Almost completeness and the Effros Open Mapping Principle in normed groups

By A. J. Ostaszewski

Abstract

We extend van Mill's version of the Effros Open Mapping Principle from analytic groups to almost complete normed groups.

Keywords: Almost completeness, Open Mapping Theorem, analytic sets, analytic Cantor theorem, non-commutative groups, group-norm, topological groups.

Classification Numbers: 26A03; 04A15; 02K20.

1 Introduction

We extend to the class of almost complete normed groups (definition below), a class which includes Polish (i.e. separable, completely metrizable) topological groups, a result that was originally proved by Effros [Eff] for Polish groups acting transitively and continuously on a non-meagre metric space. This has recently been improved by van Mill [vM1] to analytic metric groups with separately continuous, transitive action on a non-meagre metric space. Thus van Mill's variant includes meagre groups acting transitively on a non-meagre metric space (for an example see [vM1, Remark 2]). In fact van Mill's argument applies more generally to analytic *normed* groups. Below we admit (separately) continuous actions by non-analytic normed groups, but at the price of the normed groups being non-meagre (and so Baire). In this connection we note the result due to [Loy] and [HJ, Th. 2.3.6 p. 355] that a Baire analytic topological group is Polish.

A metric space is *almost complete* if it contains a dense absolute $\mathcal{G}_\delta$ (we relax the condition to 'is an absolute $\mathcal{G}_\delta$ up to a meagre subset'). The notion of 'almost completeness' is due to Frolík in [Frol] (but its name to Michael [Mich91] – see also [AL] and [BOst-N]). To state our version of the Effros Open Mapping Theorem we first recall the definition of normed groups and group actions. In the next section we recall the notion of analyticity and its connection with Cantor's Theorem; this will allow us to formulate a convergence criterion applicable under almost completeness (in lieu of the Cauchy criterion under completeness), a key tool alongside the notion of a map that is 'irreducible in category' recalled at the end of that section.

Definition (Normed groups). 1. For T an *algebraic* group with neutral element e, say that $||\cdot|| : T \to \mathbb{R}_+$ is a *group-norm* ([BOst-N]) if the following properties hold:

(i) *Subadditivity* (Triangle inequality): $||st|| \le ||s|| + ||t||$;

(ii) *Positivity*: $||t|| > 0$ for $t \neq e$ and $||e|| = 0$;

(iii) *Inversion* (Symmetry): $||t^{-1}|| = ||t||$.
Then $(T, ||.||)$ is called a *normed-group.*

2. The group-norm generates a right and a left *norm topology* via the right-invariant and left-invariant metrics $d_R^T(s,t) := ||st^{-1}||$ and $d_L^T(s,t) := ||s^{-1}t|| = d_R^T(s^{-1}, t^{-1})$. In the right norm-topology the right shift $\rho_t(s) := st$ is a uniformly continuous homeomorphism, since $d_R(sy, ty) = d_R(s,t)$, so in particular the group is a right topological group; likewise in the left norm-topology the left shift $\lambda_s(t) = st$ is a uniformly continuous homeomorphism. Since $d_L^T(t,e) = d_L^T(e, t^{-1}) = d_R^T(e,t)$, convergence at e is identical under either topology. In the absence of a qualifier, the 'right' norm topology is to be understood.

3. See Section 2 for a characterization of almost complete normed groups.

Recall that a normed group G *acts continuously* on X if there is a continuous mapping $\varphi : G \times X \to X$ such that $\varphi(e,x) = x$ and $\varphi(gh, x) = \varphi(g, \varphi(h,x))$. In view of applications, it is convenient to regard G as having the topology generated by the left invariant metric $d_L^G(g,h) := ||g^{-1}h||$. Thus $g : x \to \varphi(g,x)$ is a continuous self-map of X with a continuous inverse, and so is an autohomeomorphism, denoted $g(x)$. The action is *separately continuous* if $g : x \to \varphi(g,x)$ is continuous (so again an autohomeomorphism) and $\xi_x : g \to \varphi(g,x)$ is continuous. By a theorem of Bouziad ([Bou]), if the normed group G is Baire, as will be the case below, a separately continuous action is necessarily jointly continuous. The action is *transitive* if for any x, y in X there is $g \in G$ such that $g(x) = y$. The action of G on X is *weakly micro-transitive* if for $x \in X$ and each neighbourhood A of e_G the set

$$\mathrm{cl}(Ax) = \mathrm{cl}\{ax : a \in A\}$$

has x as an interior point (in X). Note that the neighbourhoods of e under either norm topology are the same, so the weak microaction property is a norm property, rather than a topological property of G. The action is *micro-transitive* if for $x \in X$ and each neighbourhood A of e_G the set

$$Ax = \{ax : a \in A\}$$

is a neighbourhood of x. This is again a norm property. We refer to Ax as an x *orbit* (the A-orbit of x).

