Endogenous Irreversibility With Finite Horizon Investment When Resale Is Possible

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Abstract

In this paper we consider the situation where partial reversals from a finite horizon investment are possible at discrete decision times. We are able to identify a range of circumstances in which reversal is triggered by the investment resale rate, R, falling below a critical value $R^{\#}$, assuming R to be a random variable identically and independently distributed over decision times. This result is similar in qualitative form to results derived in related infinite horizon formulations in that, once the random variable hits a key "favorable" interval, the optimal policy switches to reversal of investment. However, we also show that the alternative geometric Brownian model for the resale rate (for $R-1$) implies restrictions on the ranges over which reversal remains optimal which are significantly different in a number of ways. For instance we identify near-termination effects which do not arise in infinite horizon settings in which the favorable interval changes in magnitude as the final decision epoch approaches. Also we show that the optimal policy space is richer in that the range over which reversal is optimal can take a wider set of forms and so knowledge of a single $R^{\#}$ may be insufficient to guide optimal investment policy. In particular we show that for certain values of the parameters a twin switching policy may be optimal under which as R increases, reversal is first non optimal, then optimal and finally non optimal again.

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1 INTRODUCTION

Since many investment projects are made with a finite horizon in mind, the optimality of taking opportunities to reverse out of such a project may in practice be sensitive to end effects which arise in finite but not in infinite horizon settings. Thus, for instance, the parameter ranges over which investment / disinvestment is optimal in an infinite horizon setting may differ in important ways from the parameter ranges appropriate to a finite horizon setting.

We investigate a finite horizon model in which investment / disinvestment decisions are taken at discrete times (regarded as sampling times) and derive the parameter ranges over which a policy of reversal (disinvestment) is optimal. Our model is one in which the timing of investment is motivated by capital input price hedging considerations[1]. This arises because the capital input is assumed to be expected to rise in price as time progresses and hence it may be in the interests of the firm to overinvest (relative to current needs) in capital stock. In reality firms' investment decisions are complicated by a mix of problems such as lumpiness, changing technology and capital input price drift. In order to understand the effects of the later concern in a finite horizon setting, we shall assume that firms can purchase any quantity of capital (no lumpiness) and that the technical efficiency of capital remains constant through time. This allows us to concentrate upon pure capital hedging considerations and derive clear intuition for optimal policy in such settings. Clearly, a task for future research would be to incorporate features of the other two complications.

Rather than assume firms must use all the capital they purchase, following Abel and Eberley (1996), we allow them to reverse out of investment positions at a cost. Our model setting in discrete time is thus close in spirit to that of Eberley and Van Mieghem (1997) in that firms face 'kinked costs', but it is our costs (rather like their returns) that evolve stochastically, whereas contrarily our returns are deterministic. In fact our returns are of Cobb-Douglas form, which enables a tractable determination of what corresponds here to their 'optimal ISD control limits' (acronym for Invest/Stay put/Disinvest) and which are referred to herein as 'censors'. A similar Cobb-Douglas formulation is taken by Hartman and Hendrickson (2002), albeit in continuous time and with infinite time horizon, but though their costs and returns are stochastic, the distinguishing feature is that their resale rate R is constant in time. However, the treatment of capital is different here, in that, consistently with an 'inventory' interpretation, capital depreciates only through 'useage' (completely,when used), rather than merely 'eroding' at a constant 'geometric rate' that is totally unconnected with useage. In light of this, the 'optimal ISD control limits' here, though similar, are indeed different, and need to be derived.

We are able to identify a range of situations in which reversal is triggered by a critical value of the time varying investment resale rate which is initially assumed to be independently and identically distributed at the (sampling) decision moments. The results derived are similar in qualitative form to those derived in related infinite horizon formulations of Abel and Eberley (1996), in that once the resale rate variable enters a critical "favorable" range, the optimal policy

[1]See Hopp and Nair (1991) who discusses in detail additional elements of the hedging element of investment decisons.

switches to reversal. However, we also show that adoption, as an alternative, of a standard geometric Brownian model for the resale rate, leads qualitatively significantly different ranges over which reversal is optimal. Indeed, we show that a twin switching policy in which reversal is only optimal for a bounded range of resale rates $[R^{\#}, R^{\#\#}]$ can apply[2]. We provide intuition for the findings and then consider how the results change as we vary the length of the investment horizon. Next, in order to gain further insight into the significance of these results we compare and contrast our models characteristic properties to those of the established Abel and Eberley (1996) model[3]. We point out how the two models differ in important respects, over and above the differing horizon length assumptions and explore why adopting a finite horizon formulation can lead to a significant revision in the characterization of the optimal policy space.

The paper is organized as follows. In section 2 we present our simplest investment model, that with two periods and irreversibility. In section 3 we relax the assumption of irreversibility allowing the firm the possibility to resell part or all of the investment. In section 4 we compare and contrast our results to the Abel and Eberley (1996) model of optimally triggered partial reversibility. We present concluding comments in section 5. The general N-period model is derived in Appendix 1 and offer numerical examples for $N = 3$ to illustrate the various qualitative differences that can arise; the derivation depends on a number of technicalities which have been removed to Appendices 2A (inductive derivation of a formula), 2B (justification of certain limit operations). Appendix 3 derives the endogenous depreciation rate for a two period model in terms of a general Cobb-Douglas index.

2 TWO PERIOD MODEL WITH IRREVERSIBILITY

A critical feature of any investment model is the specification of how capital is consumed. In the simplest setting of two periods a model in which investment is depreciated purely as a function of clock time is unattractive, because management only have a passive role to play. Instead we develop a model in which capital depreciation results from conscious decisions by mangement. That is in a two period setting, given for the moment a fixed stock of capital $\overline{v}$ (that can not be added to or resold), we assume management needs to choose how optimally to split the allocation of the opening capital input $\overline{v}$ between the two periods. That is our model is one in which capital depreciates through usage. This assumption is important when hedging and resale considerations arise. For instance, when the depreciation (usage) rate is a choice variable, rather than having to resell unused capital at a discounted resale rate, it is more realistic to allow management to alternatively decide to apply it to current production at a greater rate. Similarly the desirability of hedging is influenced by the subsequent possibility to vary capital

[2]These findings are sumarized in subsection 3.5 by the M-S (mean-standard deviation) diagram.

[3]It is important to stress that our formulation is not a finite horizon version of the Abel and Eberley model. In part this arises because tractability is more difficult to maintain in a finite setting and because we argue that working within a finite horizon setting motivates one to characterize management policy choice in a different fashion. In particular, in an infinite horizon setting optimal depreciation of an investment may differ substantially from that in a finite setting. We discuss this in detail in section 5.

consumption if ex-post a position becomes over-hedged given the actual evolution of the capital input price.

In the subsection 2.1 we shall formally analze the problem of how to allocate the first period's capital stock between two periods. Initially it is assumed for simplicity that the capital can be added to in the final (second) period but not first period. In subsection 2.2 we then relax this assumption. Throughout we assume a two period setting with irreversibility of any capital investment, but relax this irreversibility assumption in the section 3.

Commencing at time t_0, we assume that a firm has v_{t_0} ($v_{t_0} \geq 0$) units of capital in stock. Given the firm can purchase some more capital in the next period the decision of how to optimally allocate capital stock between the current and latter period will ceteris paribus be driven by the capital input price process. We shall denote the price of capital as b_t. It is assumed to evolve stochasticly as a geometric Brownian motion with positive drift[4] (anticipated growth) $\mu_b > 0$, i.e.

$$db_t = b_t(\mu_b dt + \sigma_b dW_b(t))$$

where $W_b(t)$ is a standard Wiener process. The firm observes the price at discrete times, in this case at times t_0 and t_1 and purchase levels of capital at these discrete moments is denoted z_{t_0} and z_{t_1}. In order to track the stock of capital carried forward between periods we shall denote the period t_0 closing capital stock as u.

2.1 Optimizing capital stock u carried forward when no current period purchases (NCP) are allowed ($z_{t_0} = 0$)

The time order diagram reflecting this decision regime is as follows:

Time Order Line

At $t = t_0$ with a stock of v_{t_0} units of input:
Observe the current price of capital b_{t_0}.
Predict $E(b_{t_1})$ the expected price for the following period, under the assumption that $E(b_{t_1}) > b_{t_0}$.
Between t_0 and t_1 :
Apply x_{t_0} ($\leq v_{t_0}$) units of capital into production and carry forward $u = v_{t_0} - x_{t_0}$ units.

At $t = t_1$ after observing b_{t_1}, purchase z_{t_1} units at the prevailing price of b_{t_1}.

Between t_1 and t_2 :
use $x_{t_1} = u + z_{t_1}$ remaining units of capital in production.

[4]The drift is net of an implicitly assumed constant interest rate. Thus b_t is to be regarded as a depreciated price.

The question whether any stock of capital should be carried forward is particularly pertinent when we restrict consideration to the case of $z_0 = z_{t_0} = 0$, since under this regime increasing u (priced at b_{t_0}) means less is made available for the current period.

As the above time order line makes clear, all the subscripts are timings. So we will drop the t subscript when a specific time is being referenced. To simplify the presentation further it is assumed[5] that the functional form for the gross returns function is $2\sqrt{x_t}$. Given the initial value for v_0, the **risk-neutral** firm chooses the optimal first period production plan x_0^* such that:

$$x_0^* = \arg\max 2\sqrt{x_0} + F_1(v_0 - x_0, b_0), \tag{1}$$

where $F_1(.)$, is the expected optimal value function for the subsequent period (which we will compute below) and $v_0 - x_0 =_{def} u$ is the capital stock carried forward to the next period. Reformulating this production planning problem directly in terms of the carried forward capital stock we can rewrite (1) in terms of choice of u as:

$$u^* = u_1(v_0, b_0) = \underset{0 \le u \le v_0}{\arg\max} 2\sqrt{v_0 - u} + F_1(u, b_0). \tag{2}$$

Let us now turn to consider the form for $F_1(u, b_0)$. It is straight-forward to show that it is a piecewise 'dichotomous' continuous function, in that its form depends upon whether the capital carried forward u, set at t_0, turns out to be too large or, too small given the subsequently realized input price of b_1 at t_1. To see this consider the firm's production decision at t_1. Disregarding u and $z_1 = z_{t_1}$ for the moment, having observed b_1 the (direct) returns function in the second period is:

$$2\sqrt{x_1} - b_1 x_1$$

and thus the first-order conditions for the optimal production imply the firm need purchase:

$$x_1^* = \frac{1}{(b_1)^2} \tag{3}$$

units. Hence the form[6] of the indirect returns function is:

$$\begin{aligned} G_1(b_1 | u = 0) &= 2\sqrt{\frac{1}{(b_1)^2}} - b_1 . \frac{1}{(b_1)^2} \\ &= \frac{1}{b_1}. \end{aligned}$$

Thus, if we now allow u to be a free variable we can identify two different scenarios arising in the final period, as follows:

[5]In general we require concavity of the returns function. However, in our presentation we specialize to the square root for simplicity of presentation.

[6]This most simple form results from our choice of the square root returns function.

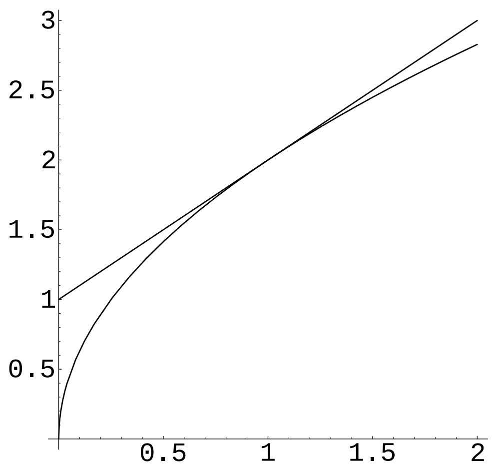

Figure 1: Final period returns function \$G_1[u,b_1]\$ against \$u\$

(i) Given (3) if the hedge is too small, that is $u < \frac{1}{(b_1)^2}$, it is optimal to purchase a further $z_1 = \frac{1}{(b_1)^2} - u$ units and earn:

$$\frac{2}{b_1} - b_1 z_1 = \frac{1}{b_1} + b_1 u, \tag{4}$$

otherwise,
(ii) Given (3) if the hedge is (weakly) too large, that is $u \geq \frac{1}{(b_1)^2}$, it is optimal to set $z_1 = 0$ and apply all u units to production and earn:

$$2\sqrt{u}.$$

Thus the period t_1 returns function $G_1(u, b_1)$ takes the form:

$$G_1(u, b_1) = \begin{cases} \frac{1}{b_1} + b_1 u & \text{if} \quad u < \frac{1}{(b_1)^2} \\ 2\sqrt{u} & \text{if} \quad u \geq \frac{1}{(b_1)^2} \end{cases} \tag{5}$$

as illustrated below.
Now we can determine the expected optimal value function. We have:

$$F_1(u, b_0) = E_{b_1|b_0}[G_1(u, b_1)]$$

and to simply the notation, denoting the density of the conditional expectation as:

$$q(b_1) =_{def} q(b_1|b_0).$$

Recalling the form of (5) let:

$$u = \frac{1}{(\widetilde{b}_1)^2}, \tag{6}$$

That is, $\widetilde{b}_1$ defines the lowest input price which, if given u (fixed at t_0) followed by observation of $b_1 = \widetilde{b}_1$ (at t_1), would result in the firm not wanting to purhase additional units; infact, this critical price is the no arbitrage equilibrium price associated with the investment u. Thus given this definition of $\widetilde{b}_1$ we have:

$$F_1(u, b_0) = \int_0^{\widetilde{b}_1} \left(\frac{1}{b_1} + b_1 u\right) q(b_1) db_1 + 2\sqrt{u} \int_{\widetilde{b}_1}^{\infty} q(b_1) db_1. \tag{7}$$

Recalling (2) we see that the first order necessary condition for optimal choice of the investment requires:

$$\frac{\partial F_1}{\partial u}(u, b_0) = \frac{1}{\sqrt{v_0 - u}}. \tag{8}$$

This equation should be regarded as defining a function of v_0 and b_0 which determines the optimal investment $u = u_0(v_0, b_0)$. Now define:

$$\Psi(\widetilde{b}_1, b_0) =_{def} \int_0^{\widetilde{b}_1} b_1 q(b_1) db_1 + \widetilde{b}_1 \int_{\widetilde{b}_1}^{\infty} q(b_1) db_1. \tag{9}$$

Notice how $\widetilde{b}_1$ in effect censors the input price distribution. We have thus arrived at the following lemma.

