Regular variation, topological dynamics, and the Uniform Boundedness Theorem.

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August 2008 Ost-knit.tex
CDAM Research Report LSE-CDAM-2008-15

Abstract

In the metrizable topological groups context, a direct product construction (mimicking the 'action groupoid') provides a multiplicative representation canonical for arbitrary continuous flows. This implies, modulo metric differences, the topological equivalence of the natural, flow setting of regular variation of [BOst13] with the Bajšanski and Karamata [BajKar] group formulation. In consequence topological theorems concerning subgroup actions may be lifted to the flow setting. Thus the Bajšanski-Karamata Uniform Boundedness Theorem (UBT), as it applies to cocycles in the continuous and Baire cases, may be reformulated and refined to hold under Baire-style Carathéodory conditions. Its connection to the Banach-Steinhaus UBT is clarified. An application to Banach algebras is given.

Classification: 26A03

Keywords: multivariate regular variation, uniform convergence theorem, category, duality, flows, cocycles, internal direct product, action groupoid, Carathéodory conditions.

1 Multiplicative action, duality and a transfer principle

We work in the category of metrizable topological groups, implying that if X and Y are isomorphic, then they are also homeomorphic. For groups T and X, with identities e_T and e_X, a (continuous) T*-flow on* X ([GoHe], [Be], or the more recent [Ell1]) is a continuous mapping $\varphi : T \times X \to X$ such that, for $s, t, \in T$ and $x \in X$,

$$\varphi(st, x) = \varphi(s, tx) \text{ and } \varphi(e_T, x) = x.$$

Write the map induced by t as $\varphi^t(x) := \varphi(t, x)$; then φ^t is a homeomorphism with (continuous) inverse $\varphi^{t^{-1}}(x)$ and for $e = e_T$, $\varphi^e = id_X$, where $id_S(s) := s$ denotes the identity self-homeomorphism of a space S. Thus $\varphi : t \to \varphi^t$ embeds T as a subgroup of $Auth(X)$, the group of self-homeomorphisms (*auto-homeomorphisms*) of X. As a *blanket assumption*: we restrict T to contain only *bounded* members, those t for which $||t|| := \sup_x d_X(t(x), x) < \infty$; this guarantees T is metrized by the supremum metric, denoted d_T.

Identifying t with φ^t, one may write $t(x)$ for $\varphi^t(x)$, which permits development of a proper *duality* between the T-flow $\varphi^T(t, x) = t(x)$ and the associated X-flow on T_X the *group of translates* $\{t_x : x \in X, t \in T\}$ with group operation $s_x \cdot t_y = st_{xy}$, where $t_x(u) := t(ux)$ and the X-flow is $\varphi^X(x, t) = t_x$ (first noted albeit in another setting in [Se1]). Point-evaluation of t_x at e_X, formally a projection on the e_X co-ordinate space, is $t_x(e_X) = t(x)$, the original T-flow. (One may write xt for t_x or even $\langle t, x\rangle$, for $t(x)$, so that t and x commute at least at the projection level; we see very pervasive consequences of this later.) Proper expression of a duality calls for embedding X in the double 'topological dual' $Auth(Auth(X))$. Alternatively, the duality may be captured by a commutative diagram of *homeomorphisms* (where $\Phi^T(t, x) = (t, tx)$ and $\Phi^X(x, t) = (t, xt)$.

$$
\begin{tikzcd}
(t,x) \arrow[r, leftrightarrow, "\Phi^T"] \arrow[d, leftrightarrow] & (t,tx) \arrow[d, leftrightarrow] \\
(x,t) \arrow[r, leftrightarrow, "\Phi^X"] & (t,xt)
\end{tikzcd}
$$

Here the two vertical maps may, and will, be used as identifications, since $(t,tx) \leftrightarrows (t,x) \leftrightarrows (t,xt)$ are bijections (more is true, see below).

However, there is a simpler, purely algebraic approach for capturing the duality. Observe first that the simplest example of a flow is a restriction of the *multiplicative action* of a group X on X to the action of a subgroup T of X on X, e.g. left translation $(t,x) \to tx$. We show that a T-flow on X and the associated X-flow on T_X may be represented *canonically* in this multiplicative form by a group structure on the phase space $T \times X$ with T and X represented by complementary normal subgroups isomorphic to T and X. We denote the group $T \bowtie X$ and call it the **phase-group**. (Anatole Beck points out that T is sometimes called the parameter space and X the state space, so their product may correctly be termed a phase space.) Albeit with more structure here, this is similar in spirit to the semi-direct product of group theory which describes a 'split extension' of a group G by a group A of automorphisms of G; see eg [As] Sect. 10. Our construction mimics the construction of the action groupoid of Lie groupoid theory (cf. [We], or [ALR] Section 1.4), but remains within group theory (appropriately to our context/category). Here again the topological structure is richer than in the groupoid setting (it contains a representation of the action groupoid, for which see below), since it also takes into account the group structure of X – see Example 2 below for further elucidation. In topological dynamics $t(x)$ is written multiplicatively as tx (cf. [GoHe]), consistently with a multiplicative representation.

The representation implies the *transfer principle* that a topological theorem about multiplicative group actions may be lifted to a theorem concerning flow actions, in fact to a primal and dual form of the theorem (see [BOst13] for a discussion of this point). Here we give the details for two such transfers which are of interest to the topological theory of regular variation: the two uniform boundedness theorems (for continuous, alternatively Baire, cocycles).

Recall that a group G is an *internal direct product* (for a topological view see [Na] Ch. 2.7; for an algebraic view see [vdW] Ch. 6, Sect. 47, [J] Ch. 9 and 10, or [Ga] Section 9.1) if it is factorizable by two normal subgroups H, K, i.e. $G = HK$ with $H \cap K = \{e_G\}$ (so that factorization in G is unique). Under these circumstances $hk = kh$ holds for $h \in H, k \in K$ (since $hkh^{-1}k^{-1}$ is in $H \cap K$, cf. [vdW] Ch. 6, Sect. 47), so this setting provides a pleasingly simple expression, when X, T are metrizable, of the inherent duality between T acting on X and X acting on T if, as can be arranged, H and K are isomorphs of X and T. We now indicate why.

Under the above circumstances K is a unique *complement* for H (for which see [As] Section 10 p. 29), and vice versa H a unique complement of K, so we may also regard them as *duals* of each other. Furthermore, suppose that G has a right-invariant metric (see Section 2 for details). If we identify an element h in H with translation by h on G (i.e. with $\tau_h(g) := hg$), then

$$d_H(h, h') := \sup_{g \in G} d_G(hg, h'g) = d_G(h, h')$$

shows that H, as a subgroup of G, is isometric with $\{\tau_h : h \in H\}$, as a subgroup of $Auth(G)$ under the supremum metric. Now, restricting φ^G, the multiplicative action of G on G, to H we obtain the H-flow on G, namely $\varphi^H(h, g) := hg$. The map induced by h is τ_h and $h \to \tau_h$ embeds H in $Auth(G)$; its image, $\varphi^H(H)$, is simply an isometric isomorph of H. The same goes for K and φ^K. Our theorem says we may identify H with T and K with X, as well having a commutative diagram of isomorphisms.

Theorem (Multiplicative Representation of dual flows on topological groups).