Theorem 1 (Main Theorem). *Let the normed group G have separately continuous and transitive action on X. If under either norm topology G is separable and almost complete (so Baire), and X is non-meagre, then the action of G is micro-transitive.*

A tribute to the importance of the Effros's result is the existence of several proofs with varying contexts, including [Anc], [Hoh], [ChMa], one attributed to Becker in [Kech-T], as well as the already cited [vM1]. The last of these describes the historical development and applications to functional analysis and continuum theory; our interest arises with its application to topological regular variation (which draws on the 'crimping property' – for the definition and its derivation from the Effros Theorem see [BOst-N] Th.3.15, and for its applications see [BOst-TI]). Our proof of the current version of the Effros Theorem blends the argument in [Anc] with that in [vM1], and relies on the convergence criterion of Lemma 1 below, valid in analytic spaces. It has recently emerged in [Ost-E] that the Effros theorem is intimately connected with the notion of shift-compactness, a fact that yields an altogether different, short proof of Theorem 1; see also [Mil-Ost] for further background.

Remarks. 1. Working still under d_L^G and with G acting transitively, if for each fixed k the shift $\rho_k : g \to gk$ is a homeomorphism of G (having a continuous inverse $\rho_{k^{-1}}$), in particular if G is a topological group, then open-ness of any one map ξ_y implies open-ness of all the other maps ξ_x. Indeed, by transitivity write $ky = x$ for some $k \in G$; then $g(x) = g(k(y)) = gk(y)$, so that $\xi_x(g) = \xi_y \circ \rho_k(g)$, and then ξ_x is open (as a composition of two open maps).

If all the maps ξ_x are open, then for any fixed x and H an open neighbourhood of e_G the set Hx is open, and so $x \in Hx$ is an open neighbourhood of x, i.e. the action is microtransitive.

2. The theorem may be generalized in such a way that, in the case of a topological group, for T almost open (i.e. Baire non-meagre) in G the set $Tt^{-1}x$ is almost open for quasi all $t \in T$. For details see [Mil-Ost].

2 Analytic Cantor Theorem and convergent sequences

Recall that Cantor's Theorem on the intersection of a nested sequence of closed (or compact, as appropriate) sets has two formulations: (i) referring to vanishing diameters (in a complete-space setting), and (ii) to (countable) compactness. Our first aim in this section is to give a topological version that is in this same spirit but appropriate to an analytic, rather than complete or compact, context. For this we need first to recall that, in a metric space, a set is *analytic* if it is the continuous image of a Polish space P (as before, a separable topologically complete metrizable space), i.e. of the form $f(P)$ for f continuous and P Polish – see [Jay-Rog] for details.

Although our concern here is with metric spaces, there are several advantages in discussing analytic sets in the broader context of Hausdorff topological spaces, arising from explicitly exposing their underlying topological nature. The brief account below will suffice here – see [Ost-AH] for a wider discussion.

For X a Hausdorff space write $\mathcal{K} = \mathcal{K}(X)$ for the family of compact subsets of a space X, and $\wp(X)$ for the power set. Following the the notation of [Jay-Rog], write I for $\mathbb{N}^{\mathbb{N}}$ endowed with the product topology (treating $\mathbb{N}$ as discrete) and $i|n := (i_1, ..., i_n)$ for $i \in I$. For X a Hausdorff space a map $K : I \to \wp(X)$ is called *compact-valued* if $K(i)$ is compact for each $i \in I$, and *singleton-valued* if each $K(i)$ is a singleton. K is *upper semicontinuous* if, for each $i \in I$ and each open U in X with $K(i) \subseteq U$, there is a neighbourhood $N = \{j \in I : j|n = i|n\}$ of i such that $K(j) \subseteq U$ for each j in N. A subset of X is K-*analytic* if it is the image $K(I)$ under an upper semicontinuous compact valued map.

The following result is implicit in a number of situations, and goes back to Frolík's characterization of completely regular Čech-complete spaces as $\mathcal{G}_\delta$ in some compactification ([Frol]; see [Eng] §3.9).