Lemma 1. (Properties of the Marginal Expected Optimal Value Function)

$$\frac{\partial F_1}{\partial u}(u, b_0) = \Psi(1/\sqrt{u}, b_0), \tag{10}$$

where $\Psi(\widetilde{b}_1, b_0)$ *is defined by (9) and has the properties that* $\Psi(0, b_0) = 0$, $\Psi(\infty) = E[b_1|b_0]$, $\Psi'(0) = 1$, $\Psi'(\infty) = 0$, $\Psi'(\widetilde{b}_1, b_0) > 0$ *and* $\Psi''(\widetilde{b}_1, b_0) \leq 0$.

Recalling (3) the first-order condition becomes:

$$\Psi(\widetilde{b}_1, b_0) = \frac{1}{\sqrt{v_0 - \frac{1}{(\widetilde{b}_1)^2}}}, \tag{11}$$

where we shall call the choice of $\widetilde{b}_1$ solving this equation the NCP-censor (there being no current purchases). Now let

$$\Phi(v_0, \widetilde{b}_1) =_{def} \frac{1}{\sqrt{v_0 - \frac{1}{(\widetilde{b}_1)^2}}}.$$

By inspection, $\Phi(v_0, \widetilde{b}_1)$ is a monotonically decreasing function of $\widetilde{b}_1$ defined for $\widetilde{b}_1 > \frac{1}{\sqrt{v_0}}$ with $\Phi(v_0, \frac{1}{\sqrt{v_0}}) = +\infty$ and $\Phi(v_0, \infty) = v_0^{-0.5}$. Thus for the NCP-censor $\widetilde{b}_1 = \widetilde{b}_1(v_0, b_0)$ to exist with:

$$\Psi(\widetilde{b}_1, b_0) = \Phi(v_0, \widetilde{b}_1, b_0)$$

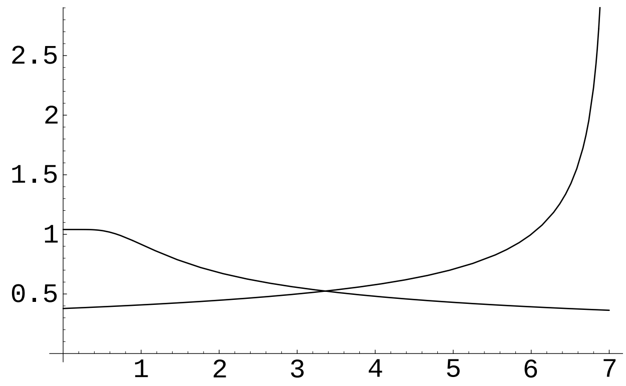

Figure 2: Ψ intersects Φ at the optimal investment level \$u\$

we require

$$E[b_1|b_0] > \frac{1}{\sqrt{v_0}},$$

that is we require:

$$v_0 > \frac{1}{E[b_1|b_0]^2} =_{def} \overline{v}_0(b_0)$$

To illustrate we provide a plot of these functions below, and summarize our findings by the following Lemma.

Lemma 2 (Minimum Opening Stock v_0 to Insure Investment ($u > 0$) is Desirable When There are No Current Purchases)
For it to be optimal to choose $u > 0$ when $z_0 = 0$ we require the initial opening stock $v_0 > \overline{v}_0(b_0)$.
Proof: Established immediately above.

The intuition for this result is as follows. If at t_0 when $v_0 > \overline{v}_0(b_0)$, some u is carried forward to period t_1, then the expected gain from having positive investment is outweighed by the lost current return at t_0 resulting from insufficient input being on hand.

In order to provide a simple interpretation for the NCP-censor $\widetilde{b}_1$ (which defines the optimal investment u^*), inspecting Figure 2 we see that applying (6) the optimal returns function $G_1(\widetilde{b}_1, b_1)$, the censor may be interpreted as choosing an optimal floor for realized returns. The optimal floor return is given by:

$$2\sqrt{u^*(\widetilde{b}_1)}$$

which is the guaranteed minimum return the firm can earn once it commits to carrying forward u^*. This is illustrated by Figure 3.

To summarize, we have shown how the optimal investment level u^* is determined by the identification of a NCP-censor $\widetilde{b}_1$. The investment level can in part be seen as providing a hedge against uncertainty since purchasing capital in advance insures that operations are not

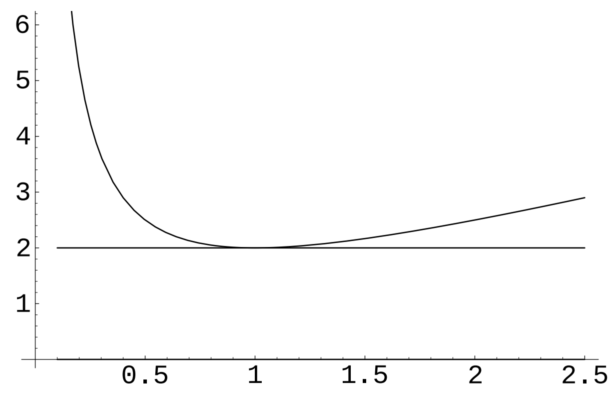

Figure 3: Plot of $G_1[u,b_1]$ against b_1

subjected to adverse price movements of a key input and this insures that profits never fall below a predetermined optimal floor. Having established the optimal investment policy in the simplest case let us now relax the restriction that $z_0 = 0$ and consider the alternative possibility of buying a quantity of input $z_0 > 0$ at price b_0.

2.2 Optimizing investment u when current period purchases are unrestricted ($z_{t_0} \geq 0$)

The firm must now maximize over both z_0 and x_0 the profit:

$$2\sqrt{x_0} - b_0 z_0 + F_1(v_0 + z_0 - x_0, b_0),$$

or, writing much as before $u = v_0 + z_0 - x_0$,

$$2\sqrt{x_0} - b_0(u + x_0 - v_0) + F_1(u, b_0), \tag{12}$$

¿From (12) we have

$$F_1'(u, b_0) = b_0 \tag{13}$$

and

$$x_0 = \frac{1}{b_0^2} \tag{14}$$

as first-order conditions, where the prime denotes differentiation with respect to u.

Let $u = \widehat{u}(b_0)$ denote the solution to equation (13). Writing

$$\widehat{u}(b_0) = \frac{1}{\widetilde{b}_1(b_0)^2}$$

and using the characterization (10), we see that $\widetilde{b}_1(b_0)$, as defined by the first-order condition, is the solution to

$$\Psi(\widetilde{b}_1, b_0) = b_0. \tag{15}$$

We call (15) the **censor equation**. The solution exists and is unique if and only if $b_0 < E[b_1|b_0]$, since, as we have already noted, $\Psi(\widetilde{b}_1, b_0)$ is increasing from zero to $E[b_1|b_0]$; we call the value of b_1 given by $\widetilde{b}_1$ which solves (15) the **censor**. We examine this solution in detail in a more general context later. Now let

$$\widehat{v}_0(b_0) =_{def} \frac{1}{b_0^2} + \widehat{u}(b_0),$$

where $\widehat{u}(b_0)$ solves (13). Hence we obtain that

$$z_0 = \frac{1}{b_0^2} + \widehat{u}(b_0) - v_0,$$

and $z_0 \geq 0$ if and only if $v_0 \leq \widehat{v}_0(b_0)$. We have just proved the following.

Lemma 3 (The optimal first period investment is determined by the censor when opening stock v_0 is not too large.)
For it to be optimal to choose $z_0 > 0$ we require the initial opening stock to satisfy $v_0 < \widehat{v}_0(b_0)$. In this case the optimal second period investment is given by

$$\widehat{u} = \frac{1}{\widetilde{b}_1(b_0)^2},$$

where $\widetilde{b}_1(b_0)$ is the censor. When $v_0 = \widehat{v}_0(b_0)$ the NCP-censor concides with the censor. Thus the maximum opening stock beyond which no current purchases are made ($z_0^* = 0$) is given by $\widehat{v}_0(b_0)$.

Proof: Established immediately above, since $\widehat{v}_0(b_0) > \overline{v}_0(b_0)$ so that the NCP-censor is defined by Lemma 2.

To summarize: provided the opening stock of the input is not too large (as defined by $\widehat{v}_0(b_0)$) in the opening period, it is optimal for the firm to increase its capital to $\widehat{v}_0(b_0)$ so as to enable it to invest $\widehat{u}(b_0)$ units of capital for carrying forward to the following period. However, if $v_0 > \widehat{v}_0(b_0)$ investment in period t_0 would not be optimal, and given this the firm would use the NCP-censor to establish the correct division of capital between the two periods.

Let us now turn to relax a further temporary restriction. Above we have assumed that once purchased, input quantities must be used at some time in production. However in reality it may be possible to resell some quantity of the input. Clearly if prior investment in v_0 were reversible, the firm may benefit from disinvestment (reselling) of some of the units. In order to incorporate this possibility, we shall in the following section amend the above analysis assuming the firm has the possibility to resell[7] at a unit price of b_1/R were $R > 1$.

[7]That $R > 1$ is standard in the literature, otherwise if $R = 1$ we would have the possibility of simple portfolio rebalancing.

3 TWO PERIOD MODEL WITH PARTIAL REVERSIBILITY

Let us now repeat the above analysis assuming that at the commencement of period t_1 the firm can (disinvest) resell input at b_1/R_1 where $R_1 > 1$. In the following subsection we will show how the optimal derived payoff schedule when R_1 is deterministic, differs from that of Figure 3 where we assumed irreversibility. In the remaining subsections we shall model R_1 as a stochastic random variable, first under the assumption that it is distributed independently of R_0 (and identically) and then more generally that R_1 is sampled from a geometric Brownian motion R_t. We will see that the form of the optimal policy space varies significantly depending upon the assumption for the R_1 process.

3.1 Analysis with a deterministic resale rate R_1

Let us commence the analysis by considering the optimal policy at t_1 assuming that the (investment) quantity u has been brought forward into the period[8]. At this time the resale rate R_1 is presumed to have been observed. Purchasing additional stock costs b_1 and selling existing stock earns b_1/R_1. Thus the firm needs to determine z_1^* for which

$$z_1^* = \arg\max 2\sqrt{u + z_1} - c_1 z_1,$$

where $c_1 = c_1(b_1, R_1)$ is given by

$$c_1 = \begin{cases} b_1, & \text{if} \quad z_1 > 0, \\ \frac{b_1}{R_1}, & \text{if} \quad z_1 < 0. \end{cases}$$

There are thus three potential solutions: the corner point solution $z_1^* = 0$ and the two internal solutions ($z_1^* > 0$, $z_1^* < 0$) given by:

$$z_1^* = \frac{1}{c_1^2} - u.$$

The case $z_1^* > 0$ corresponds to a situation in which it is optimal to add to the opening stock (given subsequently realized conditions, i.e. $u < 1/b_1^2$ or equivalently $b_1 < \widetilde{b}_1$ where $\widetilde{b}_1 = \frac{1}{\sqrt{u}}$); similarly $z_1^* < 0$ corresponds to the over-invested position $u > R_1^2/b_1^2$ (equivalently $b_1 > \frac{R_1}{\sqrt{u}} > R_1\widetilde{b}_1$). For prices b_1 between the two limits the optimal behavior is $z_1^* = 0$ which generates a return of $2\sqrt{u}$. Thus the optimal return at t_1 is trichotomous and is given by:

$$G_1(u, c_1(b_1, R_1)) = \begin{cases} \frac{1}{c_1} + c_1 u, & \text{if} \quad u \leq \frac{1}{(b_1)^2} \quad \text{or if} \quad u \geq \frac{R_1^2}{(b_1)^2}, \\ 2\sqrt{u}, & \text{if} \quad \frac{1}{(b_1)^2} \leq u \leq \frac{R_1^2}{(b_1)^2}. \end{cases} \tag{16}$$

It is interesting to note that the graph of the returns plotted against price b_1 comprises two hyperbolic curves linked smoothly by a horizontal floor as illustrated below.

[8]Recall operations end at t_2 so there is no further need for investment. We relax this assumption in a following section when we allow additional periods, that is allow $N > 2$ periods.

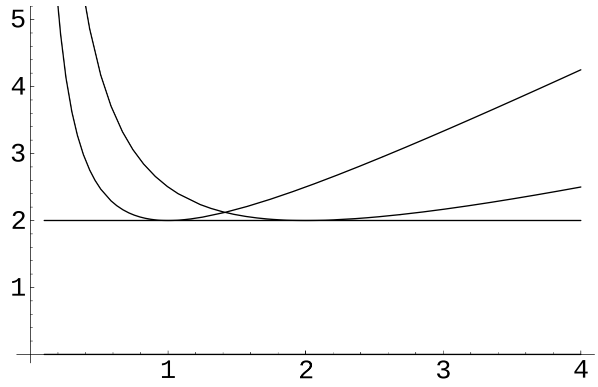

Figure 4: Plot of $G_1[u,c_1(b_1,2)]$ against b_1

Comparing Figures 3 and 4, we clearly see the value of the possibility of reversal. In particular for $u \geq \frac{R_1^2}{(b_1)^2}$ the firm earns greater return when reversal is possible since beyond this level, reselling stock in the input resale market is more profitable than being forced to apply it in production to produce final output.

3.2 General Analysis With a Stochastic Resale Rate

Let us commence by considering how u is optimally chosen at t_0. First, note that at t_0 the optimal expected value function is given by:

$$F_1(u, R_1, b_0) = \int_0^{\widetilde{b}_1} (\frac{1}{b_1} + b_1 u) q(b_1) db_1 + 2\sqrt{u} \int_{\widetilde{b}_1}^{R_1 \widetilde{b}_1} q(b_1) db_1 + + \int_{R_1 \widetilde{b}_1}^{\infty} (\frac{R_1}{b_1} + \frac{b_1}{R_1} u) q(b_1) db_1 \quad (17)$$

where $\widetilde{b}_1 = \frac{1}{\sqrt{u}}$. That is comparing this to (7) we see that the final term incorporates the return from being allowed now to resell capital. Furthermore it is helpful to classify these three price policy ranges as follows:

(**U**) the under-invested range ($0 \leq b_1 \leq \widetilde{b}_1$); in which additional investment in capital is made;
(**IO**) the (endogenously) irreversible[9] over-invested range ($\widetilde{b}_1 \leq b_1 \leq R_1 \widetilde{b}_1$), where excess capital is put into production;
(**RO**) the reversible investment range ($R_1 \widetilde{b}_1 \leq b_1 \leq \infty$); where excess capital is resold.