For φ any continuous T-flow on X with $T \subseteq Auth(X)$, there is a canonical internal direct product group $G = \Theta\Xi$ and isomorphisms $\theta : T \to \Theta, \xi : X \to \Xi$ (as between topological groups) such that the T-flow on X is represented by the multiplicative Θ-flow on G :

$$\varphi^\Theta : (\tau, g) \to \tau g, \qquad (\tau \in \Theta, g \in G),$$

as is simultaneously (mutatis mutandis) the associated X -flow on T_X. That is,

(i) *the isomorphisms θ, ξ commute: $\theta_t \xi_x = \xi_x \theta_t$;*

(ii) *there are isomorphisms such that*

$$\begin{array}{ccccccccc}
(t,x) & \longleftrightarrow & (\theta_t,\xi_x) & \longleftrightarrow & \theta_t\xi_x & \longleftrightarrow & (t,tx) & \longleftrightarrow & (t,x) \\
\updownarrow & & & & & & & & \updownarrow \\
(x,t) & \longleftrightarrow & (\xi_x,\theta_t) & \longleftrightarrow & \xi_x\theta_t & \longleftrightarrow & (xt,t) & \longleftrightarrow & (x,t)
\end{array}$$

(iii) T_X *is isomorphic to* G *under the mapping* $xt \to \xi_x\theta_t$,
(iv) *denoting* $(\theta \times \xi)(t,x) := (\theta_t, \xi_x)$ *etc., the diagrams below commute:*

$$\begin{tikzcd}
\Theta\Xi \arrow[r, "\varphi^\Theta"] & G \arrow[d, equal] \\
T \times X \arrow[u, "\theta\times\xi"] \arrow[r, "\Phi^T"] & T \times X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Xi\Theta \arrow[r, "\varphi^\Xi"] & G \arrow[d, equal] \\
X \times T \arrow[u, "\xi\times\theta"] \arrow[r, "\Phi^X"] & T \times X
\end{tikzcd}$$

as

$$\Phi^T = \varphi^T \circ (\theta \times \xi) \text{ and } \Phi^X = \varphi^\Xi \circ (\xi \times \theta);$$

(v) *moreover, if* T *is an internal direct product with* $T = UV$, *then* $\Theta = \theta(U)\theta(V)$ *is also an internal direct product; likewise, if* X *is an internal direct product with* $X = YZ$, *then* $\Xi = \xi(Y)\xi(Z)$ *is an internal direct product.*

Proof. We proceed by constructing a generalized product group (as in the Zappa-Szép product, or knit product, cf. [Sz], see Remark 3 below), i.e. a group that is factorizable by two general subgroups H, K, so that $G = HK$ with $H \cap K = \{e_G\}$. We then check that H, K are normal. For X a group and $T \subseteq Auth(X)$, we equip the Cartesian product

$$G = T \times X$$

with a group operation on G defined by

$$(s,x) \bowtie (t,y) = (st, st(s^{-1}xt^{-1}y)),$$

for which $e_G = (e_T, e_X)$. (For an interesting homeomorphic alternative see Remark 2.) An equivalent definition is by the symmetric product formula:

$$(s,sa) \bowtie (t,tb) = (st, st(ab)),$$

showing that $(t, tx)^{-1} = (t^{-1}, t^{-1}(x^{-1}))$. The latter product formula (which motivates the construction) shows that $\Phi^T : (t, x) \to (t, tx)$ describes an isomorphism from the direct product $T \times X$ to the general product $T \bowtie X$. As this is also a homeomorphism, we see that $T \bowtie X$ is a metrizable topological group, when X is metrizable. For $t \in T, x \in X$, write

$$\theta_t := (t, t(e_X)), \qquad \xi_x := (e_T, x).$$

Then X is isomorphic to

$$\Xi := \{\xi_x : x \in X\} = \{(e_T, x) : x \in X\}.$$

Also Ξ is a normal subgroup, since

$$(s, sa) \bowtie (e_T, x) \bowtie (s^{-1}, s^{-1}a^{-1}) = (e_T, axa^{-1}).$$

On the other hand, T is isomorphic to

$$\Theta := \{\theta_t : t \in T\} = \{(t, t(e_X)) : t \in T\},$$

since by the symmetric product formula

$$(s, s(e_X)) \bowtie (t, t(e_X)) = (st, st(e_X)).$$

As with Ξ, so too here Θ is a normal subgroup, since

$$(s, sa) \bowtie (t, t(e_X)) \bowtie (s^{-1}, s^{-1}a^{-1}) = (sts^{-1}, sts^{-1}(e_X)).$$

Finally, note $\Xi \cap \Theta = \{e_G\}$, since if $(t, t(e_X)) \in \Xi$, then $t = e_T = id_X$ and so $t(e_X) = e_T(e_X) = e_X$. Thus G is in fact an internal direct product.

The flow $T \times X \to X$ may now be recovered from φ^Θ, the multiplicative action of the subgroup Θ, when restricted to the subgroup Ξ via the projection $\pi : G \to X$, since

$$\theta_t \bowtie \xi_x = (t, t(e_X)) \bowtie (e_T, x) = (t, t(t^{-1}(t(e_X))x)) = (t, t(x)).$$

Indeed the equation confirms that the multiplicative action yields an isomorphic target and also that the T-flow on X is isomorphic, because

$$\theta_s \bowtie \theta_t \bowtie \xi_x = \theta_{st} \bowtie \xi_x = (st, st(x)).$$

We note that $\theta_t \bowtie \xi_x = \xi_x \bowtie \theta_t$, since

$$\xi_x \bowtie \theta_t = (e_T, x) \bowtie (t, t(e_X)) = (t, t(xe_X)) = (t, t(x)).$$

The same goes for the flow $X \times T \to T_X$ and restriction of the action φ^Ξ to the subgroup Θ. Indeed T_X is of course isomorphic to the internal direct product $X \bowtie T$ under the mapping $t_x \leftrightarrow (x, t) \to (t, t(x)) = \xi_x \bowtie \theta_t$; indeed it is a homomorphism since $(s_x \cdot t_y)(e_X) = st_{xy}(e_X) = st(xy)$, so that

$$(x, s) \cdot (y, t) = (xy, st) \to (st, st(xy) = (s, s(x)) \ltimes (t, t(y));$$

it is injective, since $t(x) = s(y)$ and $t = s$ implies $x = y$; and it is surjective since $(t, y) = (t, t(t^{-1}y))$.

Finally, suppose that T itself is an inner direct product $T = UV$, with $U \cap V = \{e_T\}$ and U, V normal. Then, since $UV = VU$ elementwise, we see that

$$\theta_u \bowtie \theta_v = (uv, uv(e_X)) = (vu, vu(e_X)) = \theta_v \bowtie \theta_u.$$

Put $\theta(U) = \{\theta_u : u \in U\}$ and $\theta(V) = \{\theta_v : v \in V\}$. Then $\theta(U)$ and $\theta(V)$ are normal subgroups of $\theta(T) = \Theta$. Since $\theta_u = \theta_v$ iff $u = v$, we see that Θ is an inner direct product of $\theta(U)$ and $v(V)$. Thus

$$\Theta = \theta(U)\theta(V),$$

Likewise, if $X = YZ$, with $Y \cap Z = \{e_X\}$ and Y, Z normal, since this time we have

$$\xi_y \bowtie \xi_z = (e_T, y) \bowtie (e_T, z) = (e_T, zy) = \xi_z \bowtie \xi_y,$$

as claimed. $\square$

Remarks

0. Note that $(s, sa)_{\bowtie}^{-1} = (s^{-1}, s^{-1}a^{-1})$, since

$$(s, sa) \bowtie (t, tb) = (st, st(ab)).$$

similarly$(s, s^{-1}a)_\star^{-1} = (s^{-1}, sa^{-1})$ since

$$(s, s^{-1}a) \star (t, t^{-1}b) = (st, (st)^{-1}(ab)).$$

1. If T is a group of self-isomorphisms of X, then $t(e_X) = e_X$ and so $\theta_t = (t, e_X)$. Here

$$(s, x) \bowtie (t, y) = (st, (sts^{-1}x) \cdot sy),$$

suggesting more general forms, appropriate to isomorphism groups, such as

$$(h_1, k_1)(h_2, k_2) = (\alpha(h_1, h_2)h_1, \beta(h_1, h_2)(k_1)h_1(k_2)),$$

with α, β homomorphisms, e.g. $\alpha(h_1, h_2) = h_1 h_2 h_1^{-1}$ and $\beta(h_1, h_2) = h_1 h_2 h_1^{-1}$.

2. An alternative product, denoted $T \star X$, derives from the group operations on G defined by

$$(s, x) \star (t, y) = (st, (st)^{-1}(sxty)),$$

and is homeomorphic to $T \bowtie X$ via inversion (with a repeated inversion on the first coordinate). An equivalent definition of the operation is by the symmetric product formula

$$(s, s^{-1}a) \star (t, t^{-1}b) = (st, (st)^{-1}(ab)).$$

Specialization of the latter formula here to the case of T a subgroup of X yields pairs (x, y) satisfying $xy = a$ etc.; thus this generalized product reflects the mechanics of a multiplicative convolution (Mellin transform). The notation of regular variation, however, prefers the earlier choice $T \bowtie X$ (see later). For

$$\bar{\theta}_s := (s, s^{-1}(e_X)) \qquad \xi_x := (e_T, x),$$

we obtain

$$\bar{\theta}_s \cdot \xi_x = (s, s^{-1}(e_X)) \star (e_T, x) = (s, s^{-1}(s(s^{-1}(e_X))x)) = (s, s^{-1}(x)).$$

3. Note that $\pi(\theta_s \cdot g) = sx$ for $g = (t, x)$, since

$$\theta_s \bowtie g = (s, s(e_X)) \bowtie (t, x) = (st, st(t^{-1}x) = (st, sx).$$

We use this observation in the Transfer Principle of the next section.

