On the Effros Open Mapping Principle

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Abstract

We give a new short proof of an improved version of the Effros Open Mapping Principle, applicable to absolutely analytic normed groups (which includes complete metrizable topological groups, separable or otherwise) via a shift-compactness theorem (also with a short proof), and establish a connection between microtransitivity and two sequential properties.

Keywords: Open Mapping Theorem, absolutely analytic sets, almost completeness, topological groups, group-norms, non-commutative groups, shift-compactness, crimping property.

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1 Introduction

Throughout this paper, without further comment, all spaces considered will be metrizable, but not necessarily separable.

We are concerned with generalizing the classic Effros Theorem ([Eff]) – see Theorem E on 'micro-transitivity' (definition below) – which in the setting of a completely metrizable *separable* and *topological* group, acting continuously and transitively on a non-meagre space X, asserts that the map $\varphi_x : g \to g(x)$ is for each x an open mapping. Here we extend this theorem to apply also beyond the separable realm (of metric spaces). Our approach relies on the non-separable theory of metrizable (absolutely) analytic spaces, recalled briefly below, and on two inter-related results involving sequential properties which have only recently come to light. Of these two the first, the *crimping property*, is motivated by the recently developed topological theory of regular variation (see e.g. [BOst-TRI]), the second is related to the notion of *shift-compactness* (see [BOst-N], [Ost-LBIII], or the survey [Ost-Sur]), but is here formulated in the language of group-actions (though such general actions can be represented as actions of subgroups on a group, cf. [Ost-knit]). The two bear a duality-like resemblance, as one involves a convergent sequence $\{g_n(x)\}$ and the other a sequence $\{g(x_n)\}$. Our theme is that, in an appropriate setting, all three results are, if not equivalent, then almost so (see Th. EC and the remarks in § 3.5).

We recall first some definitions from general topology, before turning to ones that are group related. We refer to [Eng] for general topological usage (though we prefer 'meagre' as a term). We say that a subspace S of a metric space X has a *Souslin-$\mathcal{F}(X)$ representation* if there is a 'determining' system $\langle F(i|n)\rangle := \langle F(i|n) : i \in \mathbb{N}^{\mathbb{N}}\rangle$ of sets in $\mathcal{F}(X)$ (the closed sets) with

$$S = \bigcup\nolimits_{i\in I} \bigcap\nolimits_{n\in\mathbb{N}} F(i|n), \text{ where } I = \mathbb{N}^{\mathbb{N}}$$

and $i|n$ denotes $(i_1, ..., i_n)$. We will say that a topological space is an (absolutely) *analytic* space, if it is embeddable as a Souslin-$\mathcal{F}$ set in its own metric completion; in particular, in a complete metric space $\mathcal{G}_\delta$-subsets (being $\mathcal{F}_{\sigma\delta}$) are analytic. We call a Hausdorff space *almost analytic* if it is analytic modulo a meagre set. Similarly, a space X' is absolutely $\mathcal{G}_\delta$, or an *absolute-$\mathcal{G}_\delta$*, if X' is a $\mathcal{G}_\delta$ in all spaces X containing X' as a subspace. This is equivalent to complete metrizability in the narrowed realm of metrizable spaces [Eng, Th. 4.3.24] (and to topological/Čech completeness in the narrowed realm of completely regular spaces – [Eng, §3.9].) So a metrizable absolute-$\mathcal{G}_\delta$ is analytic. A metric space is *almost complete* if it contains a dense absolute $\mathcal{G}_\delta$. The notion of 'almost completeness' is due to Frolík in [Frol-60] (but its name to Michael [Mich91] – see also [AL] and [BOst-N]).

In general under a continuous mapping an analytic space need not have an analytic image. If, additionally, the continuous map is base-σ-discrete, as defined below, the image is analytic ([Han-74] Cor. 4.2); this is the standard assumption for preservation of analyticity and holds automatically in the separable realm. Consequently, we will assume that when a group G acts on a space X each point evaluation map $\varphi_x : g \to g(x)$ is not only continuous but also base-σ-discrete. However, [Han-74] Ex. 3.12 shows that, for $D = D(2^\omega)$ the discrete space of cardinality the continuum, the projection from $D\times[0, 1] \to [0, 1]$, is an open mapping that is not base-σ-discrete (by reference to the closed discrete graph of a bijection between D and $[0, 1]$). So whilst our assumption conforms with the standard, it is not clear that it is a necessary condition for the non-separable extension of the Effros Theorem. To define this key concept recall that for an (indexed) family $\mathcal{B} := \{B_t : t \in T\}$:

(i) $\mathcal{B}$ is *index-discrete* in the space X (or just *discrete* when the index set T is understood) if every point in X has a neighbourhood meeting the sets B_t for at most one $t \in T$,

(ii) $\mathcal{B}$ is *σ-discrete* if $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where each set $\mathcal{B}_n$ is discrete as in (i), and (iii) $\mathcal{B}$ is a *base for* $\mathcal{A}$ if every member of $\mathcal{A}$ may be expressed as the union of a subfamily of $\mathcal{B}$. For $\mathcal{T}$ a topology (the family of all open sets) with $\mathcal{B} \subseteq \mathcal{T}$ a base for $\mathcal{T}$, this reduces to $\mathcal{B}$ being simply a (topological) *base*.

Definitions. 1. Call $f : X \to Y$ *base-σ-discrete* (or *co-σ-discrete*) if the image under f of any discrete family in X has a σ-discrete base in Y.
2. ([Han-74], §2) An indexed family $\mathcal{A} := \{A_t : t \in T\}$ is *σ-discretely decomposable* if there are discrete families $\mathcal{A}_n := \{A_{tn} : t \in T\}$ such that $A_t = \bigcup_n A_{tn}$ for each t.
3. ([Mich82] Def. 3.3) Call $f : X \to Y$ *index-σ-discrete* if the image under f of any discrete family $\mathcal{E}$ in X is σ-discretely decomposable in Y. (Note $f(\mathcal{E})$ is regarded as indexed by $\mathcal{E}$ so could be discrete without being index-discrete.)

Index-σ-discrete maps are base-σ-discrete ([Han-74] Prop. 3.7(i)); the latter combined with continuity preserves analyticity, as earlier mentioned. In many circumstances it is easier to work with index-σ-discrete maps: see [Ost-AB] for a brief discussion of this point and for relations with the automatic continuity of homomorphisms (noted earlier in [N]). Furthermore, if $f : X \to Y$ is injective and *closed-analytic* i.e. carries closed sets to analytic sets (or, alternatively *open-analytic*, mutatis mutandis), then f is base-σ-discrete – in fact index-σ-discrete ([Han-74] Prop. 3.14). On the other hand, if f is surjective and closed with Y metrizable, then f is base-σ-discrete ([Han-74] Prop. 3.10). So if the group action is such that each φ_x is open-analytic and *injective* (as when a vector space acts on itself), then each φ_x is index- and so base-σ-discrete. Evidently, if all evaluation maps φ_x are continuous and base-σ-discrete and A is an open subset of an analytic group G, acting on X, then A is analytic, and so $\varphi_x(A) = Ax$ is analytic; thus the point evaluations are open-analytic. So our additional hypothesis is not far from demanding that point evaluations be open-analytic.

Example. For $L : X \to Y$ a surjective, continuous linear map between normed vector spaces (regarded as additive groups) define $\varphi(x, y) := L(x) + y$ (cf. vanMill [vM1]). This is a jointly continuous action of X on Y which is transitive, but for which point evaluation $\varphi_y(x)$ is not necessarily injective (as $L(x) + y = L(z) + y$ iff $L(x - z) = 0$). If X, Y are Banach spaces, then L is open, by the classical Open Mapping Theorem. So if $\mathcal{V}$ is discrete and open then, as $\mathcal{V}y = \{L(V) + y : V \in \mathcal{V}\} = L(\mathcal{V}) + y$, the open family $L(\mathcal{V})$

and so $\mathcal{V}y$ has a σ-discrete base (since Y is metrizable). Evidently the point evaluation φ_y is base-σ-discrete in relation to discrete *open* families, but far more interesting is the question:
Are surjective, continuous linear maps between Banach spaces base-σ-discrete? If yes, then in particular so would be the evaluation maps φ_y of this example.

Definition (Normed groups). 1. For T an *algebraic* group with neutral element e, say that $||\cdot|| : T \to \mathbb{R}_+$ is a *group-norm* ([BOst-N]) if the following properties hold:

(i) *Subadditivity* (Triangle inequality): $||st|| \leq ||s|| + ||t||$;

(ii) *Positivity*: $||t|| > 0$ for $t \neq e$ and $||e|| = 0$;

(iii) *Inversion* (Symmetry): $||t^{-1}|| = ||t||$.