[9]Endogenous in the sense that though reversal is possible, it is never optimal in this setting to choose it and hence the firm acts as if the situation was irreversible.

Thus at t_0 if an investment of u is chosen the expected payoff (now taking expectations over the $R_1 \geq 1$ realization space) is:

$$\overline{F}_1(u, b_0) = \int_1^\infty F_1(u, R_1, b_0)\rho(R_1)dR_1,$$

where $\rho(R_1)$ is the probability density function for R_1 and $E[\frac{1}{R_1}]$ is finite. Hence at t_0 the firm must solve an analogous, albeit two variable, maximization problem:

$$\max_{z_1, u} 2\sqrt{v_0 + z_0 - u} - c_0 z_0 + \overline{F}_1(u, b_0).$$

Assuming $z_0 \neq 0, u \neq 0$ is optimal, the pair of first-order conditions jointly imply:

$$\frac{1}{\sqrt{v_0 + z_0 - u}} = c_0 = \overline{F}'_1(u, b_0), \tag{18}$$

where the prime denotes partial differentiation with respect to u. Similarly, as in earlier analysis, we define $\widehat{u}(c_0)$ as denoting the solution to:

$$\overline{F}'_1(u, b_0) = c_0, \tag{19}$$

and hence for both first-order conditions to be jointly satisfied we require $c_0 = b_0/R_0$:

$$v_0 \geq \widehat{u}(c_0) + \frac{1}{c_0^2} =_{def} \widehat{v}_0(c_0) \tag{20}$$

and the reverse when $c_0 = b_0$. Alternatively, when $z = 0, u \neq 0$ is optimal, as before we have $u^*(v_0) = u_1(v_0, b_0)$ as denoting the solution to:

$$\frac{1}{\sqrt{v_0 - u}} = \overline{F}'_1(u, b_0),$$

provided $v_0 > \overline{v}_0$. Next expanding the right-hand side of (18) we have:

$$c_0 = \int_1^\infty F'_1(u, R_1, b_0)\rho(R_1)dR_1$$

where by (17):

$$\begin{aligned} F'_1(u, R_1, b_0) \;\; &= \;\; \int_0^{\widetilde{b}_1} b_1 q(b_1|b_0)db_1 + \widetilde{b}_1 \int_{\widetilde{b}_1}^{R_1\widetilde{b}_1} q(b_1|b_0)db_1 + \\ &+ \frac{1}{R_1}\int_{R_1\widetilde{b}_1}^\infty b_1 q(b_1|b_0)db_1. \end{aligned} \tag{21}$$

Next we assume the input price process is a **proportional** one[10] so that:

$$b_1 = b_0 g,$$

where g has a density $\widetilde{q}(g)$ independent of b_0. In particular we shall make the common assumption that the input price is distributed log normally. Rescaling we have:

$$\begin{aligned} F_1'(u, R_1, b_0) &= \int_0^{\widetilde{b}_1/b_0} b_0 g \widetilde{q}(g) dg + \widetilde{b}_1 \int_{\widetilde{b}_1/b_0}^{R_1 \widetilde{b}_1/b_0} \widetilde{q}(g) dg \\ &+ \frac{1}{R_1} \int_{R_1 \widetilde{b}_1/b_0}^{\infty} b_0 g \widetilde{q}(g) dg \end{aligned}$$

and, letting $\widetilde{b}_1 = b_0 \widetilde{g}_1$ and recalling (9), we can now define:

$$\begin{aligned} F_1'(u, R_1, b_0) &= {}_{def} \Psi_{R_1}(\widetilde{b}_1, b_0) \qquad (22) \\ &= b_0 [\int_0^{\widetilde{g}_1} g \widetilde{q}(g) dg + \widetilde{g}_1 \int_{\widetilde{g}_1}^{R_1 \widetilde{g}_1} \widetilde{q}(g) dg + \frac{1}{R_1} \int_{R_1 \widetilde{g}_1}^{\infty} g \widetilde{q}(g) dg], \end{aligned}$$

with the first-order condition (18) becoming:

$$c_0 = b_0 \overline{F}_1' (\frac{1}{(\widetilde{g}_1)^2}, 1) \qquad (23)$$

by (6). Here it is important to notice that **infimum marginal expected return** is

$$\inf_u F_1'(u, b_0) = E[b_1] E[\frac{1}{R_1}], \qquad (24)$$

obtained by passing to the limit as $\widetilde{g}_1 \to 0$ (for details see the Appendix B). Evidently this formula as the product of two expectations reflects our assumption that the two variables R_1 and b_1 are independent.

We need now to consider the existence conditions for a solution of (19). We define $\gamma > 1$ as:

$$\gamma = E[g] = \int_0^{\infty} g \widetilde{q}(g) dg = \frac{E(b_1)}{b_0}.$$

It follows from inspection of (22) that we have bounds upon $\Psi_{R_1}(\widetilde{b}_1, b_0)$ as follows:

$$\frac{\gamma}{R_1} < \Psi_{R_1}(\widetilde{b}_1, b_0) < \gamma. \qquad (25)$$

Consider $c_0 = \frac{b_0}{R_0}$, where $R_0$is the current period resale rate, and, to develop intuition, suppose that for simplicity that the next period's resale rate R_1 is deterministic (we shall relax this in a moment). Under these circumstances for (19) to be soluble and thus reselling to be optimal we require:

$$\frac{\gamma}{R_1} < \frac{1}{R_0} \qquad (26)$$

[10]Also called 'multiplicative' in R. Merton and P.A. Samuelson, 1969.

or equivalently

$$\frac{E(b_1)}{b_0} < \frac{R_1}{R_0}. \tag{27}$$

That is, there is an increase in the resale rate and this increase must be greater than the expected increase in the input price[11]. The intuition for this result is clear. If at t_0 the input price is expected to drift upwards, and the resale rate is fixed, a higher resale price will be expected when waiting until t_1 and so for early resale to be optimal, there must be a cost of waiting. In (27) this is captured by the resale rate (discount) 'increasing' at a higher rate than the input price and hence selling later, becomes costly[12]. Clearly when (27) does not hold we can describe the situation as one of endogenous irreversibility, IO, since although resale is possible, it is not optimal.

Remark 1. Note that in the case $R_1 = \infty$ the function $\Psi_\infty(B)$ is strictly concave. In general we expect a convexity change for $\Psi_{R_1}(B)$ and there is only one point of inflection in the log-normal case. As $R_1 \to \infty$ the point of inflection recedes towards the origin. It also follows that since

$$\overline{F}'(u, 1) = \Psi(1/\sqrt{u}) \tag{28}$$

$\overline{F}'(u, 1)$ is decreasing in u. We shall need this result later.

The above analysis assumed deterministic resale rates R_0 and R_1. However a more realistic assumption in this dynamic setting is to assume that R_t evolves stochastically. Below, we shall analyze the optimal resale policy accordingly under two scenarios for the resale rate process. First we shall consider the case of each period's resale rate being independently and identically distributed through time. This simple model allows us to develop intuition before considering the more complex but realistic second scenario under which the resale rate evolves as a geometric Brownian motion.

3.3 Analysis with i.i.d. resale rate

The analysis of this scenario will proceed as follows. First we recall that when we are in the over-invested situation the firm needs to choose in the final period between putting all the (investment) stock into production (policy IO) or reselling some stock (policy RO). We show how the choice between policies is determined by the identification of a critical trigger resale rate $R_0^{\#}$ below which it is optimal to adopt RO, and above which it is optimal to adopt IO.

Recalling (26) in the stochastic R_1 case and working under the assumption (20) that $v_0 > \widehat{v}_0(c_0)$ for $c_0 = b_0/R_0$ we have that (18) is soluble if and only if

$$\gamma E[\frac{1}{R_1}] < \frac{1}{R_0} \tag{29}$$

[11]In this model the resale price is some proportion of the current market price of the input.

[12]There may exist other costs of waiting such as working capital etc.

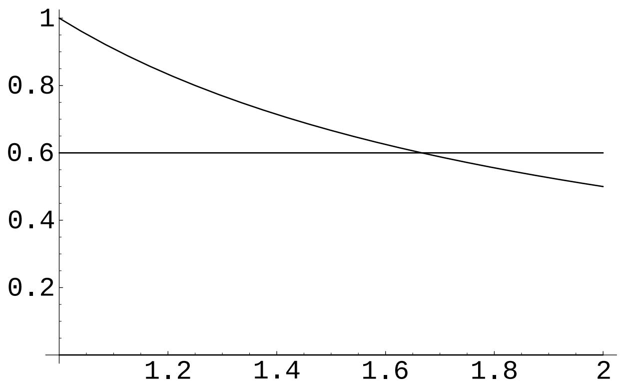

Figure 5: Determination of $R_0^#$

that is letting

$$\frac{1}{R_0^{\#}} = \frac{E(b_1)}{b_0} E(\frac{1}{R_1}) \tag{30}$$

we have when:

$$\frac{1}{R_0} \leq \frac{1}{R_0^{\#}} \quad IO \text{ is the optimal policy} \tag{31}$$

and for

$$\frac{1}{R_0} \geq \frac{1}{R_0^{\#}} \quad RO \text{ is the optimal policy} \tag{32}$$

which is illustrated below for $v_0 > \widehat{v}_0(c_0)$.
The intuition for the result is straightforward. We can interpret $R_0^{\#}$ as the resale rate ratio[13] in the sense that the right hand side of (30) compares the proportional increase in the input price to that of the inverse of the resale rate. This naturally arises since an increase in the former encourages waiting (since the base input price upon which the resale rate discount is applied has increased), while an increase in the later discourages waiting (since a larger discount is offered on the existing purchase price). We summarize this result in the following Theorem.

Theorem 1: (Optimality Conditions for the Endogenous Reversibility Policy RO when the Resale Rate is I.I.D.)
At t_0, applying (32) when *$v_0 > \widehat{v}_0(c_0)$, $R_0^{\#}$ is the trigger below which RO is the optimal policy.*

Proof: Follows from immediate argument above, by (24).

3.4 Analysis with geometric Brownian resale rate

[13] We shall formally introduce and define this ratio below.

Consider now[14] a scenario where $R_t - 1$ evolves according to geometric Brownian motion, i.e

$$d(R_t - 1) = (R_t - 1)[\mu_R dt + \sigma_R dW_R(t)],$$

where W_R is a standard Wiener process, assumed independent of the input price process $W_b(t)$. Thus $\widetilde{R}_t = (R_t - 1)$ is log-normally distributed with drift μ_R and volatility σ_R^2. So if $\widetilde{R}_0$ has been revealed the expected value of $\widetilde{R}_1$ is $\widetilde{R}_0 e^{\mu_R}$.

In order to understand the condition (26) for reversal of investments, i.e.

$$E(\frac{1}{R_1})\gamma < \frac{1}{R_0}$$

we need to recognize formally the conditional dependence of the expected R_1 value on the current value R_0 via $E[\frac{1}{R_1}|R_0]$. Under the i.i.d. assumption of the previous section the expectation $\gamma E[\frac{1}{R_1}]$ is represented by a (constant) horizontal curve which, if below unity, interects the $1/R_0$ curve exactly once, since the latter curve starts at unity and is asymptotic to the horizontal R_0-axis. In the current setting we find that the expectation $E[\frac{1}{R_1}|R_0]$ is now decreasing with R_0 and the intersection condition for the two curves is more involved. The analysis in this case uncovers a somewhat more complicated scenario in that a single switching $R_0^{\#}$ trigger policy does not fully describe the optimal policy range. Before establishing this formally we shall first provide some intuition for two initially counter-intuitive features: the absence of a reversal range commencing at $R_1 = 1$; and the possibility of a double switching regime. We begin by approximating the expectation term by its asymptotic value (derived below):

$$E[\frac{1}{R_1}|R_0] = \frac{e^{(\sigma_R^2 - \mu_R)}}{R_0}. \tag{33}$$

Now consider the condition (26) rewritten with only γ on one side:

$$1 < e^{\mu_b} = \frac{E(b_1)}{b_0} = \gamma < \frac{\frac{1}{R_0}}{\frac{e^{(\sigma_R^2 - \mu_R)}}{R_0}} = e^{(\mu_R - \sigma_R^2)}. \tag{34}$$

When the drift $\mu_R > 0$ we see that if the '**relative volatility**' σ_R^2/μ_R is close to unity then (34) need not be satisfied. In particular if the current discount $1/R_0$ is close to unity (so the resale price is close to the purchase price), then the expected discount next time, $1/R_1$ will also be close to unity (because $e^{(\sigma_R^2 - \mu_R)}$ is close to unity). Since the purchase price is expected to rise there may be advantages to waiting, in the form of a higher than current resale price occurring in the next period. The parameter values for which it is worth waiting are given, at least approximately, by the failure of (34). This explains why for R_0 close to unity non-reversal of investment is optimal under these circumstances.

[14]This subsection contains perhaps the most surprising result of our work which is that the optimal switching policy between non-reversal and reversal may be a dual switching policy. For this reason we present the analysis in some considerable detail. For the reader who wants first to directly see the implications of the analysis, subsection 3.5 presents a diagrammatic summarization of the results.

Now, even though we may have (34) failing, suggesting that irreversibility is always optimal, in that we have the dominance

$$\frac{e^{(\sigma_R^2-\mu_R)}}{R_0} > \frac{e^{-\mu_b}}{R_0},$$

the exact form of the function $P(R_0) = E[\frac{1}{R_1}|R_0]$ contains a further ingredient of convexity controlled by the volatility σ_R ; this enables the graph of $P(R_0)$ to dip below that of $\frac{e^{-\mu_b}}{R_0}$ for a range of R_0 and a range of volatilities, rather than behave for all R_0 as in the asymptotic situation. This makes possible two switches in policy from non-reversal to reversal at $R_0 = R_0^{\#} > 1$ and again from reversal to non-reversal at $R_0 = R_0^{\#\#} > R_0^{\#}$ (with $R_0^{\#\#} < \infty$).