Examples.

1. If $T \subseteq Tr(X)$ is a subgroup of translations $\tau_t : z \to tz$ and X is abelian, then

$$(\tau_u, x) \bowtie (\tau_v, y) = (\tau_{uv}, uv(u^{-1}xv^{-1}y)) = (\tau_u \tau_v, xy).$$

2. For two commuting flows U and V on X, the action $T = U \times V$ is an internal direct product and the theorem asserts that both flows on

X are representable by commuting multiplications. Analogous to this is a representation for the general linear skew-product flow π. This is defined to be (see [Se1], [Se2]) a T-flow on $X = Y \times Z$, with Y a topological space and Z a normed vector space, whereby π takes the form

$$\pi(t, y, z) = (t(y), \alpha(t, y)z).$$

Here $(t, y) \to t(y)$ is a flow in Y. Note that

$$\begin{aligned} \pi(st, y, z) &= \pi(s, \pi(t, y, z)) = \pi(s, (t(y), \alpha(t, y)z)) \\ &= (s(t(y)), \alpha(s, t(y))\alpha(t, y)z), \end{aligned}$$

so since

$$\pi(st, y, z) = (st(y), \alpha(st, y)z)$$

we have

$$\alpha(st, y)z = \alpha(s, t(y))\alpha(t, y)z,$$

or the cocycle condition:

$$\alpha(st, y) = \alpha(s, t(y))\alpha(t, y).$$

We have

$$I = \alpha(e, y) = \alpha(t^{-1}, t(y))\alpha(t, y),$$

so $\alpha(t^{-1}, t(y)) = \alpha(t, y)^{-1}$.

The one-parameter group $\Lambda(t) := \alpha(t, t(y))$ has $\Lambda(0) = I$, with $\Lambda(t)$ invertible since $\Lambda(-t)\Lambda(t) = \Lambda(0) = I$. (In fact more is true when $T = \mathbb{R}$, as the defining properties of a flow secure the continuity condition $\lim_{t\to 0} ||Q(t)z - z|| = 0$ for every z in Z; hence, if $\Lambda(t)$ is itself continuous on Z, then $\Lambda(t)$ has an exponential representation – see [Ru] Ch. 13, Semigroups of operators.) Thus a phase-group $\Theta H Z$ can be created with

$$\pi(t, y, z) = \theta_t \bowtie \eta_y \bowtie \zeta_z,$$

with $T = \mathbb{R}$, $Z = \mathbb{R}^d$, and Y as in the standard example. (Motivation and details are presented in Appendix 1 of the extended web-site version of this paper.)

3. We note here the multiplicative representation that the phase-group gives for the action-groupoid of a T-action on X. (Compare Appendix 2 of the extended website version of this paper.) In the current circumstances

the groupoid is presented as a space of points (objects) together with a space of arrows (morphisms), with the space X taken as the space of objects (we agree to call points *locations*) and $T \times X$ as the space of arrows (t, x). The arrow (t, x) has *source* x and *target* $t(x)$. The binary operation is composition of two arrows, (t, x) followed by (s, y), and is possible if and only if $y = t(x)$ (when the arrows are said to be a *composable,* ordered pair); that is, speaking intuitively, the target of the first displacement provides the location for a subsequent displacement. We term the points ξ_x in the group G the *source elements,* as they correspond to sources of arrows, and the terms θ_t *displacement elements*.

The natural embedding $\gamma : T \times X \to \Theta\Xi$ of arrows to the phase-group G is

$$\gamma(t, x) := (t, t(x)).$$

The embedding is continuous, if we agree to use the product topology on the space of arrows $T \times X$. We may call the arrow (t, e_X) a *basic displacement*, as it is represents an arrow from the base point e_X of X; this is carried to $\gamma(t, e_X) = (t, t(e_X))$, i.e. to the point θ_t of Θ. We then have the unique representation of an arrow in G as a multiplicative decomposition

$$\gamma(t, x) = \theta_t \bowtie \xi_x,$$

i.e. the product in G of a displacement θ_t and a source ξ_x.

The decomposition above induces a natural projection δ from arrows to displacements, defined from the set $T \times X$ to the subset Θ of $T \times X$ by

$$\delta(t, x) = \theta_t = (t, t(e_X)).$$

This is an idempotent when viewed as acting only on sets; however, regarding Θ as a subgroup of G, the map δ there serves further as a *disabling operation,* since it disables one of the two operations which define $\bowtie$ in G, as we see in the following computation:

$$\begin{aligned} \gamma(st, x) &= (st, stx) = (s, s(e_X)) \bowtie (t, tx) \\ &= \theta_s \bowtie \gamma(t, x) \\ &= \delta(s, tx) \bowtie \gamma(t, x). \end{aligned}$$

Thus

$$(s, tx) \circ (t, x) = \gamma^{-1}[\delta(s, tx) \bowtie \gamma(t, x)],$$

so that the binary operation of composition $\circ$ in the space of arrows is recoverable via the representation γ, from the projection δ and the binary operation $\bowtie$ of G.

4. Continuing from the last computation of Example 3, we deduce, for the composable pair of arrows $\alpha = (s, stx)$ and $\beta = (t, x)$, that

$$\gamma(\alpha \circ \beta) = \delta(\alpha) \bowtie \gamma(\beta).$$

Thus fixing α, the following relation, for any β right-composable with α, holds in $G = \Theta\Xi$

$$\delta(\alpha) = \gamma(\alpha \circ \beta)\gamma(\beta)^{-1}.$$

We are about to recognize, in Section 3, the right-hand side (independent here of β) as a cocycle σ_γ, the main concept in the Uniform Boundedness Theorems.

2 Metric aspects of duality: regular variation

In any group X we define the *group-norm* by $||x||_X := d_X(x, e_X)$. If d_X is right- or left-invariant, then we have $||x^{-1}|| = ||x||$, i.e. the group-norm is *symmetric*. Assuming either right- or left-invariant d_X, we have *the triangle inequality* in the form

$$||xy|| \leq ||x|| + ||y||,$$

since, for instance, for the (preferred) right-invariant case

$$d(xy, e) = d(x, y^{-1}) \leq d(x, e) + d(e, y^{-1}).$$

(On its own symmetry is not helpful, though easily arranged using the symmetrization $||x|| := ||x||_X + ||x^{-1}||_X$.) Normed groups are of fundamental importance to regular variation; see [BOst12] for an exploration of the theory, the earlier literature on the subject, and an alternative approach to the duality of topological flows.

If the group is abelian the defining inequality reduces to the usual triangle inequality. If the group is a vector space (e.g. $\mathbb{R}$, or $\mathbb{C}$), then the group-norm is just the usual norm. For a less obvious, but significant, example note that if T is a subgroup of the bounded elements in $Auth(X)$, with composition as the group operation, then the group-norm is symmetric, as

$$||h|| = \sup_x d(h(x), x) = \sup_y d(y, h^{-1}(y)) = ||h^{-1}||,$$

and the triangle inequality is satisfied, because

$$||h'h|| = \sup_x d(h'h(x), x) = \sup_y d(h(y), h^{-1}(y)) \leq ||h|| + ||h'||,$$

an argument which draws on the fact that the metric d_T is in fact right-invariant, since

$$d_T(hg, h'g) = \sup_x d_X(h(g(x)), h'(g(x))) = \sup_y d_X(h(y), h'(y)) = d_T(h, h').$$

We may give the phase-group $G = T \bowtie X$ the metric

$$d_G((t, x), (s, y)) = d_T(s, t) + d_X(x, y), \tag{1}$$

so that if d_X is right-invariant, then so is d_G. Here $d_T(s, t) = \sup_z d_X(s(z), t(z))$.

Before investigating metric connections between $T \bowtie X$ and $T \times X$ we note that sequential convergence is a topological notion, whereas the notions of divergence are metric. We are thus more concerned with divergence, especially so in the following cases: divergence defined in X by $||x_n|| \to \infty$, and in T by either a uniform condition $||t_n|| \to \infty$, or a pointwise condition $||t_n x|| \to \infty$, for each x. The first lemma below is concerned with $T \times X$ and is followed by a result for $T \bowtie X$.