Theorem AC (Analytic Cantor Theorem, [Ost-AH]**).** *Let X be a Hausdorff space, and let $A = K(I)$ be $\mathcal{K}$-analytic in X, where K is compact-valued and upper semicontinuous.*

Suppose that F_n is a decreasing sequence of (non-empty) closed sets in X such that

$$F_n \cap K(i_1, ..., i_n) \neq \emptyset,$$

for some $i = (i_1, ...) \in I$ and each n. Then

$$K(i) \cap \bigcap_n F_n \neq \emptyset.$$

Equivalently, if there are open sets V_n in I with $\mathrm{cl}V_{n+1} \subseteq V_n$ *and* $diam_I V_n \downarrow 0$ *such that $F_n \cap K(V_n) \neq \emptyset$, for each n, then*

(i) $\bigcap_n \mathrm{cl}V_n$ *is a singleton, $\{i\}$ say,*

(ii) $K(i) \cap \bigcap_n F_n \neq \emptyset$.

Proof. If not, then $\bigcap_n K(i) \cap F_n = \emptyset$ and so, by compactness, $K(i) \cap F_p = \emptyset$ for some p, i.e. $K(i) \subseteq X \backslash F_p$. So by semicontinuity $F_p \cap K(I(i_1, ..., i_n)) = \emptyset$ for some $n \geq p$, yielding the contradiction $F_n \cap K(I(i_1, ..., i_n)) = \emptyset$. □

We will make use of the following immediate corollary.

Lemma 1 (Convergence criterion). *In an analytic normed group $X = K(I)$, for $r_n \searrow 0$ and $\alpha_n = a_n \cdot ... \cdot a_1$ with $\mathrm{cl}B_{r_{n+1}}(a_{n+1}) \subseteq B_{r_n}(e)a_n$ and $K(i_1, ..., i_n) \cap B_{r_n}(\alpha_n) \neq \emptyset$, the sequence $\{\alpha_n\}$ is convergent.*

Proof. Indeed, $\alpha_n \to \alpha$, where

$$\{\alpha\} = K(i) \cap \bigcap\nolimits_n F_n \text{ for } F_n = \mathrm{cl}(B_{r_n}(\alpha_n)) \ \square.$$

We note that the convergence criterion may be used to derive the following characterization of almost complete normed groups. For the proof see [Ost-LBIII, Th. 2].

Characterization Theorem (Almost completeness). *In a separable normed group X under d_R^X, the following are equivalent:*

(i) *X is a non-meagre absolute $\mathcal{G}_\delta$ modulo a meagre set (i.e. is almost complete);*

(ii) *X contains a non-meagre analytic subset;*

(iii) X *is non-meagre analytic modulo a meagre set.*

Definition (cf. [Eng]). Call a map $f : X \to Y$ *irreducible on X in the sense of category* if there is no proper closed $F \subseteq X$ such that $f(F)$ equals $f(X)$ modulo a meagre set.

Equivalently: for non-empty open V in X the set $f(V)$ is non-meagre in Y. In particular, for f continuous, if W is open in Y and meets $f(X)$ then, as $V = f^{-1}(W)$ is non-empty and open in X, the set $W \cap f(V) = f(V)$ is non-meagre. For brevity, we shall say that a set S is *heavy* if $S \cap V$ is non-meagre for each open V meeting S. (We follow [BrGo] in using this term; [vM1, Prop. 2.2] calls sets that are dense and heavy 'fat'.)

The following result is the first step in [vM1, Propo. 2.2] and is inspired by a theorem of Levi [Lev]; more in fact is true – see [Ost-AH]. We repeat the proof as it is short.

Lemma 2. *For a continuous surjective map $f : X \to Y$ with X separable, there is a closed set $X' \subseteq X$ such that the restriction map $f' := f|X'$ is irreducible on X' in the sense of category.*

Proof. Let $\mathcal{U}$ be the family of sets U such that $f(U)$ is meagre; put $U := \bigcup \mathcal{U}$, $X' := X \backslash U$ and $f' := f|X'$. By separability there is a countable open family $\mathcal{V}$ with $U = \bigcup \mathcal{V}$; then $f(U) = \bigcup_{V \in \mathcal{V}} f(V)$, being a countable union of meagre sets, is meagre. Suppose that for some V open in X the set $f'(V \cap X')$ is meagre; as $V \backslash X' \subseteq U$, one has $f(V) \subseteq f'(V \cap X') \cup f(U)$, which is meagre. So $V \in \mathcal{U}$ and $V \subseteq U$, so that $V \cap X'$ is empty; so f' is irreducible. □

3 Proof of the Effros Theorem

We first give the normed-group version of a key result; that will require a definition. In what follows we use letters from the beginning of the alphabet for (open) subsets in G and letters from the end for (open) subsets of X.