Then $(T, ||.||)$ is called a *normed-group.*

2. The group-norm generates a right and a left *norm topology* via the right-invariant and left-invariant metrics $d_R^T(s,t) := ||st^{-1}||$ and $d_L^T(s,t) := ||s^{-1}t|| = d_R^T(s^{-1}, t^{-1})$. In the right norm-topology the right shift $\rho_t(s) := st$ is a uniformly continuous homeomorphism, since $d_R(sy, ty) = d_R(s,t)$, so in particular the group is a right topological group; likewise in the left norm-topology the left shift $\lambda_s(t) = st$ is a uniformly continuous homeomorphism. Since $d_L^T(t,e) = d_L^T(e, t^{-1}) = d_R^T(e,t)$, convergence at e is identical under either topology. In the absence of a qualifier, the 'right' norm topology is to be understood. One may refer to whichever topology is appropriate, since despite their differences, they are homeomorphic. Thus one may say, for instance, that separability is a *norm property*, as separability of either norm topology implies that of the other. It is in this sense that below we make topological assumptions about norm properties.

3. See [Ost-LBIII] for a characterization of almost complete normed groups.

Recall that a normed group G *acts continuously* on X if there is a continuous mapping $\varphi : G \times X \to X$ such that $\varphi(e,x) = x$ and $\varphi(gh, x) = \varphi(g, \varphi(h,x))$. (Despite the right norm-topology being implicitly used here, in view of applications, it is sometimes also convenient to simultaneously work with G under the topology generated by the left invariant metric $d_L^G(g,h) := ||g^{-1}h||$.) Thus $g : x \to \varphi(g,x)$ is a continuous self-map of X with a continuous inverse, and so is an autohomeomorphism, denoted $g(\cdot)$.

The action φ is *separately continuous* if $g : x \to \varphi(g,x)$ is continuous and $\varphi_x : g \to \varphi(g,x)$ is continuous; in such circumstances:

(i) the elements of the group G are autohomeomorphisms of X (as before), and
(ii) point-evaluation of these homeomorphisms, $\varphi_x(g) = g(x)$, is continuous. Indeed, these two properties (and the fact that G is a group under composition) are all that we demand of the actions below. However, in certain situations joint continuity of action is implied by separate (see [Bou], and the comments in §3.1).

The action is *transitive* if for any x, y in X there is $g \in G$ such that $g(x) = y$. For later purposes (§ 2 and 3), say that the action of G on X is *weakly micro-transitive* if for $x \in X$ and each neighbourhood (nhd) A of e_G the set

$$\mathrm{cl}(Ax) = \mathrm{cl}\{ax : a \in A\}$$

has x as an interior point (in X). Note that the nhds of e_G under either norm topology are the same, so the weak microaction property is a 'norm property' of G. The action is *micro-transitive* ('transitive in the small' – for details see [vM1]) if for $x \in X$ and each nhd A of e_G the set

$$Ax = \{ax : a \in A\}$$

is a nhd of x. This is again a norm property. Note that this second stronger property implies that Ux is open for U open in G (i.e. that here each φ_x is an open mapping). Indeed, if $y \in Ux$, put $y = ux$ with $u \in U$; then Uu^{-1} is an open nhd of e_G (in the right norm-topology, in which the right shift $g \to gu^{-1}$ is a homeomorphism), and so $Ux = (Uu^{-1})y$ is a nhd of y. We refer to Ax as an x *orbit* (the A-orbit of x).

Definition ([BOst-TRI]). For a normed group G acting on X say that X has the *crimping property* (property C for short) w.r.t. G if, for each $x \in X$ and each sequence $\{x_n\} \to x$, there exists in G a sequence $\{g_n\} \to e_G$ with $g_n(x) = x_n$. See [BOst-TRI] for a discussion of this property and its relation with strong local homogeneity. We note that [ChCh] study a variant of the crimping property appropriate to the setting of a semigroup of surjective self-maps.

Example. Recall that $Auth(X)$ denotes the autohomeomorphisms of a metric space X with metric d^X; this is a group under composition. $\mathcal{H}(X)$ comprises those $h \in Auth(X)$ that have bounded norm:

$$||h|| := \sup_{x \in X} d^X(h(x), x) < \infty.$$

If $\mathcal{G} \subseteq \mathcal{H}(X)$ is a (normed) subgroup of the bounded autohomeomorphisms of X, say that X has the *crimping property* w.r.t. $\mathcal{G}$ if X has the crimping property w.r.t. to the natural action $(g, x) \to g(x)$ from $\mathcal{G} \times X \to X$. (This action is continuous relative to the left or right norm topology on $\mathcal{G}$ – cf. [Dug] XII.8.3, p. 271.)