Proposition (Duality of divergence) *Let the topological group X have right-invariant metric. For s a bounded member of $Auth(X)$ and $a \in X$,*

$$||s(a)|| \leq ||s|| + ||a|| \qquad \text{and} \qquad ||a|| \leq ||s|| + ||s(a)||.$$

Hence, for s and $\{t_n\}$ bounded members of $Auth(X)$,
(i) $||x_n||_X \to \infty$ *iff* $||s(x_n)||_X \to \infty$, *and*
(ii) *if* $||t_n(x)||_X \to \infty$,*then* $||t_n||_T \to \infty$.
Moreover, if $T \subseteq X$ and the action is multiplicative, then

$$||s|| \leq ||sa|| + ||a||,$$

so that here

$$||t_n|| \to \infty \text{ iff } ||t_n(x)|| \to \infty, \text{for all/for some } x \in X.$$

Proof. All three results follow from inversion-invariance and the triangle inequality. The second and third follow from the identities: $a = s^{-1}s(a)$,

$se = saa^{-1}$. The first inequality shows (ii) because $||t_n(x)|| \leq ||t_n|| + ||x||$ and x is fixed. The third shows that $||t_n|| \leq ||t_n x|| + ||x||$. $\square$

Proposition (Triangle inequality with a parameter cf. [Ra] 2.2). *Let $G = T \bowtie X$ be metrized by* (1)*; then*

$$||\xi_x||_G = ||x||_X \qquad \text{and} \qquad ||\theta_t||_G = ||t||_T + d_X(t(e_X), e_X),$$

so that

$$||t||_T \leq ||\theta_t||_G \leq 2||t||_T \qquad \text{and} \qquad ||\theta_t \bowtie \xi_x|| \leq 2(||\theta_t|| + ||\xi_x||).$$

Hence, for $x \in X, t \in T$
(i) $||x|| \to \infty$ *iff* $||\xi_x|| \to \infty$, *and*
(ii) $||t|| \to \infty$ *iff* $||\theta_t|| \to \infty$.

Proof. Indeed $||\xi_x|| = d_G((e_T, x), (e_T, e_X)) = ||x||$. Now

$$\begin{aligned} ||(t, t(x))|| &= d_G((t, t(x)), (e_T, e_X)) = \sup_z d_X(t(z), z) + d_X(t(x), e_X) \\ &= ||t|| + d_X(t(x), e_X) \leq 2||t|| + ||x|| \end{aligned}$$

so, in particular, $||\theta_t|| = ||t|| + d_X(t(e_X), e_X)$.Thus

$$\begin{aligned} ||\theta_t &\bowtie \xi_x|| = ||(t, t(x))|| \leq 2||t|| + ||x|| + ||x|| \\ &\leq 2(||\theta_t|| + ||\xi_x||). \end{aligned}$$

Clearly the parameter 2 does not disturb divergence considerations. $\square$

Interest in divergence structures is motivated by the following.

Definitions. Given groups T, X, H and a T-flow on X, we say that the function $h : X \to H$ is *regularly varying on* T , resp. *regularly varying on* X, if the respective limit below exists. (For a development of the theory, see [BOst13]).

$$\begin{aligned} \partial_X h(s) &= \lim_{||x|| \to \infty} h(sx)h(x)^{-1}, \qquad (s \in T) \\ \partial_T h(x) &= \lim_{||s|| \to \infty} h(sx)h(s(e_X))^{-1}, \qquad (x \in X). \end{aligned}$$

In the next section we begin a study of the relation of these ideas to the phase-group.

3 Cocycles and the transfer principle

Recall (cf. [Ell2]) that for a T-flow on X, a function $\sigma : T \times X \to H$ is a *cocycle* on X if

$$\sigma(st, x) = \sigma(s, tx)\sigma(t, x). \tag{2}$$

(This says, according to Example 3 of Section 1, that σ preserves the composition of composable arrows of the action groupoid.) Let

$$\sigma_h(t, x) = h(tx)h(x)^{-1}. \tag{3}$$

Then σ_h is a *cocycle* (the h-cocycle), since

$$h(stx)h(x)^{-1} = h(stx)h(tx)^{-1}h(tx)h(x)^{-1},$$

and this permits an interleafing idempotent of H, a projection, π to be inserted into the formula for σ_h to yield the cocycle $h(tx)\pi h(x)^{-1}$. A cocycle is a *coboundary* on X if there is continuous $h : X \to H$ such that

$$h(tx) = \sigma(t, x)h(x).$$

We will then say that the cocycle is a h-coboundary on X. Thus, for h continuous on X, σ_h is a h-coboundary on X. (Equipping the space of arrows $T \times X$ of Example 3 of Section 1 with the product topology, the cocycle $\sigma_\gamma(\alpha, \beta)$ of Example 4 is a γ-coboundary, since γ is continuous.)

Before investigating boundedness properties of cocycles, we show how to lift cocycles from $T \times X$ to $T \bowtie X$.

Proposition (Transfer Principle). *Given a T-flow on X, and a function $h : X \to H$ into the group H, define its extension h_G to the phase-group G by*

$$h_G((t, x)) = h(x).$$

Then the corresponding cocycle σ_G defined on $\Theta \times G$ by $h_G(\theta_s \bowtie g)h_G(g)^{-1}$ satisfies

$$\sigma_G(\theta_s, (t, x)) = \sigma_h(s, x) \text{ and, in particular, } \sigma_G(\theta_s, \xi_x) = \sigma_h(s, x).$$

Hence, if h is regularly varying on T, then h_G is regularly varying on Θ, and likewise, if h is regularly varying on X, then h_G is regularly varying on Ξ. That is,

$$\begin{aligned} \partial_X h(s) &= \lim_x h(sx)h(x)^{-1} = \lim_g h_G(\theta_s \cdot g)h_G(g)^{-1}, \\ \partial_T h(x) &= \lim_s h(sx)h(s(e_X))^{-1} = \lim_s h_G(\theta_s \cdot \xi_x)h_G(\theta_s)^{-1}. \end{aligned}$$

Proof. Interpreting G as the internal direct product of T and X in the sense of the representation theorem, we have

$$h_G(\theta_t \bowtie \xi_x) = h_G((t, t(x))) = h(tx), \text{ and } h_G(\xi_x) = h_G((e_T, x)) = h(x),$$

and, for $g = (t, x)$, we have

$$h_G(\theta_s \bowtie g) = h_G((s, s(e_X)) \bowtie (t, x)) = h_G((st, st(t^{-1}x)) = h(sx) = h_G(\theta_s \bowtie \xi_x).$$

Also $h_G(\theta_s) = h_G((s, s(e_X))) = h(s(e_X))$. Thus

$$\sigma_h(s, x) = h(sx)h(x)^{-1} = h_G(\theta_s \bowtie g)h_G(\xi_x)^{-1} = \sigma_G(\theta_s, \xi_x),$$

Thus we do indeed have

$$\begin{aligned}
\partial_X h(s) &= \lim_x h(sx)h(x)^{-1} = \lim_x h_G(\theta_s \cdot \xi_x)/h_G(\xi_x), \\
\partial_T h(x) &= \lim_s h(sx)h(s(e_X))^{-1} = \lim_s h_G(\theta_s \cdot \xi_x)/h_G(\theta_s),
\end{aligned}$$

as asserted. Here it is important to bear in mind that $||x|| \to \infty$ iff $||\xi_x|| \to \infty$, and $||t|| \to \infty$ iff $||\theta_t|| \to \infty$. $\square$

Remark. Recall that T_X is isomorphic to G under $xt \to \xi_x\theta_t = (t, t(x))$. The natural extension of $h : X \to H$ from X to T_X is via point-evaluation as given by

$$h_{T_X}(\tau) := h(\tau(e_X)) = h(t(x)), \text{ for } \tau = t_x \in T_X.$$

This is consistent with the transfer principle, since

$$h_G(\xi_x\theta_t) = h(t(x)) = h_{T_X}(xt).$$

4 Uniform boundedness theorems for cocycles

In the theorems of the next section we will be concerned with boundedness of cocycles. We say that σ is *locally bounded* (resp., *locally essentially bounded*) at $t \in T$ if, for some open neighbourhood $U \subset T$ of t, the set $\{\sigma(s, x) : s \in$

$U, x \in X\}$ is bounded in H (resp. the set $\{\sigma(s,x) : s \in U, x \in X\backslash E\}$ is bounded in H, for a meagre set E).