We proceed with a formal analysis in the two period case. Let us define Γ_n for $n = 0, 1$ to be the set of resale rates R_n in period n for which it is optimal to resell stock given large enough stock levels v. Evidently at time t_1 any R_1 will lead to an optimal resale in the final period for large enough stock, that is since no economic activity occurs to the right of the interval $[t_1, t_2]$ any stock not used in the optimal production plan for this interval should be sold, no matter what the resale price. Thus we have $\Gamma_1 = [1, \infty)$.

We apply (34) to characterize the set of resale rates Γ_0 for which RO is the optimal policy as:

$$\Gamma_0 = \{R_0 \geq 1 : 1/R_0 > \gamma \int_{\Gamma_1} \frac{1}{R_1} \rho(R_1|R_0) dR_1\}. \tag{35}$$

In order to determine the explicit form of Γ_0 let us first consider the qualitative properties of the expectation of the inverse of the time t_1 resale rate; it will be convenient to define

$$\gamma \int_0^\infty \frac{\rho(R_1|R_0)}{1+\widetilde{R}_1} d\widetilde{R}_1 =_{def} P_1(\widetilde{R}_0).$$

We apply the substitutions:

$$\begin{aligned} w &= \frac{\ln \widetilde{R}_1 - (\ln(\widetilde{R}_0) + m_R)}{\sigma_R}, \\ m_R &= (\mu_R - \frac{1}{2}\sigma_R^2), \\ \alpha_R &= \sigma_R^2 - \mu_R, \end{aligned}$$

to give

$$P_1(R_0) = \gamma \int_{-\infty}^\infty \frac{e^{-w^2/2} dw}{(1 + \widetilde{R}_0 e^{\sigma_R w + m_R})\sqrt{2\pi}}.$$

Evidently

$$P_1(1) = \gamma,$$

and

$$P_1'(R_0) = -\gamma \int_{-\infty}^\infty \frac{e^{\sigma_R w + m_R} e^{-w^2/2} dw}{(1 + \widetilde{R}_0 e^{\sigma_R w + m_R})^2 \sqrt{2\pi}} < 0,$$

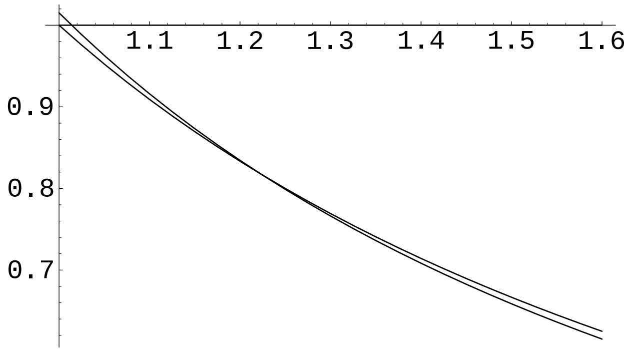

Figure 6: Determination of $R\_0\hat{}\#\$when\$\alpha_R + \mu_b < 0$

so that $P_1(R_0)$ is decreasing and behaves asymptotically for large R_0 like

$$\frac{\gamma}{R_0}\int_{-\infty}^{\infty}\frac{e^{-w^2/2}dw}{e^{\sigma_R w+m_R}\sqrt{2\pi}} = \frac{\gamma}{R_0}e^{\alpha_R}.$$

In the case where the input price process is also log-normal we write $\gamma = e^{\mu_b}$, where μ_b is the input price drift. Thus since $P_1(1) = e^{\mu_b} > 1$ (for $\mu_b > 0$) a single intersection of the curve $P_1(R_0)$ with the graph of $1/R_0$ as required in (35) is guaranteed to occur, say at $R_0^{\#}$, if the asymptotic curve of $P_1(R_0)$, namely, $e^{\alpha_R+\mu_b}/R_0$ lies below the $1/R_0$ curve, i.e. if

$$1 > e^{\alpha_R+\mu_b}$$

i.e. $\alpha_R < -\mu_b$. This condition in effect requires that $\mu_R > \mu_b + \sigma_R^2$, a marked tendency for the resale rate to rise (relative to $\mu_b + \sigma_R^2$). Alternatively, it may be read as requiring low input price inflation (assuming $\mu_R - \sigma_R^2 > 0$); but, of course, if $\mu_R = 0$ (zero resale drift), this condition cannot be satisfied (with our assumed positive μ_b). If the condition holds, then for values $R_0 > R_0^{\#}$ we have $1/R_0 > P_1(R_0)$ and the possibility of resale occurs (i.e. $\Gamma_0 = (R_0^{\#}, \infty)$). This case is illustrated by the following figure.

Given our earlier discussion it is now not surprising that the *RO* policy is not optimal for R_0 close enough to 1.

However, asymptotic dominance between the two curves, occurring when $e^{\alpha_R+\mu_b} > 1$, is not a sufficient condition for characterizing $P_1(R_0) > 1/R_0$ for all R_0, since $\sigma_R > 0$ introduces additional convexity, as we have indicated earlier.

Changing point of view with α_R fixed we rewrite the first-order condition in the format

$$1 > R_0 P_1(R_0),$$

put

$$e^{-\mu_b}R_0P_1(R_0) =_{def} Q(R_0)$$

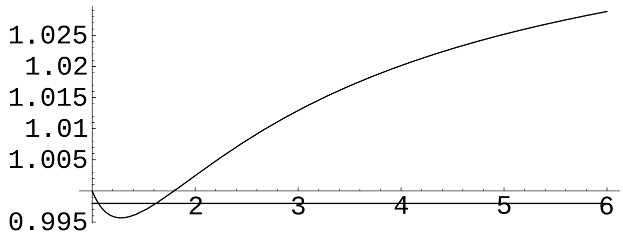

Figure 7: Two intersections when $\alpha_R \geq 0$

to factor out dependence on e^{μ_b} and investigate this function analytically[15]; evidently $Q(R_0) = E[\frac{R_0}{R_1}|R_0]$ is the expected resale ratio. A routine calculation leads to the following.

Lemma 3 (Behavior of the resale ratio). *$Q(1) = 1$ and the initial slope at $R_0 = 1$ of $Q(R_0)$ is $(1 - e^{\mu_R})$, whilst the second derivative is $2e^{\mu_R}(e^{\mu_R+\sigma_R^2} - 1)$; thus for $\mu_R > 0$ the function is initially decreasing and for $\mu_R \leq 0$ and $\sigma_R > 0$ the function is initially increasing. The asymptotic value as $R_0 \to +\infty$ is e^{α_R}.*

If $\mu_R < 0$ it is helfpul to note the limiting case when $\sigma_R = 0$, as we then have

$$Q(R) = \frac{e^{-\mu_R} R}{R - (1 - e^{-\mu_R})},$$

which is increasing in R; since $Q(1) = 1$, it follows that $Q(R) = e^{-\mu_b}$ is insoluble for $R > 1$ under our assumption that $\mu_b > 0$, that is, in this case $R_0 P_1(R_0) > 1$ for all $R_0 > 1$. Thus Γ_0 is empty. This continues to be true for $\sigma_R > 0$ with $\mu_R \leq 0$.

If $\mu_R > 0$, two forms of behavior occur according as the expected resale ratio

$$\lim_{R_0 \to +\infty} E(\frac{R_0}{R_1}|R_0) = e^{\alpha_R}$$

is greater than unity or less than unity (i.e. respectively $\alpha_R \geq 0$ or $\alpha_R < 0$).
In the first case, see Figure 7, the equation

$$Q(R_0) = e^{-\mu_b}$$

has two solutions for all μ_b with $0 < \mu_b < \mu_R^*$, where

$$\mu_R^* = -\ln\left(\inf_{R_0} Q(R_0)\right),$$

with one or no solutions according as $\mu_b = \mu_R^*$ or $\mu_b > \mu_R^*$.

In the second case, we have to distinguish between $\sigma_R > 0$ and $\sigma_R = 0$,see Figure***(i) and (ii).

[15]The definition of P_1 includes the factor γ with a later generalization in mind, see the following section.

When $\sigma_R > 0$, or equivalently $\mu_R > \sigma_R^2$, there are two critical values of μ_b. For $\mu_b < -\alpha_R$ there is only one solution, i.e. Γ_0 is a half infinite interval with left end-point to the right of $R_0 = 1$; otherwise for μ_b with $\mu_R^* \geq \mu_b > -\alpha_R$ the interval Γ_0 is bounded; and for $\mu_b > \mu_R^*$ the interval Γ_0 is empty (i.e. resale is never optimal).

When $\sigma_R = 0$, or equivalently $\mu_R > 0$, the two critical values coincide, so that $\alpha_R = -\mu_R^* = \ln(\inf_{R_0} Q(R_0))$, i.e. Γ_0 is always a half infinite interval with left end-point to the right of $R_0 = 1$. Using the lemma it is easy to compute an **under-estimate** of the left-hand end-point as

$$R_1^{\#} \approx 1 + \frac{1 - e^{-\mu_b}}{e^{\mu_R} - 1}.$$

3.5 The M-S diagram in the two period geometric Brownian resale rate setting.

For a simple diagrammatic representation of parameter values under which reversal is optimal we turn the three parameter setting into a revised two-parameter description. Choosing to hold μ_R constant and positive we can consider the dimensionless parameters of relative price drift and realtive resale variance:

$$M = \frac{\mu_b}{\mu_R}, \qquad S = \frac{\sigma_R^2}{\mu_R}.$$

Here M is a natural measure of comparison for the two drifts. This choice of parameters leads to the following Figure 8 in the positive MS quadrant. It is helpful to take note of certain extreme values.

3.5.1 Single switching boundary $S + M = 1$

We have seen that the single switching regime ($\Gamma_0 = [R^{\#}, \infty)$) occurs when $\alpha_R + \mu_b < 0$. Its boundary in revised parameter notation is $S + M = 1$. Along this boundary we have $R^{\#\#} = \infty$.

3.5.2 Zero inflation $M = 0$

The limiting case $M = 0$ has $R^{\#} = 1$ (because $P_1(1) = 1$); we have seen that for $S \leq 1$ we have $\Gamma_0 = [R^{\#}, \infty)$. For $S > 1$ and $M = 0$ by the Lemma Γ_0 is necessarily a bounded interval. The limiting case at $S = 1$ has $R^{\#\#} = \infty$.

3.5.3 Zero volatility $S = 0$

Evidently in this case we have

$$P_1(R) = \frac{e^{\mu_b}}{1 + e^{\mu_R}(R - 1)}.$$

We conclude that for $M \geq 1$ and $R \geq 1$

$$P_1(R) > \frac{1}{R},$$

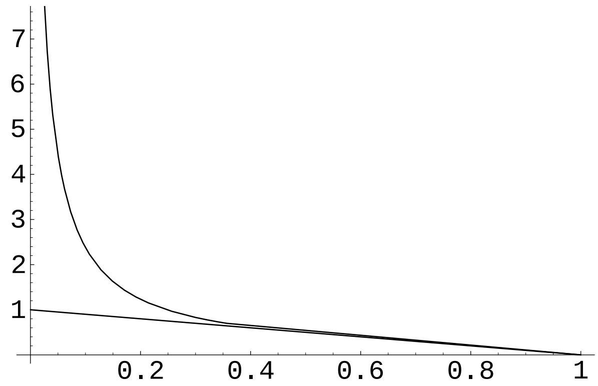

Figure 8: The M-S diagram

and so Γ_0 is empty, i.e. reversal is sub-optimal. We also conclude for $M < 1$ that

$$P_1(R) < \frac{1}{R} \text{ if and only if } R > \frac{1 - e^{-\mu_R}}{1 - e^{(M-1)\mu_R}} = R^{\#},$$

i.e. $\Gamma_0 = [R^{\#}, \infty)$ here, and this agrees since we are below the single switching boundary. We note that for large R we have approximately that

$$\frac{\partial P_1}{\partial \sigma} = \frac{2\sigma e^{\alpha_R + \mu_b}}{R} > 0$$

showing that for $M = 1$ and $S > 0$ we also have Γ_0 empty as we move vertically into the interior of the quadrant.

3.5.4 Second switching boundary

For $0 < M < 1$ there is an $S = S^* > 1 - M$ such that Γ_0 a bounded interval $[R^{\#}, R^{\#\#}]$ whenever $S < S^*$; this was deduced from Lemma 3. At $S = S^*$ we evidently have $R^{\#} = R^{\#\#}$ and for $S > S^*$ we have Γ_0 empty. The curve $S = S^*(M)$ may thus be termed the Double Switching Boundary which separates the irreversible region from the reversible. This curve is tangential to the line $S = 1 - M$ at $M = 1$ and is asymptotic to the vertical at $M = 0$.

Having now concluded our discussion of the two period policy space our next consideration is how to generalize the results for $N > 2$ periods. In Appendix 1 we present the analysis and a numerical example in the case $N = 3$. To summarize those findings intuitively we find that the results depend on the values for M and S (defined in subsection 3.5) and the value of the expected proportional price increase:

$$\gamma_i = \frac{E[b_i | b_{i-1}]}{b_{i-1}}.$$

In the case the expected proportional price drift γ_i is constant, we are able to identify the following characterization[16] of the N-period policy space:
i) (Existence of a double switching regime)
If $1 - M < S < S^*(M)$; for all $n < N - 1$, $\Gamma_{N-1} = [1, \infty)$ and the reversal range Γ_n is empty or takes the form of an interval $[R_n^{\#}, R_n^{\#\#}]$ and is **shrinking as n decreases**.
ii) (Existence of a single switching regime)
If $M + S < 1$; for all n the reversal range Γ_n takes the form of an interval $[R_n^{\#}, \infty)$ which is **expanding as n decreases** (i.e. with $R_n^{\#}$ decreasing as n decreases).
iii) (Existence of an irreversible regime)
If $M \leq 0$ or $M > 1$ or $S > S^*(M)$; **for all $n < N - 1$ there is no reversal of investment**.