We write the action of $g \in G$ on $x \in X$ as $g(x)$. In particular recall that $e_G(x) = x = id_X(x)$ and that for $g \in G$ the action of g^{-1} (inverse in the group G) on x is the inverse homeomorphism to $g(\cdot)$. We begin with a statement of a variant of the Effros Theorem. The classical version has G a Polish group; van Mill's version [vM1] requires the group G to be analytic (i.e. the continuous image of some Polish space, cf. [Jay-Rog], [Kech-Cl]). The following version improves the version given in [Ost-ACE], where the group is almost complete. (The two cited sources taken together cover the literature.)

Theorem E (Effros Theorem – Baire variant). *Let the normed group G have separately continuous and transitive action on X with all point-evaluation maps φ_x base-σ-discrete. Suppose that under either norm topology G is analytic and Baire and that X is non-meagre, then the action of G is micro-transitive. That is, for U an open neighbourhood of e_G and for arbitrary $x \in X$ the set $Ux := \{u(x) : u \in U\}$ is a neighbourhood of x, so that in particular the point-evaluation maps $g \to g(x)$ are open for each x.*

We shall say that the Effros property (or *property E*) holds for the group G acting on X if the action is micro-transitive. In his version of this theorem van Mill [vM1] proves that if the group G acting on a non-meagre space is analytic, then property E holds, and also gives an example of a meagre analytic G that has property E. We refer to this in §2.3.

We shall prove in § 2.1:

Theorem EC. *The Effros property holds for a group G acting on X iff X has the Crimping property w.r.t. G.*

Before stating the next result we need a definition.

For a subgroup $\mathcal{G} \subseteq Auth(X)$ say that X is *$\mathcal{G}$-shift-compact* (or, shift-compact under $\mathcal{G}$) if for any convergent sequence $x_n \to x_0$, any open subset U in X and any Baire set T co-meagre in U, there is $g \in \mathcal{G}$ such that

$g(x_n) \in T \cap U$ along a subsequence. Say that the space is *shift-compact* if it is $\mathcal{H}(X)$-shift-compact. In such a space, any Baire non-meagre set is locally co-meagre (co-meagre on open sets) in view of the following.

Proposition B1. *For any subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, if X is $\mathcal{G}$-shift-compact, then X is Baire.*

Proof. We argue as in [vM2] Prop 3.1 (1). Suppose otherwise; then X contains a non-empty meagre open set. By Banach's localization principle (for which see [Jay-Rog] p. 42, or [Kel] Th. 6.35), the union of all such sets is a largest open meagre set M, and is non-empty. Thus $X \backslash M$ is a co-meagre Baire set. For any $x \in M$ the constant sequence $x_n \equiv x$ is convergent and, since $X \backslash M$ is co-meagre in X, there is $g \in G$ with $g(x) \in X \backslash M$. But, as g is a homeomorphism, $g(M)$ is a non-empty open meagre set, so is contained in M, implying $g(x) \in M$, a contradiction. □

By a similar argument, one has:

Proposition B2 (cf. [vM2]). *If X is non-meagre and G acts transitively on X, then X is Baire.*

Proof. As above, refer again to M, the union of all meagre open sets, which, being meagre, has non-empty complement. For x_0 in this complement and any non-empty open U pick $u \in U$ and $g \in G$ such that $g(x_0) = u$. Now, as g is continuous, $g^{-1}(U)$ is a nhd of x_0, so is non-meagre, since every nhd of x_0 is non-meagre. But g is a homeomorphism, so $U = g(g^{-1}(U)$ is non-meagre. So X is Baire, as every non-empty open set is non-meagre. □

Remark. For a generalization see [HJ, Prop. 2.2.3].

We shall show in § 2.3 that the Effros Theorem is implied by the shift-compactness theorem below, and depending on context implies it (again see the remarks in §3.5). The proofs of S below and of the implication of E by S (especially the latter) is notable through being very short. Unlike the proof of the Effros Theorem attributed to Becker in [Kech-T, Th. 3.1], this one does not employ the Kuratowski-Ulam Theorem (the Category version of the Fubini Theorem). Our focus on various shift-compactness theorems derives from its close affinity with the literature of 'generic' automorphisms (for which see [M], also described in the introduction to [Ost-LBIII]) and from a multitude of its applications, for which see e.g. [BOst-N], and these offering a unifying sequential compactness-like, or combinatorial, perspective on measure-category duality and on other, apparently unrelated, problems.