We will invoke somewhat less than continuity, placing instead conditions on the separate behaviours of $\sigma(t,.)$ and $\sigma(.,x)$. Examples below illustrate how these conditions may arise; however, it is as well to pause and consider the general significance of the separate continuity on T of the map $t \to \sigma(t,x)$. We note it is a natural assumption in the theory of integral equations (for which see [MS]) including the renewal equation of probability (see [Le]).

Specifically, consider the situation in a multiplicative framework, when $T \subseteq X$, so that $e_T = e_X$. Since T may act on T (being a subgroup), we examine the restriction of cocycles from $T \times X$ down to $T \times T$. Let $h : T \to H$. Note that $\sigma_h(t, e_T) = h(t)h(e_T)^{-1}$, from where h may be recaptured. Observe also the standardizations

$$h(t) = \sigma_h(t, e_T)h(e), \text{ and } \sigma_h(e_T, e_T) = e_H,$$

and additionally, w.l.o.g., we may also require $h(e_T) = e_H$ (since $H(t) = h(t)h(e_T)^{-1}$ generates the same cocycle as h on T).

Now let σ be an arbitrary cocycle from $T \times T \to H$ (implying association with the multiplicative T-flow on T), save only that it satisfies $\sigma(e_T, e_T) = e_H$. Put $k(t) = k_\sigma(t) := \sigma(t, e_T)$; then $\sigma_k(s,t)$ is a k-coboundary on T provided $\sigma(., e_T)$ is continuous. But,

$$\begin{aligned} \sigma_k(s,t) &= k(st)k(t)^{-1} = \sigma(st, e_T)\sigma(t, e_T)^{-1} \\ &= \sigma(s, te_T)\sigma(t, e_T)\sigma(t, e_T)^{-1} = \sigma(s,t). \end{aligned}$$

So if $\sigma(., e_T)$ is continuous, then σ itself is a k-coboundary on T, as $k(.)$ is continuous on T (cf. [Ell2] Prop. 2.4). To go in the opposite direction by taking $T = X$ is, generally, over-restrictive. For a more searching analysis, played out in a compact space setting, see [Ell2]; there (X, T) is extendable to (M, T), a 'universal minimal set', where the extended cocycle σ is a k_σ-coboundary.

A special case of the *first uniform boundedness theorem* below, when T is a subgroup of X and $\sigma = \sigma_h$, with $t \to \sigma(t,x)$ continuous on T, was proved by Bajšanski and Karamata; they stated only conclusion (ii), but a close inspection of their proof reveals the stronger, unstated, result (i). The brief proof for their case is reproduced here, for convenience and to document a new environment and the stronger conclusion, stronger than asserted in [BajKar].

In the *second uniform boundedness theorem* we weaken the continuity hypothesis to merely the Baire property and obtain only the weaker original conclusion of Bajšanski and Karamata. We prove this in a group setting and from that deduce the more general flow version.

The paradigm is of course the Banach-Steinhaus Theorem (for which see [Ru] Th. 2.5, p. 44), where X, H are topological vector spaces and Γ is a collection of continuous linear maps $t : X \to Y$ with bounded 'orbits' $\{tx : t \in \Gamma\}$. (Embed Γ in the finitely generated subgroup T which it generates in the additive group of bounded linear maps $\mathcal{B}(X, H)$; this gives a T-flow $(t, x) \to t(x)$.) Example 1 demonstrates that the weaker hypothesis here yields in general (say in an infinite-dimensional Hilbert space) a weaker result.

We say that T is a Baire group when T is a Baire space ([Eng]; see especially p.198, Section 3.9 and Exercises 3.9.J). The three distinct conditions appearing as pairs in Theorems 1 and 2 may be called *Baire Carathéodory conditions* after the three conditions of (Co) continuity, (M) measurability and (Bo) boundedness, applied by Carathéodory to the initial value problem (for details see [Good], and for a more recent example [BB]); here, these are Baire analogues, obtained by replacing 'measurable' with 'Baire property'. Recall that $||h|| := d(h, e_H)$ and note that '*for quasi all t*' means 'for all t off a meagre set'.

Theorem 1 (**First, or Continuous, Cocycle Uniform Boundedness Theorem**, cf. [BajKar], Th. 3). *Let X and H be topological groups and T a Baire group acting on X. Suppose the cocycle $\sigma : T \times X \to H$ is such that*

(Bo) *for quasi all $t \in T$, the mapping $x \to \sigma(t, x)$ is bounded over X,*
i.e. there a meagre set E^T and function $m : T \to \omega$ such that, for all $t \in T \backslash E^T$, $||\sigma(t, x)|| \le m(t)$, for all $x \in X$;
(Co) *for quasi any $x \in X$, the mapping $t \to \sigma(t, x)$ is continuous on T.*
Then
(i) $\sigma(t, x)$ *is essentially-bounded on the unit ball of T, and so*
(ii) $\sigma(t, x)$ *is uniformly essentially-bounded for t in compact subsets K avoiding E^T.*

Moreover, replacing 'quasi all' with 'all' yields the stronger conclusion obtained by replacing 'essentially-bounded' with 'bounded' and 'compact subsets K avoiding E^T' with 'all compact subsets K'.

Proof. We give a streamlined version of the proof in [BajKar] for the group version of the theorem; the transfer principle implies the flow version (see the second step of the second theorem below for an explicit deduction of the flow version). We suppose that (Co) and (Bo) holds off the respective meagre sets E^X and E^T of exceptions. For $n \in \omega$, put $F_n = \{h \in H : ||h|| \leq n\}$. For $n \in \omega$, put also

$$K_n(x) = \{t : \sigma(t, x) \in F_n\}, \qquad K_n = \bigcap \{K_n(x) : x \in X \backslash E_X\}.$$

By assumption (Co), for each $x \in X\backslash E_X$, the mapping $t \to \sigma(t, x)$ is continuous. Hence $K_n(x)$ is closed, for each $x \in X\backslash E_X$. Hence also K_n is closed. Now, for a given $t \notin E_T$, the set $\{\sigma(t, x) : x \in X\}$, being bounded, is contained in some $F_{m(t)}$. Hence $t \in K_{m(t)}(x)$ for each $x \in X$ in fact, and so $t \in K_{m(t)}$. Thus

$$T = E^T \cup \bigcup_{n\in\omega} K_n = \bigcup_{n\in\omega} E^T_n \cup \bigcup_{n\in\omega} K_n,$$

where each E^T_n is nowhere dense. By Baire's Theorem, for some open U and some $p \in \omega$, we have $U \subset K_p$. Thus, for $t \in U$ and arbitrary $x \in X\backslash E^X$, we have

$$||\sigma(t, x)|| \leq p,$$

i.e. σ is locally uniformly-essentially bounded at t. But this local assertion is true on sU for any $s \notin E^T$, because for any $t \in U$

$$\sigma(st, x) = \sigma(s, tx)\sigma(t, x),$$

and the set $\{\sigma(s, y) : y \in X\}$ is bounded, so that $\{\sigma(st, x) : t \in U, x \in X\backslash E^X\}$ is bounded.

This last result easily implies the weaker property of uniform essential-boundedness on compact sets. Indeed, let K be compact in $T\backslash E^T$. Since $(E^T)^{-1}$ is meagre, being a homeomorphic image of E^T, we may pick $t \in U\backslash(E^T)^{-1}$; thus $t^{-1} \notin E^T$. Since $e \in t^{-1}U$ we see that $kt^{-1}U$ is an open neighbourhood of k. Thus there are finitely many points $k_1, .., k_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} k_i t^{-1} U.$$

So for $k \in K$ there is $i \leq n$ and $s \in U$ such that $k = k_i t^{-1}s$. Again applying the defining property that $\sigma(st, x) = \sigma(s, tx)\sigma(t, x)$, we obtain

$$\begin{aligned} \sigma(k, x) &= \sigma(k_i t^{-1}s, x) = \sigma(k_i, t^{-1}sx)\sigma(t^{-1}s, x) \\ &= \sigma(k_i, t^{-1}sx)\sigma(t^{-1}, sx)\sigma(s, x). \end{aligned}$$

Since $s \in U$, the set $\{\sigma(s,x) : x \in X \backslash E^X\}$ is bounded. By assumption (Bo) the set $\{\sigma(t^{-1}, y) : y \in X\}$ is bounded, and likewise, so is each of the sets $\{\sigma(k_i, z) : z \in X\}$ for $i = 1, ..., n$. Hence the set $\{\sigma(k,x) : k \in K, x \in X \backslash E^X\}$ is bounded, i.e. $\sigma(k,x)$ is bounded uniformly for $x \in X \backslash X_E$ with K ranging over compact sets in $T \backslash E^T$.