For the next result note that, since $d_L^G(g, h) = d_R^G(g^{-1}, h^{-1})$, the map $g \to g^{-1}$ from (G, d_L^G) to (G, d_R^G) is an isometry. So if G is separable in either norm topology, then it is separable in the other; likewise with almost completeness.

Theorem 2 (Effros Theorem – weak micro-transitive form). *Let the normed group G act on X transitively. If G is separable under either norm topology and X is non-meagre, then the action of G is weakly micro-transitive.*

Proof. Here it is convenient to work under d_L^G so that each left shift $\lambda_g(h) = gh$ is continuous (and so a homeomorphism), as $d_L^G(gh_n, gh) = d_L^G(h_n, h)$. If H is a neighbourhood of e, then gH is open in the left norm topology. As G is second-countable there are elements g_n in G such that $\{g_n H : n \in \omega\}$ covers G. Fix $x \in X$. If G acts transitively on X, then

$\{g_nHx : n \in \omega\}$ covers X. If X is non-meagre, then for some n the set $\mathrm{cl}(g_nHx)$ has non-empty interior.

That is, for some non-empty open set W in X the set g_nHx is dense in W. Then Hx is dense in the open set $U := g_n^{-1}(W)$; indeed for any open $V \subseteq g_n^{-1}(W)$, the set $g_n(V)$ is open (since $g_n : X \to X$ is a homeomorphism) and, being contained in W, meets g_nHx. So V meets Hx. Thus $\emptyset \neq U \subseteq \mathrm{int}(\mathrm{cl}(Hx))$.

As Hx is dense in U, for some $h \in H$, the point hx is in U, i.e. in $\mathrm{int}(\mathrm{cl}(Hx))$. So $x \in h^{-1}\mathrm{int}(\mathrm{cl}(Hx))=\mathrm{int}(\mathrm{cl}(h^{-1}Hx))\subseteq\mathrm{int}(\mathrm{cl}(H^{-1}Hx))$, since h is a homeomorphism. But sets of the form $H^{-1}H$ form a basis for the open neighbourhoods of e_G, so x is in the interior of $\mathrm{cl}(Ax)$ for any neighbourhood A of e_G. $\square$

Theorem 3 (Effros Theorem from weak micro-transitivity). *If the normed group G is almost complete under the right norm topology and the continuous action of G on X is weakly micro-transitive, then the action of G is micro-transitive.*

Proof. Let $d = d^X$ be any metric on X and for $x \in X$ denote by $\xi := \xi_x : G \to X$ the evaluation map $\xi_x(g) = g(x)$, which is continuous.

Fix x in X, and let $H_0 = K_0 = B_\varepsilon(e)$ be any ball about $e = e_G$. By Lemma 2, for some Polish space P we may write $H_0 = K_0 = G(P) \cup N$ with N meagre and G an irreducible continuous map, i.e. with the property that $G(A)$ is heavy for each non-empty open subset A of P. Without loss of generality, $G_0 = G(P)$ is dense. (Otherwise expand N to an $\mathcal{F}_\sigma$. Then $G_0\backslash N$ is a $\mathcal{G}_\delta$ and comeagre; then G being a Baire space $G_0\backslash N$ is dense.) We put $P_0 = Q_0 = P$ and without loss of generality assume that $\mathrm{diam}(P) = 1$.

Pick U_0 open with $x \in U_0 \subseteq\mathrm{cl}(H_0x)$. Let $y \in U_0$; we will show that $y = gx$ for some $g \in H_0^{-1}H_0$.

We work inductively. We begin with the (rather long) first step in the induction. We then set out the general inductive step (which follows the same pattern). What follows is a 'back and forth' argument, performed within successively smaller sub-orbits of the orbit H_0x of x and sub-orbits of the orbit K_0y with the intention of showing that the limiting sub-orbits meet (i.e. $hx = ky$ for some $h \in H_0$ and $k \in K_0$, so that $g = k^{-1}h \in H_0^{-1}H_0$).

Put $x_0 = x$ and $y_0 = y$. We thus have

$$x \in U_0 \subseteq \mathrm{cl}(H_0x) \text{ with } y \in U_0,$$

where the diameter of U_0 is less than 1 without loss of generality.