Theorem S (Shift-compactness Theorem). *For T a Baire non-meagre subset of a metric space X and G a normed group, analytic and Baire in its right norm topology (e.g. almost complete and non-meagre in the norm topology), acting separately continuously and transitively on X, with all point-evaluation maps φ_x base-σ-discrete:*

for every convergent sequence x_n with limit x and any Baire non-meagre $A \subseteq G$ with $e_G \in A$ such that $Ax \subseteq T$, there are $\alpha \in A$ and an integer N such that $\alpha x \in T$ and

$$\{\alpha(x_n) : n > N\} \subseteq T.$$

2 Proofs

2.1 Proof that E $\iff$ C

In [BOst-N] Th. 3.15 we showed that if the Effros property holds in regard to the action of a group G on X, then X has the crimping property w.r.t. G. We recall the argument, as it is short. For each n, take $U = B^G_{1/n}(e_G)$; then the set $Ux := \{u(x) : u \in U\}$ is an open nhd of x, and so there exists $h_{n,m} \in U$ with $h_{n,m}(x) = x_m$ for all m large enough, say for all $m > m(n)$. W.l.o.g. we may assume that $m(1) < m(2) < ....$. Put $h_m := e_G$ for $m < m(1)$, and for $m(k) \leq m < m(k+1)$ take $h_m := h_{k,m}$. Then $h_m \in B^G_{1/k}(e_G)$, so h_m converges to e_G and $h_m(e_G) = x_m$.

For the converse, suppose that the Effros property fails for G acting on X. Then for some open nhd U of e_G and some $x \in X$ the set $Ux := \{u(x) : u \in U\}$ is not an open nhd of x. So for each n there is a point $x_n \in B_{1/n}(x)\backslash Ux$. As x_n converges to x there are homeomorphisms h_n converging to the identity such that $h_n(x) = x_n$. As U is an open nhd of e_G and since h_n converges to e_G there is N such that $h_n \in U$ for $n > N$. In particular, for any $n > N$ one has $h_n(x) = x_n \in Ux$, a contradiction.

2.2 Proof of S

For clarity we begin by proving a simplified version of Th. S, and then we prove Th. S itself. We will need to refer twice to the following result, which generalizes one that for separable groups G is usually a first step in proving the weakly microtransitive variant of the classical Effros Theorem (cf. Ancel [Anc] Lemma 3, cf. [Ost-ACE] Th. 2). Indeed, one may think of it as giving a form of 'very weak microtransitivity':

Lemma. *Let G be a normed group, acting transitively on a non-meagre space X, with each point evaluation map $\varphi_x : g \to g(x)$ base-σ-discrete. Then for each non-empty open U in G and each $x \in X$ the set Ux is non-meagre in X.*

Proof. We work in the right norm topology first. Suppose that $u \in U$ and so without loss of generality assume that $U = B_\varepsilon(u) = B_\varepsilon(e_G)u$ (for some $\varepsilon > 0$); then put $y := gx$ and $W = B_\varepsilon(e_G)$. Then $Ux = Wy$. Next work, exceptionally, in the left norm topology (for which $W = B_\varepsilon(e_G)$ is a nhd of e_G); as each set hW for $h \in G$ is now open (since now the left shift $g \to hg$ is a homeomorphism), the open family $\mathcal{W} = \{gW : g \in G\}$ covers G. As G is metrizable (and so has a σ-discrete base) the cover $\mathcal{W}$ has a σ-discrete refinement say $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ with each $\mathcal{V}_n$ discrete. Put $X_n := \bigcup\{Vy : V \in \mathcal{V}_n\}$, then $X = \bigcup_{n\in\mathbb{N}} X_n$, as $X = Gy$ and so X_n is non-meagre for some n, for $n = N$ say. Since φ_y is base-σ-discrete, $\{Vy : V \in \mathcal{V}_N\}$ has a σ-discrete base, say $\mathcal{B} = \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$. with each $\mathcal{B}_m$ discrete. Then, as $\mathcal{B}$ is a base for $\{Vy : V \in \mathcal{V}_N\}$, one has

$$X_N = \bigcup\nolimits_{m\in\mathbb{N}} \left(\bigcup \{B \in \mathcal{B}_m : (\exists V \in \mathcal{V}_N) B \subseteq Vy\} \right).$$

So for some m, say for $m = M$ the set

$$\bigcup \{B \in \mathcal{B}_m : (\exists V \in \mathcal{V}_N) B \subseteq Vy\}$$

is non-meagre. But as $\mathcal{B}_M$ is discrete, by Banach's Category Theorem ([Oxt] Ch. 16, [Kel] Th. 6.35, [Jay-Rog] p. 44-5), there is $B \in \mathcal{B}_M$ and $V \in \mathcal{V}_N$ with $B \subseteq Vy$ such that B is non-meagre; say this is so for $B = \hat{B} \in \mathcal{B}_M$ and $V = \hat{V} \in \mathcal{V}_N$. By refinement, there is some $\hat{g} \in G$ such that $\hat{V} \subseteq \hat{g}W$, so $\hat{B} \subseteq \hat{V}y \subseteq \hat{g}Wy$, and so $\hat{g}Wy$ is non-meagre. As $\hat{g}^{-1}$ is a homeomorphism of X, the set $Wy = Ux$ is also non-meagre in X. □