4 RE-INTERPRETING THE ABEL AND EBERLEY MODEL IN OUR FRAMEWORK

Abel and Eberley (1996; hereafter A&E) provide an infinite horizon model of optimal investment reversibility in which investment is made (or reversed) at each moment of time t at a unit cost of say p_t (respectively p_t/R with $R > 1$) with constant depreciation δ; suppose that at time t an amount, say v_t, of (investment) input is consumed in production to obtain revenue. In the next period before the managers decide to increase or decrease investment by an amount z_t there is already available for consumption an amount $v_t(1 - \delta)$ for the period. When interpreting the form of their investment policy within in a discrete time framework, one must therefore recognize that a decision to hold v_t at time t makes available for future consumption (in the production process) inputs of $(1 - \delta)v_t$, $(1 - \delta)^2 v_t$,... respectively at times $t + 1, t + 2, ...$. Let us define a **unit standardized** investment **annuity** with parameter δ to be a contract for the supply of the following discrete inputs:

$$1, (1 - \delta), (1 - \delta)^2, ... \quad ,$$

(an input stream) supplied at the discrete decision times all the way to termination of production. It is easy to interpret their model in terms of our variables provided input price is measured relative to output prices[17] and assuming for just a moment an infinite horizon; if p_t is the price of a unit standardized annuity contracted at time t we set our input price b_t equal to the effective price per unit input in the total stream supplied by the annuity. This leads to

$$p_t = \frac{b_t}{\delta}$$

giving the simple scaling relation $b_t = \delta p_t$ between the prices of the two formulations. Of course, in the finite horizon N-period setting of our model we shall have for each time $t = 0, 1, ..., N-1$

[16] Recall $S^*(M)$ is defined in subsubsection 3.5.4.

[17] Actually, A&E model the input price as constant and have the production function with a geometric Brownian coefficent, raised to an appropriate power dictated by homogeneity concerns; the inverse of which constitutes an input price p_t and that too therefore remains geometric Brownian.

the slightly more awkward relation

$$p_t = b_t A_t(\delta),$$

where

$$A_t(\delta) = 1 + (1-\delta) + (1-\delta)^2 + ... + (1-\delta)^{N-t-1},$$

so that b_t will be geometric Brownian if p_t is, albeit with a shift in drift caused by the deterministic multiplier $A_t(\delta)$.

In the A&E model the manager needs to purchase input optimally given the depreciation rate δ. One issue left open is how the choice of δ is made. Our own model prescribes a choice for δ on grounds of depreciation that could easily be interpreted in terms of optimal 'physical' useage. We have previously defined optimal investment in terms of optimal choice over u^*, however, this can be reinterpreted in terms of a choice of an optimal depreciation rate δ^* so as to allow for more direct comparison between the models. Thus, if v_t is the opening stock of any period and z_t is the additional optimal purchase at the beginning of this period, then $v_t + z_t$ is split up into two portions $v_t + z_t - u_t$ for immediate use and u_t for subsequent use. In contrast to the A&E model we are in effect depreciating the first part of the acquired input at a rate $\delta = 1$ and the second at a rate $\delta = 0$. However, such a dichotomous depreciation policy is equivalent to an implied (average) depreciation rate of

$$\delta_t^*(v_t) = \frac{v_t + z_t - u_t}{v_t + z_t}$$

between the current and next period. In the two-period model ($N = 2$) with an opening stock $v_0 = 0$ and with $b_0 = 1$ it is optimal to choose z_0 so that $z_0 = 1 + u$ where u is the optimal investment for use in period two; this leads to the formula

$$\delta_0^*(0) = \frac{1}{1+u} > \frac{1}{2}$$

(we note that $u < 1$ as $\widetilde{g} > 1$) which secures optimal decision-taking for that model. The formula continues to be valid for $v_0 < 1 + u$ by Lemma 3. Evidently, if $v_0 > 1 + u$ we have by (8) that

$$\delta_0^*(v_0) = \frac{v_0 - u_1(v_0, 1)}{v_0},$$

with $u_1(.,.)$ defined as in section 2.1. Thus in a three period model ($N = 3$) after choosing the initial optimal depreciation rate the manager observes b_1 and at this stage must reselect a depreciation rate; the new rate is de facto a two-period initial rate δ_0^* the value of which is dependent on the stock in hand, i.e. need not be time invariant. However, we find that at any time the optimal depreciation rate decreases with the volume of stock in hand to a limiting value. This illustrates the qualitative significance of adopting a finite horizon model.

Lemma 4. (Optimality of asymptotic straight-line depreciation.) *In the N-period model with a time-invariant Cobb-Douglas production function (say x^θ/θ), a geometric Brownian input price and equal-lengthed periods we have for $t = 0, 1, ..., N-1$*

$$\lim_{v_t \to \infty} \delta_t^*(v_t) = \frac{1}{N-t},$$

and in particular, for the two period model with $N = 2$

$$\lim_{v_0 \to \infty} \delta_0^*(v_0) = \frac{1}{2}.$$

Proof. See [Gietzmann and Ostaszewski (1999b)].

We may call $\delta_0^* = \delta_0^*(0)$ the optimal endogenous rate and it is evident that this useage-depreciation rate is determined by the production technology[18]. We shall establish below that it is the optimal depreciation rate in the sense that (in the two-period model) the return is maximised over δ if and only if it takes the value $\delta = \delta_0^*$.

In order to gain intuition we analyse the two-period irreversible model in the following subsection. Then in the next subsection we consider the arbitary N-period setting and consider the generalized reversible setting.

4.1 The Irreversible Case

In this section we study the two period model with the aim of proving the following.

Proposition 1. *In the two-period irreversible model when a manager commences by contracting an optimal number z_0 of units of standardized annuity with depreciation parameter δ, the return is maximised over δ if and only if it is fixed at*

$$\delta = \delta_0^*.$$

Proof. In this subsection assuming $v_0 = 0$ we have v_1 given by

$$(1-\delta)z_0 =_{def} v_1$$

where we select z_0 initially for use in the first period (begining at $t = 0$). In the two period irreversible model since revenue generation ends at $t = 2$, the annuity value of a standardised unit is:

$$\begin{aligned} A_0(\delta) &= 1 + (1-\delta) \\ &= 2 - \delta. \end{aligned}$$

Now at the beginning of the final period when b_1 is revealed, assuming irreversibility, the firm needs to solve the usual re-investment ($z \geq 0$) decision problem of maximizing:

$$2\sqrt{v_1 + z} - b_1 z,$$

[18] We have computed (see Appendix 3) that in the two period model when the production function is Cobb-Douglas of form x^θ/θ the rate is

$$\delta_0^*(\theta) = (1 + \widetilde{g}^{1/(\theta-1)})^{-1},$$

where $\widetilde{g}$ is the solution to equation (15), i.e.

$$1 = \Psi(\widetilde{g}, 1)$$

and so depends only on μ_b, σ_b.

where, as before, we shall have $z > 0$ provided

$$\frac{1}{b_1^2} - v_1 > 0.$$

The final period return is equal to

$$b_1 v_1 + \frac{1}{b_1},$$

obtained by replacing u by v_1 in (4), giving the expected return for the final period as:

$$\begin{aligned} F(v_1, b_0) &= \int_0^{1/\sqrt{v_1}} \left(b_1 v_1 + \frac{1}{b_1} \right) q(b_1|b_0) db_1 \\ &\quad + \int_{1/\sqrt{v_1}}^{\infty} 2\sqrt{v_1} q(b_1|b_0) db_1. \end{aligned}$$

Thus, as before,

$$F'(v_1, b_0) = \int_0^{1/\sqrt{v_1}} b_1 q(b_1|b_0) db_1 + \frac{1}{\sqrt{v_1}} \int_{1/\sqrt{v_1}}^{\infty} q(b_1|b_0) db_1.$$

Consequently the optimal choice of z_0 is obtained by solving the maximization problem

$$2\sqrt{z_0} - b_0 z_0 (2 - \delta) + F(v_1, b_0)$$

for which the first-order condition is

$$(2 - \delta) b_0 - \frac{1}{\sqrt{z_0}} = (1 - \delta) F'(v_1, b_0),$$

or writing $\widetilde{b}_1^\delta = 1/\sqrt{v_1}$ we have

$$(2 - \delta) b_0 - \widetilde{b}_1^\delta \sqrt{1 - \delta} = (1 - \delta) \left(\int_0^{\widetilde{b}_1^\delta} b_1 q(b_1|b_0) db_1 + \widetilde{b}_1^\delta \int_{\widetilde{b}_1^\delta}^{\infty} q(b_1|b_0) db_1 \right),$$

or, making the natural reference to the opening prices as a numeraire through the substitution $b_1 = b_0 g$, $\widetilde{b}_1^\delta = b_0 \widetilde{g}_1^\delta$,

$$\begin{aligned} \frac{(2 - \delta) - \widetilde{g}_1^\delta \sqrt{1 - \delta}}{1 - \delta} &= \int_0^{\widetilde{g}_1^\delta} g \widetilde{q}(g) dg + \widetilde{g}_1^\delta \int_{\widetilde{g}_1^\delta}^{\infty} \widetilde{q}(g) dg \\ &= {}_{def} \Psi(\widetilde{g}_1^\delta, 1) = F'(v_1, 1). \end{aligned}$$

This implies that $\widetilde{g}_1^\delta$ is defined as an intersection point of the rising graph of $\Psi(\widetilde{g}_1^\delta)$ with the linear graph of the function $\left((2 - \delta) - \widetilde{g}_1^\delta \sqrt{1 - \delta}\right)/(1 - \delta)$ which has therefore slope $-1/\sqrt{1 - \delta}$ and horizontal intercept at $(2 - \delta)/\sqrt{1 - \delta}$. The intersection point $\widetilde{g}_1^\delta$ strictly increases with

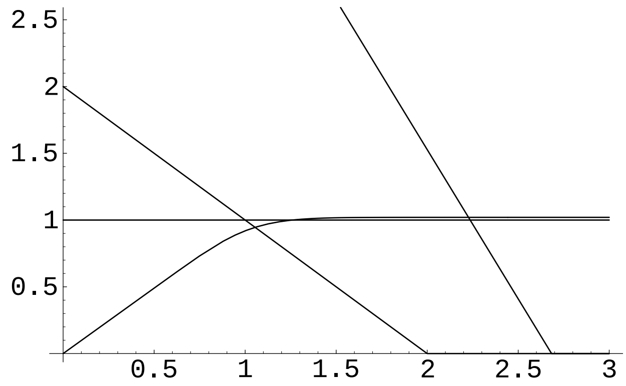

Figure 9: The irreversible A&E annuitized censor

δ from values below $\widetilde{g}$ to infinity. For an appropriate choice of δ we have $\widetilde{g}_1^\delta = \widetilde{g}$, where $\widetilde{g}$ is defined by $\Psi(\widetilde{g}) = 1$. Hence

$$\frac{(2-\delta) - \widetilde{g}\sqrt{1-\delta}}{1-\delta} = \Psi(\widetilde{g}) = 1,$$

so $\widetilde{g}\sqrt{1-\delta} = 1$, so that

$$\delta = \frac{1}{1+u} = \delta^*, \tag{36}$$

where $\widetilde{u} = 1/\widetilde{g}^2 = 1/(\widetilde{g}_1^\delta)^2 = v_0^\delta = v_0(1-\delta)$ is the optimal second period investment via either approach; since $\widetilde{u} = v_0(1-\delta)$ the equation (36) implies that $v_0 = 1$. For other values of δ we have $\widetilde{g}_1^\delta \neq \widetilde{g}$ and so both $v_0 \neq 1$ and $\widetilde{u} \neq v_0(1-\delta)$ simultaneously. But $v_0 = 1$ optimizes $2\sqrt{v_0} - v_0$ and $u = \widetilde{u}$ optimizes

$$F(u,1) - u.$$

It follows that unless $\delta = \delta^*$ we have

$$2\sqrt{v_0} - v_0 + F(v_0(1-\delta), 1) - v_0(1-\delta) < 1 + F(\widetilde{u}, 1) - \widetilde{u}.$$

4.2 The Reversible Case

Let us now comment briefly on the reversible case. Our aim here is to derive the A&E range of inactivity corresponding to the optimality of the *IO* policy in our terms. Recall that in our model, if the resale rate is constant no resale occurs until the last period. We now show that in the model where only annuities may be purchased and the resale rate is constant, say $R_t = \overline{R}$ for all t, there is a range of values of δ for which resale is optimal. The time t optimization problem takes the form

$$2\sqrt{v_t + z_t} + F_t((1-\delta)(v_t + z_t), b_t) - A_t(\delta)c_t z_t,$$

where $c_t = b_t$ when $z_t > 0$ and $c_t = b_t/\overline{R}$ when $z < 0$, corresponding to buying or selling back an annuity. The first order condition in z_t is thus

$$A_t(\delta)c_t - \frac{1}{\sqrt{v_t + z_t}} = (1-\delta)F_t'((1-\delta)(v_t+z_t), b_t).$$

Writing $\widetilde{b}_t^\delta = 1/\sqrt{(1-\delta)(v_t+z_t)}$ and again making the natural reference to the numeraire through substitution $b_{t+1} = b_t g$, $\widetilde{b}_t^\delta = b_t \widetilde{g}_t^\delta$ we have

$$\frac{A_t(\delta)/R - \widetilde{g}_t^\delta(R)\sqrt{1-\delta}}{(1-\delta)} = \Psi_{\overline{R},t}(\widetilde{g}_t^\delta(R)),$$

where $R = b_t/c_t$ is either 1 or $\overline{R}$ i.e. $R \in \{1, \overline{R}\}$.

To characterize $\Psi_{\overline{R},t}(\widetilde{g})$ we need to generalize the result (37) above for any constant $\overline{R}$ on the assumption that resale is possible in each period (for large enough v). In this case we have

$$\begin{aligned} F_{t-1}^{ae\prime}(v, b_{t-1}, \overline{R}) \quad = \quad & A_t(\delta)\int_0^{B(v)} b_t q(b_t|b_{t-1})db_t \\ & + \int_{B(v)}^{B(v,R)} \left(\frac{1}{\sqrt{v}} + (1-\delta)F_t^{ae\prime}((1-\delta)v, b_t)\right) q(b_t|b_{t-1})db_t \\ & + A_t(\delta)\int_{B(v,R)}^{\infty} \frac{b_t}{\overline{R}} q(b_t|b_{t-1})db_t. \end{aligned}$$

To understand the formula, observe that the first and third term are the usual investment and disinvestment terms as in (22) now adjusted for the annuity effects $A_t(\delta)$; the middle term arises through differentiation with respect to v from the revenue in the period beginning at time $t-1$ and the expected subsequent revenue when only depreciation occurs (i.e no new investment or disinvestment), i.e. differentiation of

$$2\sqrt{v} + F_t^{ae}((1-\delta)v, b_t).$$

The range denoted generically by the limits $[B(v), B(v,R)]$, indicates that we assert that there is a price range for which the stock in hand of v is too large for additional investment and too small for disinvestment.