We first work within the y orbit: by weak microaction pick open V_0 with diameter less than $1 = 2^0$ such that $y_0 \in V_0 \subseteq\mathrm{cl}(K_0y_0)$. Thus

$$x \in U_0 \subseteq \mathrm{cl}(H_0x) \text{ with } y \in U_0 \text{ and } y_0 \in V_0 \subseteq \mathrm{cl}(K_0y_0). \qquad \text{(ind-0)}$$

Combining the information about y_0, we have $y_0 \in U_0 \cap V_0$, so that $U_0 \cap V_0$ is non-empty open; furthermore $U_0 \cap V_0 \subseteq U_0 \subseteq\mathrm{cl}(H_0x)$. So the x orbit H_0x in particular meets $U_0 \cap V_0$, i.e. there is $h_1' \in H_0$ such that

$$x_1' := h_1'x_0 \in U_0 \cap V_0 \subseteq V_0. \qquad (1')$$

As $x_1' = h_1' x_0 \in U_0 \cap V_0$, we have $h_1' \in W_0' = \xi^{-1}(U_0 \cap V_0) \cap H_0$, so this open set is non-empty. (Recall that ξ is continuous.) As $G_0 = G(P)$ is dense and heavy, $W_0' \cap G_0$ is non-empty, and so by continuity $G^{-1}(W_0')$ is a non-empty open subset of P_0. There is thus a closed subset P_1, with $\text{diam}(P_1) < \text{diam}(P_0)/2 = 2^{-1}$, such that $G(P_1) \subseteq W_0'$ and $G(P_1)$ is heavy. So for some non-empty open $W_0 \subseteq W_0'$ the set $G(P_1)$ is dense and heavy on W_0. For $h_1 \in W_0 \subseteq W_0' \subseteq \xi^{-1}(U_0 \cap V_0)$ we have $\xi(h_1) = h_1(x) \in U_0 \cap V_0$, so

$$x_1 := h_1 x_0 \in U_0 \cap V_0 \subseteq V_0. \tag{1}$$

Now there exists a ball H_1 about e of diameter at most $\varepsilon/2$ such that $H_1 h_1 \subseteq W_0 \subseteq H_0$, and $G_1 := G(P_1)$ is dense and heavy on $H_1 h_1$.

Now we work in the orbit of $h_1 x_0$: by weak microaction, for some U_1 open and of diameter less than 2^{-1} in X,

$$x_1 = h_1 x_0 \in U_1 \subseteq \text{cl}(H_1 h_1 x_0). \tag{2'}$$

By (1) and (2′), $x_1 \in U_1 \cap V_0 \subseteq V_0 \subseteq \text{cl}(K_0 y_0)$. So here too the orbit $K_0 y_0$ meets $U_1 \cap V_0$ and so there is $k_1' \in K_0$ such that

$$y_1' := k_1' y_0 \in U_1 \cap V_0.$$

Write $\tilde{\xi} := \xi_y$. Since $k_1' \in \tilde{W}_0' = \tilde{\xi}^{-1}(U_1 \cap V_0) \cap K_0$, this open set is non-empty. As G_0 is dense and heavy in G, $\tilde{W}_0' \cap G_0$ is non-empty and so $G^{-1}(\tilde{W}_0')$ is a non-empty subset of Q_0. There is thus a closed subset Q_1, with $\text{diam}(Q_1) < \text{diam}(Q_0)/2 = 2^{-1}$, such that $G(Q_1) \subseteq \tilde{W}_0'$ and $G(Q_1)$ is heavy. So for some non-empty open $\tilde{W}_0 \subseteq \tilde{W}_0'$ the set $G(Q_1)$ is dense and heavy on $\tilde{W}_0$. For $k_1 \in \tilde{W}_0 \subseteq \tilde{W}_0' \subseteq \tilde{\xi}^{-1}(U_1 \cap V_0)$ we have $\tilde{\xi}(k_1) = k_1(y) \in U_1 \cap V_0$, so

$$y_1 := k_1 y_0 \in U_1 \cap V_0 \subseteq V_0. \tag{2}$$

Now there exists a ball K_1 about e of diameter less than $\varepsilon/2$ such that $K_1 k_1 \subseteq \tilde{W}_0 \subseteq K_0$, and $\tilde{G}_1 := G(Q_1)$ is dense and heavy on $K_1 k_1$.