Separation Lemma. *Let G be a normed group, acting separately continuously and transitively on a non-meagre space X, with each point-evaluation map $\varphi_x : g \to g(x)$ base-σ-discrete. Then for any fixed point x and F closed* *nowhere dense the set $W_{x,F} := \{\alpha : \alpha(x) \notin F\}$ is dense open. In particular, G separates points from nowhere dense closed sets.*

Proof. The set $W_{x,F}$ is open, as it takes the form $\varphi_x^{-1}(X \backslash F)$ and φ_x is continuous (by assumption). By Lemma 1, for U any non-empty open set in G, the set Ux is non-meagre, and so $Ux \backslash F$ is non-empty, as F is meagre. But, then for some $u \in U$ we have $u(x) \notin F$. □

Proposition 1. *Let T be a Baire non-meagre subset of a metric space X and G a normed group, Baire in its right norm topology (e.g. almost complete and non-meagre in the norm topology), acting separately continuously and transitively on X, with each point evaluation map $\varphi_x : g \to g(x)$ base-σ-discrete. Then, for every convergent sequence x_n with limit x there is $\tau \in G$ and integer N such that $\tau x \in T$ and*

$$\{\tau(x_n) : n > N\} \subseteq T.$$

Proof. Write $T := M \cup U \backslash \bigcup_n F_n$ with U open, M meagre and each F_n closed and nowhere dense in X. Let $u_0 \in T \cap U$. By transitivity there is $\sigma \in G$ with $\sigma x_0 = u_0$. Put $u_n := \sigma x_n$. Then $u_n \to u_0$.

As G is Baire the set

$$\{\alpha : \alpha(u_0) \in U\} \cap C, \text{ where } C := \bigcap\nolimits_{m,n} \{\alpha : \alpha(u_m) \notin F_n\} \text{ is a dense } \mathcal{G}_\delta,$$

is non-empty. For α in the above set we have: $\alpha(u_0) \in U \backslash \bigcup_n F_n$. Now $\alpha(u_n) \to \alpha(u_0)$, by continuity of α, and U is open. So for some N we have for $n > N$ that $\alpha(u_n) \in U$. Since $\{\alpha(u_m) : m = 1, 2, ..\} \in X \backslash \bigcup_n F_n$, we have for $n > N$ that $\alpha(u_n) \in U \backslash \bigcup_n F_n \subseteq T$.

Finally put $\tau := \alpha\sigma$ then $\tau(x) = \alpha\sigma(x) \in T$ and $\{\tau(x_n) : n > N\} \subseteq T$. □

The argument above in fact proves and refines the following generalization of the Lemma above, a result that was observed by van Mill in the case of metric topological groups ([vM2, Prop. 3.4]).

Proposition 2. *Let G be a normed group, Baire in the norm topology and acting transitively and separately continuously on a non-meagre space X space, with each point-evaluation map $\varphi_x : g \to g(x)$ base-σ-discrete, and T co-meagre in X, then the set $\{g : g(D) \subseteq T\}$ is a dense $\mathcal{G}_\delta$.*

Proof. Without loss of generality, the comeagre set is of the form $T = U \backslash \bigcup F_n$ where each F_n is closed and nowhere dense, and U is open. Then, by the Lemma and as G is Baire, the set

$$\{g : g(D) \subseteq T\} = \bigcap_n \{g : g(D) \cap F_n = \emptyset\} = \bigcap_{d \in D, n \in \omega} \{g : g(d) \notin F_n\}$$

is a dense $\mathcal{G}_\delta$, as asserted. □

The setting in Th. S is exactly as in the preceding Proposition. Here we will show that the translator τ of the sequence can be chosen in a Baire non-meagre subset of the group provided that subset satisfies a consistency condition (a necessary condition).

Proof of the Shift-compactness Theorem. Here we work in the right norm topology. As $e_G \in A$ and A is Baire non-meagre, we may w.l.o.g. write $A = B_\varepsilon(e_G) \backslash \bigcup_n G_n$ where each G_n is closed nowhere dense with $e_G \notin G_n$.