It is now easy to see that the expected marginal revenue has lowest value

$$\frac{A_t(\delta)}{\overline{R}} E[b_t|b_{t-1}]$$

and highest value

$$A_t(\delta)E[b_t|b_{t-1}].$$

For each value of $R \in \{1, \overline{R}\}$ we have a potential intersection point of the $\Psi_{R,t}(\widetilde{g})$ curve and the line with slope $-1/\sqrt{1-\delta}$.

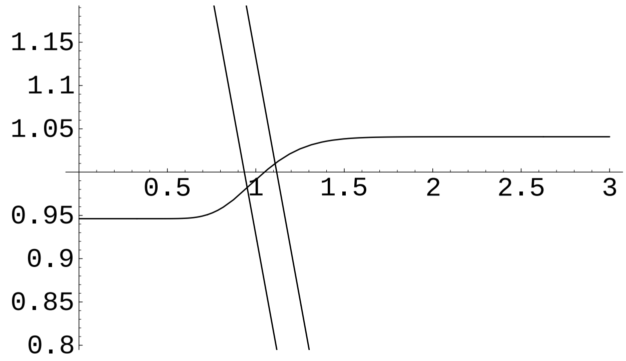

Figure 10: Reversible model: The A&E price-censors for investment (upper) and disinvestment (lower); their 'reciprocals' determine the low and high v triggers.

The two lines corresponding to the two possible values of R are evidently parallel and have vertical intersect at

$$\frac{A_t(\delta)}{R(1-\delta)}.$$

When $R = \overline{R}$ the intersect is below the infimum of the $\Psi_{R,t}(\widetilde{g})$ curve if

$$\frac{A_t(\delta)}{\overline{R}(1-\delta)} < \frac{A_t(\delta)}{\overline{R}} E[b_t|b_{t-1}]$$

i.e. if $1 < (1-\delta)E[b_t|b_{t-1}]$, whereupon there is no trigger for a sell-back. However, if $1 > (1-\delta)E[b_t|b_{t-1}]$, the trigger exists, so for consistency we assume this to hold.

For $R = 1$ the interesect $A_t(\delta)/(1-\delta)$ is above the infimum provided $\overline{R} > (1-\delta)E[b_t|b_{t-1}]$, so if the trigger is to exist this second condition is automatically satisfied and so we have recovered the A&E result in qualitative form, namely that if the stock in hand at time t is v_t and falls in the range

$$\frac{1}{(\widetilde{g}_t^\delta(\overline{R})b_t)^2(1-\delta)} \le v_t \le \frac{1}{(\widetilde{g}_t^\delta(1)b_t)^2(1-\delta)},$$

then no adjustment of investment occurs, but for v_t below this range additional investment is required, whilst for v_t above this range disinvestment is required.

5 SUMMARY AND CONCLUSION

We have developed a model of the optimal investment and divestment policy which takes place within a finite horizon setting. Unlike infinite horizon formulations our finite horizon model is able to capture end period effects which firms may in practice face.

For one class of simple distributions we then show that the optimal reversibility policy is characterized as a (single) trigger policy similar in spirit to that derived in infinite horizon settings. However, we then show how sensitive this comparability result is to the specific distributional assumption. This arises because applying another often used distributional assumption (geometric Brownian motion), the parameter range over which reversal is optimal is qualitatively different since a twin trigger policy may then be optimal.

The intuition for optimality of a twin switching policy space is as follows. Given that in our model setting, the investment asset price is expected to rise, there is ceteris paribus always a tendency to hold back with resale in order to gain as much as possible from a future resale. That is if the resale rate is a fixed proportion of the current market rate, delaying resale increases the return to the firm. We describe this tendancy as the hoarding pressure. However, in our view it does not seem reasonable to assume that the resale rate will remain a constant proportion of the asset price and so we have explicitly allowed for the resale rate to evolve stochastically as well. If the resale rate is expected to fall through time we describe this situation as the capital over supply pressure. In such a setting then a firm has to balance potential hording benefits against potential over supply costs.

In the simplest context of an iid resale rate, the balance is simply between the two growth rates - one in price and one in the discount rate (for resale). There exists a simple switching regime which agrees with simple intuition: resell when the resell rate is sufficiently close to unity, that is when the sale price for the asset is close to the market rate. Our technical analysis makes precise what is precisely meant by sufficient.

However we find this simple intuitive result does not carry over to the case where we assume that the discount rate, $R_t - 1$ follows a geometric Brownian motion. following the same comparative argument as above we commence by comparing the current resale price $1/R_0$ to that expected in the next period, inclusive of price inflation. However, critically this time the expectation is a convex function of R_0, whose value is asymptotically a constant multiple of the current 'resale price'.

$$\frac{b_0}{R_0} > \frac{1}{b_0} E[\frac{b_1}{R_1}|R_0, b_0] = \frac{E[b_1|b_0]}{b_0} \cdot E[\frac{1}{R_1}|R_0] \approx \frac{e^{\mu_b}}{R_0 e^{\mu_R - \sigma_R^2}}$$

The convexity of $E[\frac{1}{R_1}|R_0]$ as a function of R_0 is influenced by the variance σ_R^2 in such a way that the dominance of current resale occurs in quite different circumstances than in the iid case. That is we have shown that dominance may be altogether absent, it may be present only for an intermediate interval of values of R_t, or it may be present only for very large R_t.

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6 Appendix 1: The general N-period model

We begin by observing how formula (21) generalizes beyond two periods. We expect two regimes in the time period between t_n and t_{n+1}: one in which resale (reversal) of part of the opening stock is optimal, when the resale rate R_n lies in the set Γ_n and a complementary one in which resale is never optimal, i.e. for R_n lying in $\widetilde{\Gamma}_n$, the complement of Γ_n. Thus given observation of a resale rate $R_n \in \Gamma_n$ the optimal value function is made up of three integrals over the **U**, **IO** and **RO** ranges respectively:

$$
\begin{aligned}
F'_n(v_n, R_n, b_{n-1}) \;=\; & \int_0^{\widetilde{b}_n(v_n,1)} b_n q(b_n|b_{n-1}) db_n + \\
& \int_{\widetilde{b}_n(v_n,1)}^{\widetilde{b}_n(v_n,R_n)} \overline{F}'_{n+1}(u_{n+1}(v_n,b_n), b_n, R_n) q(b_n|b_{n-1}) db_n + \\
& \frac{1}{R_n} \int_{\widetilde{b}_n(v_n,R_n)}^{\infty} b_n q(b_n|b_{n-1}) db_n.
\end{aligned}
\tag{37}
$$

Here $u_{n+1}(v,b)$ is the direct generalization of the function solving the investment problem at time t_n (for use in the next period) when the opening stock v is held and the current input price is b. The bar notation in the second term implies that the future marginal revenue term is an expectation over future resale rates R_{n+1} given the current R_n value. It is important to notice the behavior of this expression as $v_n \to \infty$. We shall show that

$$
F'_n(\infty, R_n, b_{n-1}) = \frac{1}{R_n} \int_0^{\infty} b_n q(b_n|b_{n-1}) db_n = \frac{E[b_n|b_{n-1}]}{R_n}.
$$

In contrast given observation of an $R_n \in \widetilde{\Gamma}_n$ resale is not optimal and the optimal value function is made up of two integrals over the **U** and **IO** ranges, giving:

$$
\begin{aligned}
F'_n(v_n, R_n, b_{n-1}) \;=\; & \int_0^{\widetilde{b}_n(v_n,1)} b_n q(b_n|b_{n-1}) db_n + \\
& \int_{\widetilde{b}_n(v_n,1)}^{\infty} \overline{F}'_{n+1}(u(v_n,b_n), b_n, R_n) q(b_n|b_{n-1}) db_n.
\end{aligned}
\tag{38}
$$

Likewise, appealing to some homogeneity, we shall prove that in this regime

$$
\begin{aligned}
F'_n(\infty, R_n, b_{n-1}) &= \int_0^{\infty} \overline{F}'_{n+1}(\infty, 1, R_n) b_n q(b_n|b_{n-1}) db_n \\
&= \overline{F}'_{n+1}(\infty, 1, R_n) E[b_n|b_{n-1}].
\end{aligned}
$$

Thus in computing $\overline{F}'_n$ we need to mix the marginal revenues F'_n occurring in the two regimes as two integrals over R_n, one in the range Γ_n and the other in the range $\widetilde{\Gamma}_n$. This yields the following **infimum marginal expected return** formula.

$$b_{n-1} \inf_{v_n} \overline{F}'_n(v_n, 1, R_{n-1}) = E[b_n|b_{n-1}] \cdot \{E[\frac{1}{R_n}|R_n \in \Gamma_n, R_{n-1}] + E[\overline{F}'_{n+1}(\infty, 1, R_n)|R_n \in \widetilde{\Gamma_n}, R_{n-1}]\}. \quad (39)$$

This is the generalization of (21) and evidently, as before, its form reflects the independence of the R_t and b_t processes. The proof of this generalization depends on a detailed recursive analysis which we give in Appendix 2A. We confine ourselves to a very brief outline of the main ideas here; we then clarify the recursive nature of (39) drawing general conclusion about the nature of reversability and follow this with a study of the log-normal case and an illustrative example for the three period model.

6.1 Key points in the derivation of the formula.

The key observation is that the function $u = u_n(v, b)$ solving the investment problem at time t_n (for use in the next period)

$$\frac{1}{\sqrt{v-u}} = \overline{F}'_n(u, b) \quad (40)$$

may be rescaled; if we write $\widetilde{u} = ub^2, \widetilde{v} = vb^2$ we obtain in 'numeraire-form' the equation

$$\frac{1}{\sqrt{\widetilde{v} - \widetilde{u}}} = \overline{F}'_n(\widetilde{u}, 1). \quad (41)$$

Letting the solution of (41) be $\widetilde{u}_n(\widetilde{v})$ the carry forward hedge given a prevailing price b is then

$$u_n(v, b) = \widetilde{u}(vb^2)/b^2.$$

¿From here one proves a **monotonicity lemma** that the function $u_{n+1}(v, b)$ is monotonically increasing in the input stock v. The argument depends only on the inductive hypothesis that $\overline{F}'_n(u, b)$ is decreasing in u and has some (restricted) homogeneity in b. The other ingredient is to prove lemmas justifying the ranges for the optimality of each of the three regimes U,IO,RO.

6.2 Recursive structure of the reversal ranges Γ_n

We wish to show that the sets Γ_n are defined explicitly by inductive application of (39). Recall that the price sequence in the N-period model is $b_0, ..., b_{N-1}$ and that $F_N = 0$ (since there is no further economic activity after t_N). Let us write γ_i for the ratio $E[b_i|b_{i-1}]/b_{i-1}$, which we assume, for simplicity, to be deterministic, and let us put $P_i(R) = \bar{F}'_i(\infty, 1|R)$; we have of course $P_N = 0$. We may rewrite the recursion (39) in the form:

$$P_i(R_{i-1}) = \gamma_i \left\{ E[\frac{1}{R_i}|R_i \in \Gamma_i, R_{i-1}] + E[P_{i+1}(R_i)|R_i \notin \Gamma_i, R_{i-1}] \right\}. \quad (42)$$

Since $\Gamma_{N-1} = [1, \infty)$ we have

$$P_{N-1}(R) = \gamma_{N-1} \int_1^\infty \frac{\rho(R', R)}{R'} dR' \le \gamma_{N-1}, \tag{43}$$

the equality being strict unless all the probability resides at $R = 1$. We need this inequality for the Appendix 2A. We notethat when $N = 2$ this notation coincides with our earlier notation for $P_1(R)$and so expalins the reason for including the factor γ.

Once P_{N-1} is defined we may obtain Γ_{N-2} via (54) as the set $\{R \ge 1 : 1 > RP_{N-1}(R)\}$. Suppose for example that $\Gamma_{N-2} = [R_1^*, \infty)$ then

$$P_{N-2}(R) = \gamma_{N-2} \int_{R_1^*}^\infty \frac{\rho(R', R)}{R'} dR' + \gamma_{N-2}\gamma_{N-1} \int_1^{R_1^*} \int_1^\infty \frac{\rho(R'', R')\rho(R', R)}{R''} dR'' dR'.$$

This function in turn allows us to determine Γ_{N-3} and so on. Thus the sequence P_i is well-defined.

Note that since $1 < R_i P_i(R_i)$ for $R_i \notin \Gamma_i$ (i.e. $P_i > 1/R_i$) we have by (42) in general that

$$P_i(R) \ge \gamma_i \int_1^\infty \frac{\rho(R', R)}{R'} dR' = \frac{\gamma_i}{\gamma_{N-1}} P_{N-1}(R).$$

6.3 The log-normal model

We investigate this sequence for the log-normal distribution. We re-write the recursion thus

$$P_i(R) = \gamma_i \int_{w\Gamma_i} \frac{\varphi(w)dw}{1 + \tilde{R}e^{\sigma w + m}} + \gamma_i \int_{w\tilde{\Gamma}_i} P_{i-1}(1 + \tilde{R}e^{\sigma w + m})\varphi(w)dw,$$

using the transformation

$$w = w_R(R') =_{def} \frac{\ln(\widetilde{R'}/\tilde{R}) - m}{\sigma}, \tag{44}$$

where $\varphi(w)$ denotes the standard normal density, $m = m_R = \mu - \frac{1}{2}\sigma^2$ and we have suppressed the subscript i on m, μ and σ; recall that $\tilde{R} = (R - 1)$. Evidently $P_{N-1}(1) = \gamma_{N-1}$ and more generally $P_i(1) \ge \gamma_i$ with equality iff $\tilde{\Gamma}_i$ is bounded. To see this notice that one of $\Gamma_i, \tilde{\Gamma}_i$ is bounded, and observe that if Δ is a bounded interval then, in the limit as $R \to 1+$, its boundary $\partial\Delta$ is taken under w to one point at infinity:

$$w\partial\Delta \to +\infty,$$

and is similarly taken to -∞ as $R \to \infty$. If Γ_i is unbounded we obtain $\gamma_i P_{i-1}(1) \ge \gamma_i \gamma_{i-1} > \gamma_i$.