Working again in the y_1 orbit: by weak microaction, for some V_1 open with diameter less than 2^{-1}

$$y_1 \in V_1 \subseteq \text{cl}(K_1 y_1).$$

This completes the first step in the induction, as we now have closed sets P_1, Q_1 in P, open neighbourhoods H_1, K_1 of the identity in G of diameter less than $\varepsilon/2$, points h_1, k_1 in G, points x_1, y_1 in X, and open sets U_1, V_1 in X with diameter less than $1/2$ such that

$$x_1 \in U_1 \subseteq \text{cl}(H_1 x_1) \text{ with } y_1 \in U_1 \text{ and } y_1 \in V_1 \subseteq \text{cl}(K_1 y_1). \tag{ind-1}$$

and

$$\begin{aligned} G_1 &:= G(P_1) \text{ is dense and heavy on } H_1 h_1, \\ \tilde{G}_1 &:= G(Q_1) \text{ is dense and heavy on } K_1 k_1. \end{aligned}$$

In general suppose that we now have closed sets P_1, Q_1 in P, open neighbourhoods H_{n-1}, K_{n-1} of the identity in G of diameter less than $\varepsilon 2^{-(n-1)}$, points $h_1, ..., h_{n-1}$ and $k_1, ..., k_{n-1}$ in G, points $x_1, .., x_{n-1}$ and $y_1, ..., y_{n-1}$ in X, and open sets U_{n-1}, V_{n-1} in X with diameter less than $2^{-(n-1)}$ such that

$$x_{n-1} \in U_{n-1} \subseteq \mathrm{cl}(H_{n-1}x_{n-1}) \text{ with } y_{n-1} \in U_{n-1} \text{ and } y_{n-1} \in V_{n-1} \subseteq \mathrm{cl}(K_{n-1}y_{n-1}). \tag{ind-$(n-1)$}$$

and

$$\begin{aligned} G_{n-1} &: = G(P_{n-1}) \text{ is dense and heavy on } H_{n-1}\eta_{n-1}, \\ \text{where } \eta_{n-1} &: = h_{n-1} \cdot ... \cdot h_1 \cdot h_0, \\ \tilde{G}_{n-1} &: = G(Q_{n-1}) \text{ is dense and heavy on } K_{n-1}\kappa_{n-1}, \\ \text{where } \kappa_{n-1} &: = k_{n-1} \cdot ... \cdot k_1 \cdot k_0, \end{aligned}$$

where $\mathrm{diam}(P_{n-1}) < 2^{-(n-1)}$ and likewise $\mathrm{diam}(P_{n-1}) < 2^{-(n-1)}$.

Then $y_{n-1} \in U_{n-1} \cap V_{n-1} \subseteq U_{n-1} \subseteq \mathrm{cl}(H_{n-1}x_{n-1})$, so as above there is $h'_n \in H_{n-1}$ such that

$$x'_n := h'_n x_{n-1} \in V_{n-1}. \tag{$1' : n$}$$

Write $\xi_{n-1} := \xi_{x(n-1)}$. As $x'_n = h'_n x_{n-1} \in U_{n-1} \cap V_{n-1}$ we have $h'_n \eta_{n-1} \in W'_{n-1} = \xi_{n-1}^{-1}(U_{n-1} \cap V_{n-1}) \cap H_{n-1}\eta_{n-1}$, and this open set is non-empty (as ξ_{n-1} is continuous). As $G_{n-1} := G(P_{n-1})$ is dense and heavy on W_{n-1} we have that $W'_{n-1} \cap G_{n-1}$ is non-empty and so $G^{-1}(W'_{n-1})$ is a non-empty subset of P_{n-1}. There is thus a closed subset P_n, with $\mathrm{diam}(P_n) < \mathrm{diam}(P_{n-1})/2 < 2^{-n}$, such that $G(P_n) \subseteq W'_{n-1}$ and $G(P_n)$ is heavy. So for some non-empty open $\tilde{W}_0 \subseteq \tilde{W}'_0$ the set $G(P_n)$ is dense and heavy on W_{n-1}. For $h_n \in W_{n-1} \subseteq W'_{n-1} \subseteq \xi_{n-1}^{-1}(U_{n-1} \cap V_{n-1})$ we have $\xi_{n-1}(h_n) = h_n(x_{n-1}) = h_n \eta_{n-1} \in U_{n-1} \cap V_{n-1}$, so

$$x_n := h_n x_{n-1} \in U_{n-1} \cap V_{n-1} \subseteq V_{n-1}. \tag{$1 : n$}$$

Now there exists a ball H_n about e of diameter at most $\varepsilon 2^{-n}$ such that $H_n h_n \eta_{n-1} \subseteq W_{n-1} \subseteq H_{n-1}\eta_{n-1}$, and one has that $G(P_n)$ dense and heavy on $H_n \eta_n$.