As $Ax_0 \cap T$ is non-empty, there is $\alpha \in A \subseteq B_\varepsilon(e_G)$ such that $\alpha x_0 \in U$. Putting $\alpha = \beta\sigma$, we have

$$\begin{aligned} \beta &= \alpha\sigma^{-1} \in B_\varepsilon(e_G)\sigma^{-1} \cap \{\alpha : \alpha(x_0) \in U\}\sigma^{-1} = B_\varepsilon(e_G)\sigma^{-1} \cap \{\beta : \beta(\sigma x_0) \in U\} \\ &= B_\varepsilon(e_G)\sigma^{-1} \cap \{\beta : \beta(u_0) \in U\}. \end{aligned}$$

That is, since $Ax_0 \cap T$ is non-empty, the set $\{\beta : \beta(u_0) \in U\} \cap B_\varepsilon(e_G)\sigma^{-1}$ is open and non-empty. Now $G_n\sigma^{-1}$ is closed and nowhere dense in G (as the right shift $g \to g\sigma^{-1}$ is a homeomorphism). But G is a Baire space in the right norm topology, the set $(C \backslash \bigcup_n G_n\sigma^{-1}) \cap \{\beta : \beta(u_0) \in U\} \cap B_\varepsilon(e_G)\beta^{-1}$ is non-empty. So there is β such that $\beta(u_0) \in U$ with $\alpha = \beta\sigma \in B_\varepsilon(e_G) \backslash \bigcup_n G_n = A$. That is, $\alpha x_0 = \beta\sigma x_0 \in U$. Now $\alpha x_0 = \lim \alpha x_n = \lim \beta\sigma x_n = \lim \beta u_n$ and so $\beta(u_n) \in U$ for large n , for $n > N$ say. So since $\{\beta(u_m) : m = 1, 2, ..\} \in X \backslash \bigcup_n F_n$ we have for $n > N$ that $\beta(u_n) \in U \backslash \bigcup_n F_n \subseteq T$.

Finally $\alpha(x) = \beta\sigma(x) \in T$ and $\{\alpha(x_n) : n > N\} \subseteq T$. □

2.3 Proof that S $\Longrightarrow$ E

Assume G acts transitively on X and that X is non-meagre. Let $B := B_\varepsilon(e_G)$ and suppose that for some x the set $T := Bx$ is not a nhd of x. Then there is $x_n \to x$ with $x_n \notin Bx$ for each n. Take $A := B_{\varepsilon/2}(e_G)$ and note first that A is a symmetric open set ($A^{-1} = A$, by the inversion axiom), and secondly that, as G is metrizable and analytic, A is analytic (since open sets are $\mathcal{F}_\sigma$ and Souslin-$\mathcal{F}$ subsets of analytic sets are analytic, cf. [Jay-Rog]). By the Lemma, since X is non-meagre and point-evaluations are base-σ-discrete and A is open, Ax is non-meagre. Since the evaluation map $g \to g(x)$ is continuous and base-σ-discrete, Ax is analytic (Hansell's Theorem), so Souslin-$\mathcal{F}$, and so Baire by Nikodym's Theorem (on the preservation of the Baire property by the Souslin operation, cf. [Jay-Rog]). So by Theorem S, there are $a \in A$ (being open, has the Baire property) and a co-finite $\mathbb{M}_a$ such that $ax_m \in Ax$ for $m \in \mathbb{M}_a$. For any such m choose $b_m \in A$ with $ax_m = b_m x$. Then $x_m = a^{-1}b_m x \in A^2 x \subseteq Bx$, a contradiction (note that $a^{-1} \in A$, by symmetry).

3 Concluding remarks

1. Assumptions in Theorem S. With regard to the assumption of separate continuity, note that a theorem of Bouziad ([Bou, Th. 3]) implies that a separately continuous action by a metrizable left-topological Baire group acting on a metric space is in fact jointly continuous; we retain the only apparently weaker hypothesis of separate continuity, because there are variants of Theorem S, where joint continuity is absent (and the group is not Baire): see [Mil-Ost]. The result is connected with van Mill's Separation Property: say that X has SP ([vM2]) with respect to a group of homeomorphisms $\mathcal{G}$ if for any countable set D in X and any meagre set M in X, there is a homeomorphism $g \in \mathcal{G}$ such that $g(D) \cap M = \emptyset$. Stated equivalently, for D countable and T co-meagre, there is $g \in \mathcal{G}$ with $g(D) \subseteq T$ (see also the next remark). Compared to this restatement, Theorem S refers on the one hand to a smaller class of countable sets (convergent sequences, or their co-finite parts), but on the other hand asserts embeddability into a larger class of sets – sets T that are 'locally' rather than globally co-meagre; for further information see also [Mil-Ost].