Taking $E^T = E^X = \varnothing$, a re-reading of the arguments above yields the asserted strengthenings. $\square$

The assumption (Co) is weakened in the following theorem and consequently the conclusion is also weaker. The proof is more involved as it employs the Category Embedding Theorem, a result that we quote below after a definition from [BOst11] (to which we refer also for its proof).

Definition (**weak category convergence**). A sequence of homeomorphisms ψ_n satisfies the *weak category convergence* condition (wcc) if for any non-empty open set U, there is an non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n \geq k} V \backslash \psi_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ \psi_n(t) \in V.$$

Category Embedding Theorem. *Let X be a Baire space. Suppose given homeomorphisms $\psi_n : X \to X$ for which the weak category convergence condition* (wcc) *is met. Then, for any non-meagre Baire set T, for locally quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{\psi_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Example. In any metrizable group with invariant metric d, for any sequence tending to the identity $z_n \to e$, the mappings defined by $\psi_n(x) = z_n x$ satisfy the (wcc) holds. For a proof see [BOst13].

Theorem 2 (**Second, or Baire, Cocycle Uniform Boundedness Theorem**, cf. [BajKar], Th. 3). *Let X and H be topological groups and T a Baire group acting on X. Suppose the cocycle $\sigma : T \times X \to H$ is such that* (Ba) *for each fixed $x \in X$, the mapping $t \to \sigma(t,x)$ is Baire on T,*

(Bo) *for quasi all* $t \in T$, *the mapping* $x \to \sigma(t,x)$ *is bounded over* X, *i.e. there a meagre set* E^T *and function* $m : T \to \omega$ *such that, for all* $t \in T \backslash E^T$, $||\sigma(t,x)|| \leq m(t)$, *for all* $x \in X$.
Then
$\sigma(t,x)$ *is uniformly bounded for* t *on compact subsets* K *avoiding* E^T.

Moreover, replacing 'quasi all' with 'all' yields the stronger conclusion obtained by replacing 'compact subsets K *avoiding* E^T*' with 'all compact subsets* K*'.*

Proof. Our first step is to prove the result for T a subgroup of X. As a second step we infer the result for flows.

We suppose that (Bo) is satisfied off a meagre set E^T of exceptions. Suppose, by way of contradiction, that $t_n \to t_0 \notin E^T$ and $\{\sigma(t_n, x_n) : n \in \omega\}$ is unbounded. We may assume that $t_0 = e$; indeed

$$\sigma(t_0^{-1}t_m, x_m) = \sigma(t_0^{-1}, t_m x_m)\sigma(t_m, x_m),$$

and by assumption (Bo), the set $\{\sigma(t_0^{-1}, z) : z \in X\}$ is bounded, hence $\{\sigma(t_0^{-1}t_n, x_n) : n \in \omega\}$ is unbounded and here $t_0^{-1}t_n \to e$.

For each n, the mapping $h_n(.) = \sigma(., x_n)$ is Baire. Let $Y := \{x_i : i \in \omega\}$. On a co-meagre set $S \subset T$ each function $h_n(.)$ is continuous on S. We may suppose that S is complementary to E^T. We now adapt the proof in [BajKar] by working with S and Y in place of T and X. Recalling that, as usual, $||h|| = d(h, e_H)$, put $F_n = \{h \in H : ||h|| \leq n\}$ and

$$K_n(x_i) = \{t \in S : \sigma(t, x_i) \in F_n\}, \qquad K_n = \bigcap\{K_n(x_i) : i \in \omega\}.$$

Thus K_n is Baire. Now, for a given $t \in S$, the set $\{\sigma(t,x) : x \in Y\}$, being bounded, is contained by some $F_{m(t)}$. Hence $t \in K_{m(t)}(x)$ for each x, and so $t \in K_{m(t)}$. Thus

$$S = \bigcup_{n \in \omega} K_n.$$

Now for some p, K_p is non-meagre. By the category embedding theorem [BOst11], for some $s \in S$ (implying that $s \notin E^T$) and some infinite $\mathbb{M}$, the set $\{st_m : m \in \mathbb{M}\} \subset K_p$. Thus, in particular,

$$|\sigma(st_m, x_m)| \leq p.$$

But

$$\sigma(st_m, x_m) = \sigma(s, t_m x_m)\sigma(t_m, x_m).$$

Now again by assumption (Bo), the set $\{\sigma(s,z) : z \in X\}$ is bounded, as $s \notin E^T$. But this contradicts the unboundedness of $\{\sigma(t_m, x_m) : m \in \mathbb{M}\}$.

Taking $E^T = E^X = \varnothing$, a re-reading of the arguments above again yields the asserted strengthenings. $\square$ (group setting)

Our second step is to deduce the theorem from its group formulation. For $h : X \to H$, and with $G = T \times X$, define the extension $h_G : G \to H$ by

$$h_G((t,x)) = h(x).$$

Then, interpreting G as the internal direct product of T and X in the sense of the representation theorem, we have

$$h_G(\sigma_t \bowtie \xi_x) = h_G((t, t(x))) = h(tx), \text{ and } h_G(\xi_x) = h_G((e_T, x)) = h(x),$$

and so

$$\sigma(t,x) = h_G(\sigma_s \bowtie \xi_x) h_G(\xi_x)^{-1} = h(tx)h(x)^{-1}.$$

Now apply the group version of the theorem established in the first step. $\square$

Theorem 3 (**Third, or Asymptotic, Cocycle Uniform Boundedness Theorem**, cf. [BGT], Th. 2.0.1). *Let X and H be topological groups with right-invariant metric. Let T a Baire group acting on X. Suppose the cocycle $\sigma : T \times X \to H$ is such that*
(Ba) *for each fixed $x \in X$, the mapping $t \to \sigma(t,x)$ is Baire on T,*
(ABo) *for quasi all $t \in T$, the mapping $x \to \sigma(t,x)$ is asymptotically bounded over X,*
i.e. there a meagre set E^T and functions $m, k : T \to \omega$ such that, for all $t \in T \backslash E^T$, $||\sigma(t,x)|| \leq m(t)$, for all x with $||x|| \geq k(t)$.
Then
$\sigma(t,x)$ is uniformly bounded for t on compact subsets K avoiding E^T.

Proof. We argue as in Theorem 2 and but now specifically suppose $||\sigma(u_n, x_n)|| > n$ for chosen sequences $\{u_n\}$ in T and $\{x_n\}$ in X with $u_n \to u$ and $||x_n|| \to \infty$. Now boundedness at t implies that, for all $n > k(t)$, we have

$$||\sigma(t, x_n)|| < m(t) < \frac{1}{2}n.$$

Put

$$T = E^T \cup \bigcup_k T_k \text{ with } T_k = \bigcap_{n \geq k} \{t : ||\sigma(t, x_n)|| < \frac{1}{2}n\}.$$

By (Ba), for each k, the set T_k is Baire. For some K, we see that T_K is non-meagre, so there is s and an infinite $\mathbb{M}_s > K$ such that

$$\{su_m : m \in \mathbb{M}_s\} \subseteq T_K.$$

This gives, for $m \in \mathbb{M}_t$, that

$$||\sigma(su_m, x_m)|| < \frac{1}{2}m.$$

We claim that $||u_m x_m|| \to \infty$; otherwise, by inversion-invariance, $||u_m^{-1}|| = ||u_m||$ is bounded, so boundedness of $||u_m x_m||$ would imply boundedness of $||x_m||$ from

$$||x_m|| = ||u_m^{-1} u_m x_m|| \leq ||u_m^{-1}|| + ||u_m x_m||.$$

Now, for $m \in \mathbb{M}_s$ such that $||u_m x_m|| > k(s)$, we have $||\sigma(s, u_m x_m)|| \leq m(s)$. But, by the defining property of a cocycle,

$$\sigma(su_m, x_m) = \sigma(s, u_m x_m)\sigma(u_m, x_m),$$

which implies that

$$||\sigma(u_m, x_m)|| = ||\sigma(s, u_m x_m)^{-1}\sigma(su_m, x_m)|| \leq ||\sigma(s, u_m x_m)^{-1}|| + ||\sigma(su_m, x_m)||.$$