We differentiate to obtain (since the integrand is continuous across the boundary of Γ_i) the result:

$$P_i'(R) = -\gamma_i \int_{w\Gamma_i} \frac{\varphi(w)e^{\sigma w + m}dw}{\left(1 + \tilde{R}e^{\sigma w + m}\right)^2} + \gamma_i \int_{w\tilde{\Gamma}_i} P_{i-1}'(1 + \tilde{R}e^{\sigma w + m})e^{\sigma w + m}\varphi(w)dw.$$

Hence by induction $P_i'(R) < 0$ and in particular the function is bounded on $[1, \infty)$.

The earlier observation about the transformation of bounded intervals allows us to show inductively that $P_i(R)$ is asymptotic to A_i/R as $R \to \infty$ for some constant A_i. Thus if $\tilde{\Gamma}_i$ is bounded we have

$$P_i(R) \sim \frac{e^{\alpha}\gamma_i}{R}$$

where $\alpha_R = \sigma_R^2 - \mu_R$; whereas if Γ_i is bounded we have

$$P_i(R) \sim \frac{A_{i-1}}{R}\gamma_i \int_{w\tilde{\Gamma}_i} e^{-\sigma w - m}\varphi(w)dw = \frac{A_{i-1}\gamma_i e^{\alpha}}{R}.$$

The calculation in the former case relies on the boundedness of P_{i-1} and the following application of l'Hôpital's rule to $R\int_{w\tilde{\Gamma}_i} P_{i-1}'\varphi(w)dw$. Let $W = W(R) = w_R(R^*)$ where R^* is the right end-point of $\tilde{\Gamma}_i$, then since $R = R^* e^{-\sigma W - m}$, by (44), we have

$$\lim_{R\to\infty} \frac{\int_{-\infty}^{W} \varphi(w)dw}{1/R} = \lim_{R\to\infty} -R^2\varphi(W)W' = \frac{R^*}{\sigma} \lim_{W\to\infty} e^{-\sigma W - m}\varphi(W) = 0,$$

as $W' = 1/(\sigma\tilde{R})$. The latter case is treated similarly.

We note the over-estimate

$$P_{N-1}(R) < \gamma_{N-1} \int_{-\infty}^{\infty} \frac{\varphi(w)dw}{\tilde{R}e^{\sigma w + m}} = \frac{\gamma_{N-1}e^{\alpha}}{\tilde{R}} = \frac{\gamma_{N-1}e^{\alpha}}{R-1}. \tag{45}$$

6.3.1 Situation with $e^{\alpha}\gamma < 1$ $(\alpha_R + \mu_b < 0)$:

In the case when $e^{\alpha}\gamma_{N-1} < 1$ it is possible to solve the inequality $RP_{N-1}(R) < 1$ by solving the stronger inequality

$$\frac{Re^{\alpha}\gamma_{N-1}}{R-1} < 1$$

in view of (45) and this which holds when

$$\frac{1}{1 - e^{\alpha}\gamma_{N-1}} < R.$$

Thus $R_1^* < 1/(1 - e^{\alpha}\gamma_{N-1})$.

We may also usefully over-estimate of P_{N-2} when $\Gamma_{N-2} = [R_1^*, \infty)$. Letting $W_1 = W_1(R) =_{def} w_R(R_1^*)$, we have

$$\begin{aligned} \frac{1}{\gamma_{N-2}}P_{N-2}(R) &= \int_{W_1}^{\infty} \frac{\varphi(w)dw}{1 + \tilde{R}e^{\sigma w + m}} + \int_{-\infty}^{W_1} P_{N-1}(1 + \tilde{R}e^{\sigma w + m})\varphi(w)dw, \\ &< \int_{W_1}^{\infty} \frac{\varphi(w)dw}{\tilde{R}e^{\sigma w + m}} + \gamma_{N-1}\int_{-\infty}^{W_1} \frac{e^{\alpha}}{\tilde{R}e^{\sigma w + m}}\varphi(w)dw \\ &= \frac{e^{\alpha}}{\tilde{R}}\{\Phi(-W_1 + \sigma) + e^{\alpha}\gamma_{N-1}\Phi(W_1 - \sigma)\}. \end{aligned}$$

Thus

$$P_{N-2}(R) < \frac{e^{\alpha}\gamma_{N-2}}{\tilde{R}}\{(1 - e^{\alpha}\gamma_{N-1})\Phi(-W_1 + \sigma) + e^{\alpha}\gamma_{N-1}\}.$$

Now $W_1 = W_1(R)$ is decreasing with R, so if $e^{\alpha}\gamma_{N-1} < 1$, the function $\{(1 - e^{\alpha}\gamma_{N-1})\Phi(-W_1 + \sigma) + e^{\alpha}\gamma_{N-1}\}$ is increasing with R from $e^{\alpha}\gamma_{N-1}$ to unity. Hence

$$P_{N-2}(R) < \frac{e^{\alpha}\gamma_{N-2}}{R-1}. \tag{46}$$

Then just as in the previous stage it is possible to solve the inequality $RP_{N-2}(R) < 1$ by solving the stronger inequality

$$\frac{Re^{\alpha}\gamma_{N-2}}{R-1} < 1,$$

which holds when $R > 1/(1 - e^{\alpha}\gamma_{N-1})$. This gives $R_2^* < 1/(1 - e^{\alpha}\gamma_{N-1})$.

This method of overestimation may be applied inductively. Thus in this case for all i we have Γ_i unbounded. We note the case $\mu_R = 0$ is not included here since $\sigma_R^2 > 0$ and $\mu_b > 0$.

6.3.2 Situation with $e^{\alpha}\gamma > 1(\alpha_R + \mu_b > 0)$:

Here $\gamma > e^{\mu_R - \sigma_R^2}$, i.e. selling price inflation may be considered 'sufficiently strong'. Assuming Γ_{N-2} bounded, we shall have $P_{N-2}(R) \sim (e^{\alpha}\gamma)^2/R$ so whilst in principle Γ_{N-3} will again be bounded it is possible for $\Gamma_{N-3} = \emptyset$ depending on parameter choices. In any event we shall have $P_{N-3}(R) \sim (e^{\alpha}\gamma)^3/R$. It is thus likely for irreversibility to obtain throughout.

6.3.3 General conclusions

When the γ_i are all equal we find that there are several scenarios depending on the three parameters.

i) The N-period double switching regime: $1 - M < S < S^*(M)$; for all $n < N - 1$, $\Gamma_{N-1} = [1, \infty)$ and the reversal range Γ_n is empty or takes the form of an interval $[R_n^{\#}, R_n^{\#\#}]$ **shrinking as n decreases**.

ii) The N-period single switching regime: $(M + S < 1)$; for all n the reversal range Γ_n takes the form of an interval $[R_n^{\#}, \infty)$ which is **expanding as n decreases** (i.e. with $R_n^{\#}$ decreasing as n decreases).

iii) The N-period irreversible regime: $M \leq 0$ or $M > 1$ or $S > S^*(M)$; **for all $n < N - 1$ there is no reversal of investment**.

We give some worked examples in the case $N = 3$ illustrating these phenomena.

6.4 Examples for $N = 3$.

Recalling the opening remarks of section 4 that we need to take a mix of the two formats (37) and (38); from (39) we now need to solve

$$\begin{aligned}\frac{1}{R_0} &= E[\frac{e^{\mu_b}}{R_1}|R_0, R_1 \in \Gamma_1] \\ &+ \int_1^{R_1^\#} \left(\int_1^\infty \frac{e^{2\mu_b}\rho(R_2, R_1)}{R_2} dR_2 \right) \rho(R_1, R_0) dR_1.\end{aligned}$$

Here with two periods each of unit duration remaining (i.e. $\gamma_1 = \gamma_2 = e^{\mu_b}$), on the left we have the current resale input price (the buying price being normalized to unity); on the right the first term corresponds to the discounted resale price in one period's time, assuming resale may occur, and the second term corresponds to the discounted price in two period's time, both suitably weighted by transition probabilities. The formula assumes that Γ_1 is an infinite interval. The corresponding formula when $\Gamma_1 = [R_1^\#, R_1^{\#\#}]$ is

$$\begin{aligned}\frac{1}{R_0} &= E[\frac{e^{\mu_b}}{R_1}|R_0, R_1 \in \Gamma_1] \\ &+ \int_1^{R_1^\#} \left(\int_1^\infty \frac{e^{2\mu_b}\rho(R_2, R_1)}{R_2} dR_2 \right) \rho(R_1, R_0) dR_1 \\ &+ \int_{R_1^{\#\#}}^\infty \left(\int_1^\infty \frac{e^{2\mu_b}\rho(R_2, R_1)}{R_2} dR_2 \right) \rho(R_1, R_0) dR_1.\end{aligned}$$

We have computed Examples 1 and 2 with $\mu_R = 0.07$ and $\mu_b = 0.02$ for two close values of σ_R namely 0.24 and 0.2468, showing in the latter case endogenous irreversibility at time $t = 0$; in the former case there is an expanding reversal range as time progresses.

7 Appendix 2A: Formula for the infimum value

In this section we prove the formula (39) which is at the heart of any resale. This will be done by backwards induction standard to the dynamic programming approach. The calculations also provide how much resale needs to be effected. We also identify the optimal investment policy.

7.1 Inductive hypotheses

We make the following **inductive hypotheses** on $\overline{F}'_n(\widetilde{u}, b_n)$ which we verify for $n = N-1$ and then show that the validity of the hypotheses at n implies validity at $n-1$. In course of doing this we will establish, also by backwards induction, the formula (39).

i) **Homogeneity hypothesis**:

$$\overline{F}'_n(u, b_n) = b_n \overline{F}'_n(ub_n^2, 1). \tag{47}$$

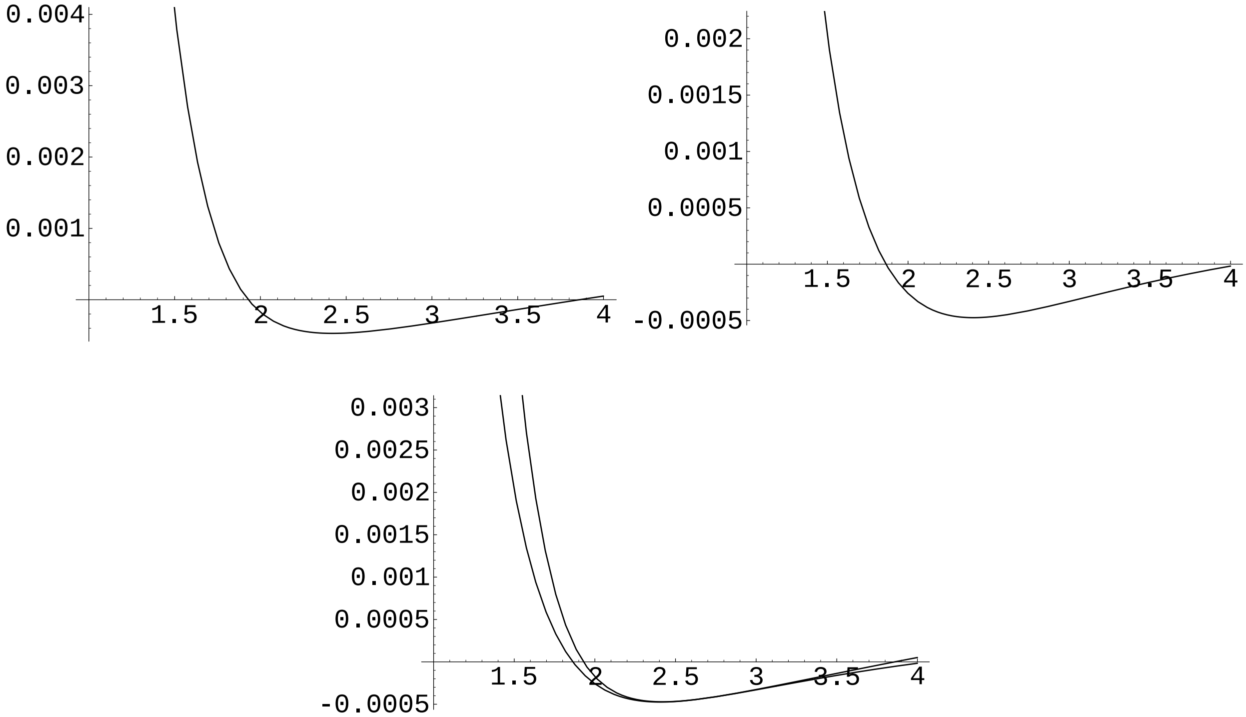

Figure 11: Three-period Example 1: (i) Two periods to go and Γ is a bounded interval; (ii) One period to go and Γ is a bounded interval; (iii) Reversible region Γ is a shrinking bounded interval. ($\mu_R = 0.07, \mu_b = 0.02, \sigma_R = 0.24$.)