Now we work in the orbit of $h_n x_{n-1}$: by weak microaction, for some U_n open and of diameter less than 2^{-n} in X,

$$x_n = h_n x_{n-1} \in U_n \subseteq \mathrm{cl}(H_n h_n x_{n-1}). \tag{$2' : n$}$$

By (1) and (2′), $x_n \in U_n \cap V_{n-1} \subseteq V_{n-1} \subseteq \mathrm{cl}(K_{n-1}y_{n-1})$. So here too the orbit $K_{n-1}y_{n-1}$ meets $U_n \cap V_{n-1}$ and so there is $k'_n \in K_{n-1}$ such that

$$y'_n := k'_n y_{n-1} \in U_n \cap V_{n-1}.$$

Write $\tilde{\xi}_{n-1} := \xi_{y(n-1)}$. Since $k'_n \in \tilde{W}'_{n-1} := \tilde{\xi}_{n-1}^{-1}(U_n \cap V_{n-1}) \cap K_{n-1}\kappa_{n-1}$, this open set is non-empty. As $\tilde{G}_{n-1}$ is dense and heavy in $K_{n-1}\kappa_{n-1}$, $\tilde{W}'_{n-1} \cap \tilde{G}_{n-1}$ is non-empty and so $G^{-1}(\tilde{W}'_{n-1})$ is a non-empty subset of Q_{n-1}. There is thus a closed subset Q_n, with $\mathrm{diam}(Q_n) < \mathrm{diam}(Q_{n-1})/2 < 2^{-n}$, such that $G(Q_n) \subseteq \tilde{W}'_{n-1}$ and $G(Q_n)$ is heavy. So for some non-empty open $\tilde{W}_{n-1} \subseteq \tilde{W}'_{n-1}$ the set $G(Q_n)$ is dense and heavy on $\tilde{W}_{n-1}$. For $k_n \in \tilde{W}_{n-1} \subseteq \tilde{W}'_{n-1} \subseteq \tilde{\xi}_{n-1}^{-1}(U_n \cap V_{n-1})$ we have $\tilde{\xi}_{n-1}(k_n) = k_n(y_{n-1}) \in U_n \cap V_{n-1}$, so

$$y_n := k_n y_{n-1} \in U_n \cap V_{n-1} \subseteq V_{n-1}. \qquad (2:n)$$

Now there exists a ball K_n about e of diameter less than $\varepsilon 2^{-n}$ such that $K_n \kappa_n \subseteq \tilde{W}_{n-1} \subseteq K_{n-1}\kappa_{n-1}$, and $\tilde{G}_n := G(Q_n)$ dense and heavy on $K_n\kappa_n$.

Working again in the y_n orbit: by weak microaction, for some V_n open with diameter less than 2^{-n}

$$y_n \in V_n \subseteq \mathrm{cl}(K_n y_n).$$

This completes the general induction step, as we now have sets H_n, K_n neighbourhoods of the identity in G, points x_n, y_n and sets U_n, V_n in X such that $x_n \in U_n$ with $y_n \in U_n$ and $y_n \in V_n \subseteq \mathrm{cl}(K_n y_n)$.

By Lemma 1, the products $\eta_n = h_n h_{n-1}...h_1$ and $\kappa_n = k_n k_{n-1}...k_1$ are convergent sequences, with limit say h and k resp. Thus $h \in \mathrm{cl}(H_0)$ and $k \in \mathrm{cl}(K_0) = \mathrm{cl}(H_0)$, which are closed balls. Thus $k^{-1}h \in \mathrm{cl}(H_0)^{-1}\mathrm{cl}(H_0)$. But sets of the form $B^{-1}B$ with B a closed ball are a base for the topology at e, so $k^{-1}h$ is as small as we wish.

For fixed x the map $(g, x) \to gx$ is continuous and $h_n h_{n-1}...h_1 \to h$, so $x_n = h_n h_{n-1}...h_1 x \to hx$, and likewise $y_n = k_n k_{n-1}...k_1 y \to ky$. (For instance, if $G \subseteq Auth(X)$ has the supremum metric derived from d^X, then $d^X(h_n h_{n-1}...h_1 x, hx) \leq \hat{d}(h_n h_{n-1}...h_1, h) \to 0$.)