So, using inversion-invariance and the triangle inequality of the group-norm, we have, for $m \in \mathbb{M}_s$ such that $||u_m x_m|| > k(s)$ that

$$m < ||\sigma(u_m, x_m)|| \leq \frac{1}{2}m + m(s) \leq \frac{1}{2}m + \frac{1}{2}m \leq m,$$

a contradiction. $\square$

Remarks

1. When H is the real line there is the opportunity to interpret unboundedness in two directions.

2. There is an implicit affinity between Theorem 3 and extensions of the Karamata Theory of regular variation (for which see [BGT] Ch. 2). The classical context places the *asymptotic boundedness assumption* on $h : X \to H$, which at its simplest requires that there exists $m^* : T \to \omega$, such that

$$\lim_n \sup_{||x|| \geq n} ||h(tx)h(x)^{-1}|| < m^*(t).$$

From this hypothesis, in the case when $T = H = X = \mathbb{R}$, one deduction of [BGT] Th. 2.0.1 p. 62 is a Uniform Asymptotic Boundedness Theorem, that for K compact

$$\lim_n \sup_{||x|| \geq n} \sup_{t \in K} ||h(tx)h(x)^{-1}|| < \infty.$$

This is implied by Theorem 3. In the classical one-dimensional case, UABT in turn yields a regularly varying function of t dominating $h(tx)h(x)^{-1}$ for all large x and t. I conjecture that the theorem generalizes to a multivariate form with varying indices in the various flow directions. It would be interesting to see whether these indices would remain bounded when X is locally compact (presumably so in the abelian case).

Illustrative Example (Euclidean equivalence of UBT with Uniform Convergence Theorem). For $h : X \to H$ and a given T-flow on X, the map $t \to \sigma_h(t, x)$ is continuous/Baire, if the function h is continuous/Baire since $(t, x) \to tx$ is continuous ('iff' when $T = X$).

Suppose now that X, H are normed vector spaces and T is a subspace of X acting on X by translation. Assume first that h is linear. Reverting to the abelian additive notation, we have

$$\sigma_h(t, x) = h(tx) - h(x) = h(t),$$

so that for fixed t the map $x \to \sigma_h(t, x)$ is bounded. More generally, assume that h is Baire and regularly varying on T, that is, (Section 2 or [BOst13]), the limit function

$$\partial_X h(t) := \lim_{||x|| \to \infty} \sigma_h(t, x) \tag{4}$$

exists for all t. Indeed, according to the Uniform Convergence Theorem (see [BOst13] for the general metrizable topological group setting of UCT, and [BGT] for the special case of $X = \mathbb{R}$), convergence to dh is uniform for t restricted to compact sets. We take up this point in a later step.

For now **fix** t; then, for all x with $||x||_X$ large enough, for simplicity say for $||x||_X > 1$,

$$||\sigma_h(t, x)||_H \leq ||\partial_X h(t)||_H + ||\sigma_h(t, x) - \partial_X h(t)||_H. \tag{5}$$

If X is finite-dimensional (Euclidean) and additionally h is continuous, then $||\sigma_h(t, x)||_H$ is bounded on the unit ball $||x||_X \leq 1$ and so again, for fixed t, the map $x \to \sigma_h(t, x)$ is bounded. Here both Theorem 1 and 2 assert that $\sigma_h(t, x)$ is bounded on the unit ball of T.

Here is an alternative proof from UCT. Observe that $\partial_X h$ is additive by (2), and, being Baire (4), is linear (by the Banach-Mehdi Theorem, see e.g. [Ban] 1.3.4, p. 40 in collected works, cf. [Meh], or the literature cited in [BOst14], or [BOst13]), since the Euclidean space T is Baire. Thus $\partial_X h$ here is continuous, so has bounded operator norm, and hence $||\partial_X h(t)||_H \leq ||\partial_X h|| ||t||_X$. This together with the UCT applied to (5) confirms that, for t restricted to the unit ball in T, i.e. when $||t||_X \leq 1$, the function $\sigma_h(t,x)$ remains bounded as x varies arbitrarily. We have just shown the following new result.

Proposition. *For h continuous, the* UCT *and the* UBT *are equivalent in the Euclidean setting.*

Remark. A close inspection of the proofs above, shows that they depend on the cocycle property and the convergence of the sequence tz_n to t when $z_n \to e$. It seems plausible that the proof of Theorem 2 could be carried out in a metrizable groupoid setting.

5 Applications in functional analysis

We give two examples of applications of the UBT to functional analysis. The first clarifies the relationship between UBT for cocycles and the Banach-Steinhaus Theorem. The other views group characters corresponding to maximal regular ideals as cocycles.

Example 1. (Adaptation of the 'equicontinuity example' of [BajKar].) Let V and H be topological vectors spaces regarded as additive groups, with V Baire (e.g. a Banach space). For simplicity, we consider a countable family of continuous linear mappings from V to H, presented for convenience as $\{L_m : m \in \mathbb{Z}\}$. Suppose that, for each $x \in V$, the set $\{L_m(x) : m \in \mathbb{Z}\}$ is bounded in H. We deduce that the family is uniformly bounded on compact subsets of V.

Form the direct product $X = V \times \mathbb{Z}$ of V with the additive group of integers. Take $T := \{(x, 0) : x \in V\}$, a subgroup of X isomorphic to V, hence a Baire group. Define the additive function $h : X \to H$ by

$$h((x, n)) = L_n(x).$$

Consider the h-cocycle $\sigma_h : T \times X \to H$, defined as in (3). Then, with $g = (y, m)$ and $t = (x, 0)$, we have

$$\begin{aligned}\sigma_h(t, g) &= \sigma_h((x, 0), (y, m)) = h((x, 0) + (y, m)) - h((y, m)) \\ &= L_m(x + y) - L_m(y) = L_m(x).\end{aligned}$$

Hence,
(i) for fixed g, the map $t \to \sigma_h(t, g)$ is Baire; indeed, for fixed m, the map $x \to L_m(x)$ is continuous;
(ii) for fixed $t = (x, 1)$, the map $g \to \sigma_h(t, g)$ is bounded in H; indeed, for fixed $x \in V$, the map $(y, m) \to L_m(x)$ is bounded on X.

Theorem 2 above asserts that $\{L_m(x) : m \in \mathbb{Z}\}$ is uniformly bounded in H for x in any compact subset of V. On the other hand, Theorem 1, with its stronger assumption that each map $x \to L_m(x)$ is continuous, implies that σ_h is locally uniformly bounded, so that $\{L_m(x) : ||x|| < 1, m \in \mathbb{Z}\}$ is bounded.

Example 2. We refer to [Loo] for standard terminology used here. When $X = \mathcal{C}(T)$ is the Banach algebra of continuous, complex-valued functions on a locally compact group T, consider the familiar continuous action of T on X given by $(t, x) \to tx$, where

$$(tx)(s) = x(t^{-1}s).$$

Thus if $h : G \to \mathbb{C}$ is an algebra homomorphism (multiplicative, as well as homogenous and additive), then, for any $x \notin \mathcal{N}(h)$, the formula $\alpha_h(t) := \sigma_h(t, x) = h(tx)/h(x)$ defines a character on T corresponding to the kernel $\mathcal{N}(h)$, viewed as a maximal regular ideal of functions (see e.g. [Loo] p. 135). The notation for α_h reflects the known fact that $h(tx)/h(x)$ is independent of x. Here h is continuous and, as in Example 1, $x \to \sigma_h(t, x)$ is trivially bounded as a function of x. As an immediate corollary we see that $\alpha_h(t)$ is uniformly bounded on compact subsets of T; indeed, in view of the continuity, it is locally uniformly bounded. In fact of course the cocycle equation (2) implies that $\alpha_h(t)$ is multiplicative (reducing in this case to Cauchy's functional equation). The conclusion here is a special case of the Uniform Convergence Theorem (UCT) of regular variation (see [BGT] for the classical setting of functions $h : \mathbb{R} \to \mathbb{R}$ and [BOst13] for a topological setting); the UCT asserts that the limit function $\partial_X h(t) := \lim_x \sigma_h(t, x)$, if it exists, is multiplicative (with uniform convergence on compacts), thus providing a representation for

$\partial_X h(t)$ in the classical setting via Cauchy's functional equation, or in the topological setting via a Riesz Representation Theorem.