2. Shift-compactness and the SP property. In view of the strength of Th. S and to place our results in context, we briefly summarize some of the relevant results of [vM2]. A (separable metric) space with SP is Baire, by the proof which we imported for Propositions B1 and B2 in §1; likewise, an almost complete non-meagre space with SP is completely Baire. Since an absolutely co-analytic space is Polish iff it is 'completely Baire', i.e. closed-hereditarily Baire (closed subspaces are Baire), for which see Kechris [Kech-Cl, Cor.

21.21], it follows that an absolutely Borel space with the SP is Polish ([vM2, Th. 1.1]). More generally, if an analytic group G acts on a space X and SP holds w.r.t. G, then X is Polish. Van Mill also shows from his Prop 3.4 (cf. our Prop. 2) that a locally compact homogeneous space has the SP. It seems likely that, just as with Proposition B, more of these arguments can be copied across in the language of Th. S.

It is noted in [BOst-TRI] that a Polish space which is strongly locally homogeneous has SP.

3. *Topological and semitopological groups.* When a group G equipped with a topology such that the (action) map $(g_1, g_2) \to g_1 g_2$ from G^2 to G is separately continuous (i.e. left and right shifts are continous), then it is said to be a *semitopological group*. Thus a topological group is semitopological. If also the topology is metrizable and G is Baire, Theorem S applies and asserts here that if T is non-meagre in G and $g_n \to g$, then for some $t \in T$ the point tg is in T and almost all of the sequence tg_n is in T. As in Theorem S, under these additional assumptions, group multiplication in G is jointly continuous; indeed, Bouziad [Bou] proves that a semitopological Baire p-space (and metric spaces are p-spaces) has jointly continuous multiplication (is 'paratopological').

4. *Specializing the proof of Theorem S to semitopological groups.* The argument proving Th. S is particularly transparent in the case of a semitopological group and follows a standard real-line argument, as follows. Suppose $x_n \to x_0$ and consider $T := U \backslash \bigcup F_m$ with F_m closed and nowhere dense, and U open. For any $u_0 \in T \cap U$, take $\sigma(x) = x x_0^{-1} u_0$, which is a continuous right-shift, so that $u_n = \sigma(x_n) = x_n(x_0^{-1} u_0) \to u_0 = \sigma(x_0)$. Now

$$u_0^{-1} U \cap \bigcap\nolimits_{n,m} u_n^{-1}(X \backslash F_m)$$

is a dense $\mathcal{G}_\delta$, since left-shifts are homeomorphisms. As G is Baire, there is g with

$$g \in u_0^{-1} U \cap \bigcap\nolimits_{n,m} u_n^{-1}(X \backslash F_m).$$

Then $u_0 g \in U$ and $u_0 g \notin F_m$ for all m, so $u_0 g \in T$. By continuity of right-shifts, $u_n g \to u_0 g$, so for large n, say for $n > N$, we have

$$u_n g \in U.$$

For each such n we have $u_n g \notin F_m$ for all m and so $u_n g \in T$. Thus, as $u_n g = x_n(x_0^{-1}u_0)g$, we conclude the existence for some $g \in G$ and $u_0 \in T$ with $t := u_0 g \in T$ that, as in Theorem S

$$x_n x_0^{-1} t \in T \text{ for } n > N.$$

5. From E back to S. In Proposition 1 density of the set $W_{x,F}$ was deduced from a Lemma which, as noted, asserts a very weak microtransitivity. Unsurprisingly, Property E also implies density of that set. However, Property E does not imply that G is Baire, since (see Introduction) there do exist meagre analytic groups acting on a non-meagre space and by van Mill's result have Property E notwithstanding. So we can go no further with the density argument of Proposition 1. Of course, as in Proposition 1, if we know that G is Baire, then E implies S.

As for the density claim, consider $\beta \in G$. Suppose first that $\beta(x) \in F$. By the Effros property, the set $B_\varepsilon(\beta)x = B_\varepsilon(e_G)\beta x$ is an open nhd of βx. As F is nowhere dense and closed, there is $y \in$int$(B_\varepsilon(e_G)\beta x)\backslash F$. So there is $\gamma \in B_\varepsilon(e_G)$ such that $y = \alpha x := \gamma\beta x \notin F$. So $a = \gamma\beta \in B_\varepsilon(\beta) \cap \{\alpha : \alpha(x) \notin F\}$.

If, on the other hand, $\beta(x) \notin F$, then $\beta \in B_\varepsilon(\beta) \cap \{\alpha : \alpha(x) \notin F\}$. This proves the claim.

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