A = \begin{pmatrix} R & 0 \\ f & g \end{pmatrix}, \text{ or } A = \begin{pmatrix} a & 0 \\ f & R \end{pmatrix}, \text{ or } A = \begin{pmatrix} R + b\xi & b \\ \xi(g - R) & g \end{pmatrix}.$$

Now under Case 1 we have

$$c \neq 0 \text{ and } h = c(2u_1^{-1}u_2 + \xi).$$

So if $b = 0$ then h is arbitrary. But otherwise $\xi = \frac{(a-R)}{b}$ from (36) and we compute $u_1^{-1}u_2$ to be $-(a - R)/b$ from

$$(a - R)u_1 + bu_2 = 0,$$

so that

$$h = -c\frac{a - R}{b}.$$

We thus arrive at

$$A = \begin{pmatrix} R & 0 \\ f & g \end{pmatrix}, \text{ or } A = \begin{pmatrix} a & 0 \\ f & R \end{pmatrix}, \text{ with } \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix}, \text{ where } c \neq 0 \text{ and } h = 0 \text{ if } a = R.$$

$$\text{or } A = \begin{pmatrix} R + b\xi & b \\ \xi(g-R) & g \end{pmatrix}, \text{ with } \mathbf{b} = c\begin{pmatrix} 1 \\ -\frac{a-R}{b} \end{pmatrix}, \text{ where } b \neq 0 \text{ and } c \neq 0.$$

Next we consider Case 2. Here

$$\bar{g}_A^2(\kappa,\xi) := \begin{vmatrix} a - R - \kappa & b - \kappa\xi \\ f & g - R \end{vmatrix} \equiv 0,$$

or

$$\begin{aligned} \bar{g}_A^2(\kappa,\xi) &= (a-R)(g-R) - \kappa(g-R) - bf + \kappa f\xi \equiv 0, \\ &= [(a-R)(g-R) - bf] + \kappa[f\xi - (g-R)] \equiv 0, \\ &= \kappa[f\xi - (g-R)] \equiv 0, \end{aligned}$$

using (35). Hence for some ξ we have

$$f\xi = (g-R), \tag{37}$$

so either $f = 0$ (and hence by (35) again $a = R$ or $g = R$) or

$$\xi = \frac{g-R}{f}.$$

Also under (37)we have that (35) is equivalent to

$$(a-R)f\xi - bf = 0$$

so either $f = 0$ or

$$b = \xi(a-R) = \frac{(a-R)(g-R)}{f}.$$

Now by Theorem 2

$$h \neq 0 \text{ and } c = hu_2^{-1}\xi$$

i.e. c is arbitrary. We thus arrive at

$$A = \begin{pmatrix} a & b \\ 0 & R \end{pmatrix} \text{ or } \begin{pmatrix} R & b \\ 0 & g \end{pmatrix} \text{ with } \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix}, \text{where } h \neq 0.$$

Alternatively

$$A = \begin{pmatrix} a & \frac{(a-R)(g-R)}{f} \\ f & g \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} c \\ h \end{pmatrix}, \text{where } f \neq 0 \text{ and } h \neq 0.$$

Note that here $|A - RI| = 0$ identically.

11.4 Connection with the principal entry

We note in passing that from the point of view of having the condition $a_{11} = R$ hold, the 2×2 matrices are especially easy. We offer an easy argument that identifies explicitly the aggregation of variables leading to the accounting-canonical form equivalent to A (that is with R in its top-left corner (11-entry) and zero in the top right corner (the 12 entry) and confirms that this is possible iff R is an eigenvalue of A. Starting from

$$\begin{aligned} x_{t+1} &= \omega_{11}x_t + \omega_{12}y_t, \\ x_{t+1} &= \omega_{21}x_t + \omega_{22}y_t, \end{aligned}$$

we let $w_t = x_t + \alpha y_t$ for some α, then we have

$$w_{t+1} = (\omega_{11} + \alpha\omega_{21})(x_t + \alpha y_t)$$

iff the following quadratic equation in α holds:

$$\alpha(\omega_{11} + \alpha\omega_{21}) = \omega_{12} + \alpha\omega_{22}. \tag{38}$$

Since we wish to have $R = \omega_{11} + \alpha\omega_{21}$, the choice of α is therefore

$$\alpha = (R - \omega_{11})/\omega_{21}. \tag{39}$$

The necessary and sufficient condition that R be realizable as the principal entry followed by a zero entry in its row may be obtained from (39) by substituting for α into (38). This yields

$$(R - \omega_{11})R = \omega_{21}(\omega_{12} + \alpha\omega_{22}) = \omega_{21}\omega_{12} + \omega_{22}(R - \omega_{11}),$$

or

$$(R - \omega_{11})(R - \omega_{22}) - \omega_{21}\omega_{12} = 0.$$

Thus the necessary and sufficient condition for transforming the first row to $(R, 0)$ is confirmed to be that R is an eigenvalue of the matrix of coefficients.

12 Appendix 2: Invertibility as between $\omega_{\mathbf{div}}$ and $(d_1, ..., d_n)$

It transpires from our analysis that we do not need to place any further restriction on the data: A, b, z_0, d_0, to require that the initial segment of the dividend sequence $d = (d_1, ..., d_n)$ be permitted to take any vector value in R^n (or for that matter any generic value, i.e. one not lying on some finite set of hyperplanes determined by the data). We consider in this section the issue of invertibility as between ω_{div} and d, and we ask how the explicit form

$$d_t = \sum_{i=1}^{n+1} l_i \kappa_i^t$$

permits the coefficient vector $(l_1, ...l_{n+1})$ to vary arbitrarily subject to the information concerning the initial dividend d_0,namely that

$$\sum_{i=1}^{n+1} l_i = d_0.$$

Observe that the equation

$$z_{t+1} = Az_t + d_t b$$

implies

$$z_t = A^t z_0 + d_0 A^{t-1} b + .. + d_{t-1} b$$

Put $d_- = (d_0, d_1, d_{n-1})$, then since

$$d_{t+1} = wz_t + \beta d_t,$$

we have

$$d - \beta d_- = w(A^t z_0 + d_0 A^{t-1} b + .. + d_{t-1} b),$$

or

$$d - \beta d_- = wM, \tag{40}$$

where

$$\begin{aligned} M &= M(d_1, ..., d_{n-2}) \\ &= \left[\begin{array}{cccccc} z_0 & Az_0 + d_0 b & A^2 z_0 + d_0 Ab + d_1 b & ... & ... & A^{n-1} z_0 + d_0 A^{n-2} b + .. + d_{n-2} b \end{array} \right]. \end{aligned}$$

Given β to solve uniquely for w the equation (**??**) for given arbitrary d it is necessary and sufficient that the rank of M be n for all $(d_1, ..., d_{n-2})$. This amounts to saying that d must not lie on a hyperplane determined by the data. Note that the condition fails if say z_0 and b are both multiples of an eigenvector of A.

Assuming the condition is fulfilled we have

$$w = w(\beta) = (d - \beta d_-) M^{-1}$$

for some β. But

$$d_1 = w(\beta) z_0 + \beta d_0$$

so the value of β is uniquely determined by the equation

$$\beta(d_0 - d_- M^{-1} z_0) = d_1 - d M^{-1} z_0$$

provided d does not lie on the hyperplane determined by the data

$$d_0 = d_- M^{-1} z_0.$$