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6 Appendix -1 : Skew-product flows

A fixed $f \in \mathcal{C}(T \times Z, Z)$, with $T = \mathbb{R}$ and Z a normed vector space (e.g. $\mathbb{R}^d$), gives rise to the non-autonomous differential equation

$$\dot{u}(t) = f(t, u(t)) \text{ with } u(0) = z.$$

For simplicity assume f is such that, for each z, the solution exists uniquely and globally. Then, regarding f as a parameter, we may write the solution in the form

$$u(t) = \phi(t, f, z).$$

The system is said to be linear if $\phi(t, f, z) = \alpha(t, f)z$ with $\alpha(t, z)$ a linear operator. Note that for the system $f(t, z) = Az$, with A a constant matrix, the operator $\alpha(t, f) = e^{tA}$ is *invertible*, a feature to which we return.

The non-autonomous system was reformulated in flow terms by G. R. Sell as follows (see [Se1], [Se2]). Put $sf(t, z) := f(s + t, z)$ and let $Y = \{y \in \mathcal{C}(T \times Z, Z) : y = sf$ for some $s \in T\}$, then the binary operation on Y defined by

$$sf \cdot s'f = (s + s')f$$

turns Y into a group. If Y is given a topology such that $s \to sf$ is continuous Y becomes a topological group.

Time-shifting the d.e by s we have $s\dot{u}(t) = f(s + t, su(t)) = sf(t, su(t))$ with $su(0) = z$. Thus

$$su(t) = \phi(s + t, sf, z),$$

and the analysis is reduced to the study of the T-flow π on $Y \times Z$ given by:

$$\pi(s, y, z) = (sy, \phi(s, y, z)),$$

known as a skew-product flow.

There are two components in π. Now, for two commuting flows U and V on X, the action $T = U \times V$ is an internal direct product and the theorem asserts that both flows on X are representable by commuting multiplications. Here we show that an analogous representation can be coaxed out for the general linear skew-product flow π defined as T-flow on $X = Y \times Z$ with Y a topological space and Z a normed vector space in which π takes the form

$$\pi(t, y, z) = (t(y), \alpha(t, y)z).$$

Here $\alpha(t,y) \in L(Z)$ is an *invertible* linear operator from Z to Z with $\alpha(e_T, e_Y) = I$ whereas $(t,y) \to t(y)$ is a flow in Y. We may obtain a multiplicative representation starting from the equation

$$(s, sa, \Lambda(s,sa)u) \bowtie (t, tb, \Lambda(t,tb)v) = (st, st(ab), \Lambda(st, st(ab))(u+v)),$$

since with $v = -u$ and $b = a^{-1}$ and $t = s^{-1}$ we have

$$(s, sa, \Lambda(s,sa)u) \bowtie (t, tb, \Lambda(t,tb)v) = (e, e, 0).$$

But this requires that we work with the product $\{(s,y,q) : s,y,q\}$ with the equation $q = \Lambda(s,y)v$ in mind. We put

$$(t,x,p) \bowtie (s,y,q) = (ts, ts(t^{-1}xs^{-1}y), \Lambda(ts, ts(t^{-1}xs^{-1}y))[\Lambda(t,x)^{-1}p + \Lambda(s,y)^{-1}q].$$

Here

$$(t, t(e), u) \bowtie (e, y, 0) = (t, t(y), \Lambda(t, t(y))u) = \pi(t,y,u),$$

and moreover, since $\Lambda(e,y) = I$ (see A spectral Theory p. 324 property (2)) we have

$$\begin{aligned}
\theta_s &\bowtie \theta_t = (s, s(e), 0) \bowtie (t, t(e), 0) = (st, st(e), 0) = \theta_{st}, \\
\theta_t &\bowtie \eta_y = (t, t(e), 0) \bowtie (e, y, 0) = (t, t(y), 0), \\
\theta_t &\bowtie \zeta_u = (t, t(e), 0) \bowtie (e, e, u) = (t, t(e), u), \\
\eta_y &\bowtie \zeta_u = (e, y, 0) \bowtie (e, e, u) = (e, e(y), \Lambda(e,y)u), \\
\theta_t &\bowtie \eta_y \bowtie \zeta_u = \theta_t \bowtie (e, e(y), \Lambda(e,y)u) = (t, t(e), 0) \bowtie (e, e(y), \Lambda(e,y)u) \\
&= (t, t(y), \Lambda(t, ty)[0 + \Lambda(e,y)^{-1}u]) \\
&= (t, t(y), \Lambda(t, ty)u) = (t, \pi(t,y,u)).
\end{aligned}$$

Here the first and last lines confirm that the mutiplicative flow $\varphi(t,g) = \theta_s \bowtie g$ is isomorphic to the T-flow (homomorphic by the first lien with image corresponding to the T-flow image.)

$$\zeta_u \bowtie \zeta_v = (e,e,u) \bowtie (e,e,v) = (e,e,u+v).$$

$$\theta_s \bowtie \theta_t \bowtie \eta_y \bowtie \zeta_u = \theta_{st} \bowtie \eta_y \bowtie \zeta_u$$
$$\sigma(st, x) = \sigma(s, tx)\sigma(t, x)$$

$$
\begin{aligned}
(st, st(y), \Lambda(st, st(y))u) &= \pi(st, y, u)\\
(t, t(y), \Lambda(t, t(y))u) &= \pi(t, y, u)\\
(t, t(y), \Lambda(t, t(y))u) &= \pi(s, \pi(t, y, u)) = (st, st(y), \Lambda(st, st(y))u)
\end{aligned}
$$

$$(s, sa, \Lambda(s, sa)u) \bowtie (t, tb, \Lambda(t, tb)v) = (st, st(ab), \Lambda(st, st(ab))(u + v)),$$

$$
\begin{aligned}
(s, se, \Lambda(s, se)0) &\bowtie (t, tb, \Lambda(t, tb)v) = (st, st(b), \Lambda(st, st(b))(0 + v))\\
\pi(s, \pi_2(t, b, v)) &= (s, tb, \Lambda(t, tb)v)\\
&= \pi(st, b, v)
\end{aligned}
$$

$$(s, t(y), \Lambda(t, t(y))v) \bowtie (t, t(y), \Lambda(t, t(y))u) = (st, (s^{-1}ty)y$$

7 Appendix - 2 : action groupoids

We offer a brief explanation of an action-groupoid, side-stepping, as is possible for T-flows on X, the language of category theory. For our purposes a groupoid arises as a structure which resembles a group but with two deficiencies: its binary operation is not necessarily defined on all pairs and there are both left and right identities for each element of the groupoid, and corresponding inverses. In principle a groupoid is presented as a space of points (objects) together with a space of arrows (morphisms), but here we can ignore the former. Indeed, in the case of a T-flow on a topological group X, with the space X taken as the space of objects (let's agree to call points *locations*) and $T \times X$ as the space of arrows (t, x) with *source* x and *target* $t(x)$, objects are superfluous, since the source map here is freely available as a projection from $T \times X$ (so, the objects form the set of sources).

Adopting instead the vectorial language of linear algebra, we regard an arrow as comprising a *displacement* together with a location to which the displacement is applied (yielding the *target*). The binary operation is composition of two arrows, (t, x) followed by (s, y), and is possible if only if $y = t(x)$ (then the arrows are said to be a *composable,* ordered pair); that is, the target of the first displacement provides the location for a subsequent displacement. The *composition* $(s, tx) \circ (t, x)$ is then (st, x) with target $st(x)$. Here e_T may be regarded as providing null displacements; specifically, $\xi_y = (e_T, y)$ provides the right identity (under composition) for the source y of (s, y), while (e_T, y) with $y = tx$ provides the left identity under composition for the target $y = tx$ of (t, x). Accordingly we term the point ξ_y in the group G a *source.*

This intuition leads to the natural embedding $\gamma : T \times X \to \Theta\Xi$, of arrows to the phase-group G is

$$\gamma(t, x) := (t, t(x)).$$

We call (t, e_X) a *basic displacement*, as it is applied to the base point e_X of X; this is carried to $\gamma(t, e_X) = (t, t(e_X))$, i.e. to the point θ_t of Θ. We therefore call θ_t a *displacement*. In consequence, we have the unique representation of an arrow in G as a multiplicative decomposition

$$\gamma(t, x) = \theta_t \bowtie \xi_x,$$

i.e. the product in G of a displacement θ_t and a source ξ_x.