But x_n and y_n have a common limit (since $x_n, y_n \in U_n$ and d^X-diam$(U_n) \to 0$), so $hx = ky$. Thus $y = k^{-1}hx$, as promised. □

Theorems 2 and 3 now yield the Main Theorem.

References

[AL] J. M. Aarts and D. J. Lutzer, *Completeness properties designed for recognizing Baire spaces*. Dissertationes Math. (Rozprawy Mat.) 116 (1974), 48.

[Anc] F. D. Ancel, *An alternative proof and applications of a theorem of E. G. Effros*, Michigan Math. J. 34 (1987), no. 1, 39–55.

[BOst-TI] N. H. Bingham and A. J. Ostaszewski, *Topological regular variation: I. Slow variation*, Topology App. 157 (2010), 1999-2013.

[BOst-N] N. H. Bingham and A. J. Ostaszewski, *Normed versus topological groups: dichotomy and duality*, Dissertationes Math., to appear.

[Bou] A. Bouziad, *Continuity of separately continuous group actions in p-spaces*, Topology & its App. 71 (1966), 119-124.

[BrGo] J. C. Bradford and C. Goffman, *Metric spaces in which Blumberg's theorem holds*, Proc. Amer. Math. Soc. 11 1960 667–670.

[ChMa] J. J. Charatonik and T. Maćkowiak, *Around Effros' theorem*, Trans. Amer. Math. Soc. 298.2 (1986), 579–602.

[Eff] E. G. Effros, *Transformation groups and C^*-algebras*. Ann. of Math. (2) 81 1965 38–55.

[Eng] R. Engelking, *General topology*, Heldermann Verlag, 1989.

[Frol] Z. Frolík, *Generalizations of the $\mathcal{G}_\delta$-property of complete metric spaces*, Czechoslovak Math. J 10 (85) (1960) 359–379.

[HJ] J. Hoffmann-Jørgensen, *Automatic continuity*, Section 3 of [THJ].

[Hoh] A. Hohti, *Another alternative proof of Effros' theorem*, Top. Proc. 12 (1987), 295-298.

[Jay-Rog] J. Jayne and C. A. Rogers, *Analytic sets,* Part 1 of [Rog2].

[Kech-T] A. S. Kechris, *Topology and descriptive set theory*, Topology Appl. 58 (1994), no. 3, 195–222.

[Lev] S. Levi, *On Baire cosmic spaces*, General topology and its relations to modern analysis and algebra, V (Prague, 1981), 450–454, Sigma Ser. Pure Math., 3, Heldermann, Berlin, 1983.

[Loy] Loy, Richard J. *Multilinear mappings and Banach algebras.* J. London Math. Soc. (2) 14 (1976), no. 3, 423–429.

[Mich91] E. Michael, *Almost complete spaces, hypercomplete spaces and related mapping theorems*, Topology Appl. 41 (1991), no. 1-2, 113–130.

[Mil-Ost] H. I. Miller and A. J. Ostaszewski, *Group-action and Shift-compactness*, preprint.

[Ost-AH] A. J. Ostaszewski, *Analytically heavy spaces: Analytic Baire and analytic Cantor theorems*, preprint.

[Ost-LBIII] A.J. Ostaszewski, *Beyond Lebesgue and Baire III: analyticity and shift-compactness*, preprint.

[Ost-Joint] A. J. Ostaszewski, *Continuity in groups: one-sided to joint*, preprint.

[Ost-AB] A. J. Ostaszewski, *Analytic Baire spaces*, preprint.

[Ost-E] A. J. Ostaszewski, *On the Effros Open Mapping Principle*, preprint.

[THJ] F. Topsøe, J. Hoffmann-Jørgensen, *Analytic spaces and their applications*, Part 3 of [Rog2].

[Rog2] C. A. Rogers, J. Jayne, C. Dellacherie, F. Topsøe, J. Hoffmann-Jørgensen, D. A. Martin, A. S. Kechris, A. H. Stone, *Analytic sets,* Academic Press, 1980.

[vM1] J. van Mill, *A note on the Effros Theorem*, Amer. Math. Monthly 111.9 (2004), 801-806.

[vM2] J. van Mill, *Analytic groups and pushing small sets apart,* Trans. Amer. Math. Soc. 361 (2009), no. 10, 5417–5434.

Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE
a.j.ostaszewski@lse.ac.uk