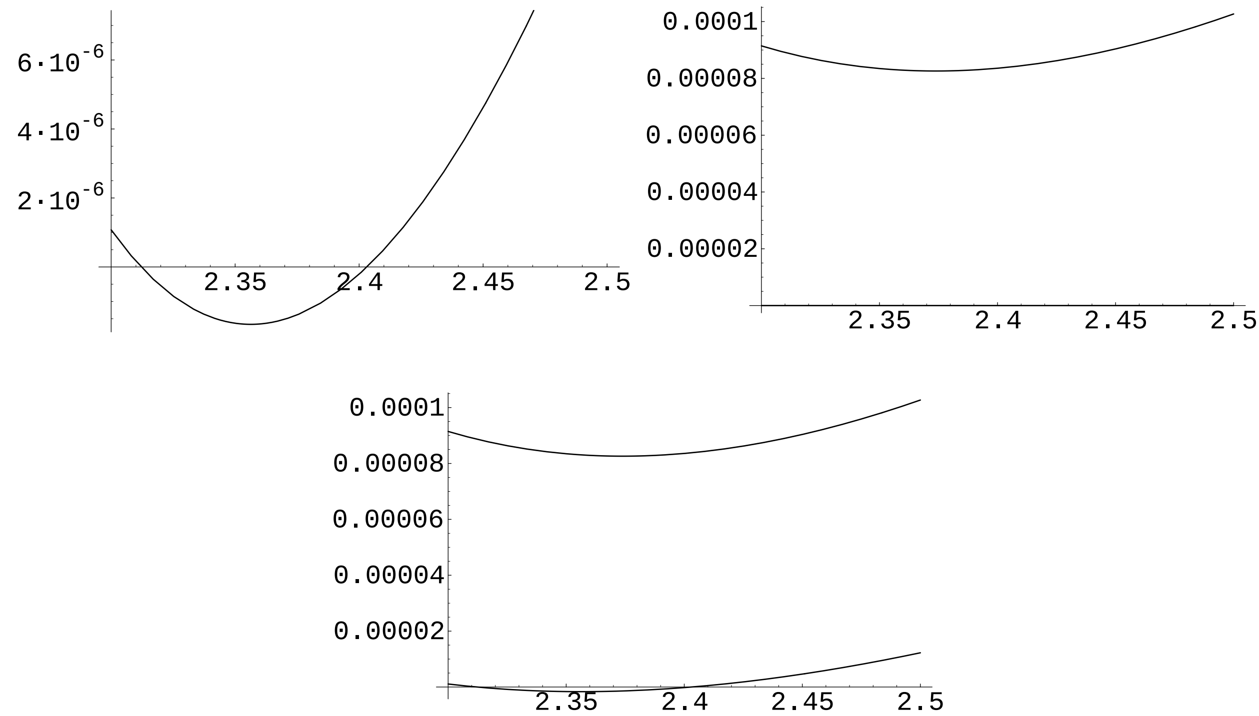

Figure 12: Three-period Example 2: (i) Two periods to go and Γ is a small bounded interval: close up; (ii) One period to go and Γ is empty: close up; (iii) Reversible region Γ is a shrinking bounded interval.($\mu_R = 0.07, \mu_b = 0.02, \sigma_R = 0.2468$.)

It now follows that if in the carry-forward equation

$$\frac{1}{\sqrt{v-u}} = \overline{F}'_n(u, b_n). \tag{48}$$

we write $\widetilde{u} = ub_n^2, \widetilde{v} = vb_n^2$ we obtain its 'numeraire-form':

$$\frac{1}{\sqrt{\widetilde{v}-\widetilde{u}}} = \overline{F}'_n(\widetilde{u}, 1). \tag{49}$$

Letting the solution be $\widetilde{u}_n(\widetilde{v})$, the carry forward hedge in (48) given a prevailing price b_n is then

$$u_n(v, b_n) = \widetilde{u}_n(vb_n^2)/b_n^2. \tag{50}$$

ii) **Concavity hypothesis:**

$$\overline{F}''_n(u, 1) < 0,$$

i.e. $\overline{F}'_n(u, b_n)$ is differentiable and decreasing in u.

7.2 Definitions

Next we need to formulate some definitions.

We begin by noting that just as in section 2.2 at time t_n the firm must maximize over both z_n and x_n the profit:

$$2\sqrt{x_n} - c_n z_n + \overline{F}_n(v_n + z_n - x_n, b_n),$$

where $c_n = b_n/R$ where $R \in \{1, 1/R_n\}$,or

$$2\sqrt{x_n} - c_n(u_n - v_n + x_n) + F_n(u_n, b_n),$$

where $u_n = v_n + z_n - x_n$. We can re-write the first order condition for $u = u_n$

$$c_n = \overline{F}'_n(u, b_n)$$

divided through by b_n as:

$$1/R = \overline{F}'_n(b_n^2 u, 1). \tag{51}$$

Let $u = \widehat{u}_n(R)$ denote the solution to (51) for $b_n = 1$ whenever it exists (i.e. since $\overline{F}'_n(u, 1)$ is decreasing in u,this occurs when $1/R > \overline{F}'_n(\infty, 1)$); we note that the solution $u = \widehat{u}_n(R, b_n)$ to

$$\overline{F}'_n(u, b_n) = b_n/R$$

is then evidently

$$\widehat{u}_n(R, b_n) = \frac{\widehat{u}_n(R)}{b_n^2},$$

which is yet another manifestation of the homogeneity of the returns function. Its monotonicity in b_n is of paramount importance.

The manager selects either $z_n = 0$, or the one of $z_n = z(R_n)$, or $z_n = z(1)$ whichever is positive, where

$$z(R) = \frac{R^2}{b_n^2} + \frac{\widehat{u}_n(R)}{b_n^2} - v,$$

or writing $\widehat{u}_n = \frac{1}{\widetilde{g}_n(R)^2}$ (as in (23))

$$z(R) = \frac{1}{b_n^2}(R^2 + \frac{1}{\widetilde{g}_n(R)^2}) - v.$$

It is natural to invoke the Cobb-Douglas duality between input price and input quantity and to write $\widehat{u}_n(R)/R^2 =_{def} 1/B_n(R)^2$ (so that the hedge is compared against buying price when $R = 1$ and against selling price when $R = R_n$). Thus $z(R) = \frac{R^2}{b_n^2}(1 + \frac{1}{B_n(R)^2})$. We shall see in Appendix B that $\widehat{u}_n(R)/R^2$ has a limit as $R \to \infty$.

7.3 Base step of the induction

We verify the hypothesis when $n = N - 1$.

Recalling (22) from the two period setting, we have the following calculation at time t_{N-1} when R_{N-1} is just known but not b_{N-1}. (since $\frac{\widetilde{b}_{N-1}}{b_{N-1}} = \frac{1}{\sqrt{b_{N-2}^2 u}}$):

$$\begin{aligned} F'_{N-1}(u, R_{N-1}, b_{N-2}) &= \int_0^{1/\sqrt{b_{N-1}^2 u}} b_{N-1} g \widetilde{q}(g) dg + \frac{b_{N-1}}{\sqrt{b_{N-1}^2 u}} \int_{1/\sqrt{b_{N-1}^2 u}}^{R_1/\sqrt{b_{N-1}^2 u}} \widetilde{q}(g) dg \qquad (52) \\ &\quad + \frac{1}{R_1} \int_{R_1/\sqrt{b_{N-1}^2 u}}^{\infty} b_{N-1} g \widetilde{q}(g) dg \\ &= b_{N-1} F'_1(b_{N-1}^2 u, R_{N-1}, 1), \end{aligned}$$

showing the required homogeneity. See the Remark at the end of section 3.2 for the concavity property.

7.4 A lemma

We now prove from our hypotheses the following.

Monotonicity Lemma. The carry-forward function $u_n(v, b_n)$ is monotonically increasing in the input stock v.

Proof. Observe that from (50) we have:

$$\frac{\partial}{\partial v} u_n(v, b_n) = \widetilde{u}'_n(v b_n)$$

and we now show that

$$\widetilde{u}_n' > 0.$$

By (49) we have

$$\widetilde{v} = \widetilde{u} + 1/\overline{F}_n'(\widetilde{u}, 1)^2$$

so differentiating with respect to $\widetilde{u}$ we have

$$\widetilde{v}' = 1 - 2\overline{F}_n'(\widetilde{u}, 1)^{-3}\overline{F}_n''(\widetilde{u}, 1) > 0$$

since by (28) $\overline{F}_n'(\widetilde{u}, 1)$ is decreasing.

7.5 Optimal investment policies

Returning to our analysis, we note that working with the first-order conditions we have identified two policies which satisfy the internal first-order conditions (investment or reversal) and one which is a corner solution (zero-investment). These generalize our earlier three policies.

For ease of notation we will write F for F_n whenever convenient.

It now remains to verify the circumstances under which the specific policy type dominates the others.

7.5.1 Deterministic resale rate

In this subsection only we assume that for all n the discount rate R_n is constant and equal to $\overline{R}$. (The results of this section should be seen in contrast to those of the following subsection where the rate is stochastic.) We now prove:

Dominance Lemma (Deterministic case). With a single discount $R_{n+1} = \overline{R}$ and for $v > (\overline{R}/b_n)^2$ we have

$$\begin{aligned} 2\sqrt{v - u(v, b_n)} + F(u, b_n) &> \frac{b_n}{\overline{R}}v + \frac{\overline{R}}{b_n} + F(0, b_n) \\ &> 2\sqrt{v} + F(0, b_n), \end{aligned}$$

that is the "no resale and zero-investment" policy dominates a policy of "efficient resale with zero carry-forward" which in turn dominates "exhausting the stock without resale".

We prove the result in a series of steps.

Observation 1. If $b_n < 1/\sqrt{v}$, then we have

$$2\sqrt{v} + F(0, b_n) < b_n v + \frac{1}{b_n} + F(0, b_n),$$

i.e. the acquisition strategy U dominates "immediate exhaustion of the stock".

This is clear from completing the square.

Sublemma 1. In the constant resale ratio case we have for $v > \overline{v} = 1/(\gamma b_n)^2$ that

$$2\sqrt{v - u(v, b_n)} + F(u, b_n) > 2\sqrt{v} + F(0, b_n),$$

i.e. it is better to carry forward than to utilize all resources.

Sublemma 2. In the constant resale ratio case we have for $v > v^* = 1/c_n$ where $c_n = \frac{b_n}{R}$, that

$$2\sqrt{v - u(v, b_n)} + F(u, b_n) > \left(c_n v + \frac{1}{c_n} + F(0, b_n) \right),$$

i.e. it is better to carry forward than to sell off resources according to a first-order condition.

Proof of Sublemma 1. Define

$$\Delta_1(v) = \left(2\sqrt{v-} + F(u(v, b_n), b_n)\right) - \left(2\sqrt{v} + F(0, b_n)\right)$$

Differentiate $\Delta_1(v)$ with respect to v to obtain

$$\Delta_1'(v) = \left(\frac{1 - u'}{\sqrt{v - u(v, b_n)}} + F'(u, b_n)u' \right) - \frac{1}{\sqrt{v}}$$

and noting that the first-order condition (48) simplifies this to

$$\Delta_1'(v) = \frac{1}{\sqrt{v - u(v, b_n)}} - \frac{1}{\sqrt{v}}.$$

Define $\overline{v}$ to be the largest v such that $u(\overline{v}, b_n) = 0$. Evidently the first order condition gives in the limit as v tends to $\overline{v}$ from above that

$$\frac{1}{\sqrt{\overline{v}}} = F'(0, b_n) =_{def} \beta(b_n),$$

so

$$\overline{v} = 1/\beta(b_n)^2.$$

Remark. We shall see that the definition above yields a generalization to any function F representing the optimal future expected profit and irrespective of the number of periods to expiry we shall have

$$\beta_F(b_n) = \int_0^\infty b_n \widetilde{q}(g) dg = b_n \gamma.$$

This result is already obvious with one period to expiry from (22).

Thus $\Delta_1'(v)$ is positive as soon as $u(v, b_n) > 0$ i.e. for $v > \overline{v}$. Thus for $v > \overline{v}$ we have $\Delta_1(v) > 0$ as $\Delta_1(\overline{v}) = 0$. We have just proved Sublemma 1.

Proof of Sublemma 2.

Let

$$\Delta_2(v) = \left(2\sqrt{v - u(v, b_n)} + F(u(v, b_n), b_n)\right) - \left(c_n v + \frac{1}{c_n} + F(0, b_n) \right)$$

Differentiate $\Delta_2(v)$ with respect to v to obtain

$$\begin{aligned} \Delta_2'(v) &= \frac{1}{\sqrt{v - u(v, b_n)}} - c_n \\ &= F'(u(v, b_n), b_n) - c_n \end{aligned}$$

which is positive since for any w we have by (25) that $F'(w,b_n) > c_n$. Let $v^* = 1/c_n$, then we have $v^* > \overline{v}$ since

$$\frac{\overline{R}^2}{b_n} > \frac{1}{\gamma^2 b_n}$$

is equivalent to $\gamma\overline{R} > 1$ (valid under our inflation assumption $\gamma > 1$). Thus from $\Delta_1(v^*) > 0$ we obtain

$$\begin{aligned}\left(2\sqrt{v^* - u(v^*, b_n)} + F(u(v^*, b_n), b_n)\right) &> \left(2\sqrt{v^*} + F(0, b_n)\right) \\ &= c_n v^* + \frac{1}{c_n} + F(0, b_n),\end{aligned}$$

i.e. $\Delta_2(v^*) > 0$ and so $\Delta_2(v) > 0$ for $v > v^*$. This proves Sublemma 2.
Proof of Lemma: We note that for $v \geq 1/c_n$ we have (completing the square) that

$$2\sqrt{v} \leq c_n v + \frac{1}{c_n}$$

with equality when $v = 1/c_n$. Thus for $v > 1/c_n$ we have

$$\begin{aligned}2\sqrt{v - u(v, b_n)} + F(u(v^*, b_n), b_n) &> c_n v + \frac{1}{c_n} + F(0, b_n) \\ &> 2\sqrt{v} + F(0, b_n).\end{aligned}$$

7.5.2 Stochastic resale rate

We contrast this result with the following.

Dominance Lemma (Stochastic case). With a stochastic discount, if $R_n < R_n^* = 1/(\gamma E[1/R_{n+1}])$ then for

$$w(b_n, R_n) = \frac{\widehat{u}(R_n)}{b_n} + \frac{R_n^2}{b_n} = \frac{R_n^2}{b_n}\left(\widehat{u}(R_n) + 1\right),$$

we have dominance of RO iff $v \geq w(b_n, R_{n)},$ i.e.

$$c_n(v - \widehat{u}) + \frac{1}{c_n} + \overline{F}(\widehat{u}, b_n) \geq 2\sqrt{v - u(v, b_n)} + \overline{F}(u(v), b_n),$$

with equality when $v = w(b_n, R_n)$. Here $\widehat{u} = \widehat{u}_{R_1}(b_n)$ solves $\overline{F}'(\widehat{u}, b_n) = b_n/R_n$.

Proof of lemma. We begin as usual by considering the difference

$$\begin{aligned}\Delta_3(v) &= \left(c_n(v - \widehat{u}) + \frac{1}{c_n} + \overline{F}(\widehat{u}, b_n)\right) \\ &\quad - \left(2\sqrt{v - u(v, b_n)} + \overline{F}(u(v), b_n)\right)\end{aligned}$$

and note that as before

$$\begin{aligned}\Delta_3'(v) &= c_n - \left(\frac{1-u'}{\sqrt{v-u(v,b_n)}} + \overline{F}'(u(v),b_n)u'\right)\\ &= c_n - \frac{1}{\sqrt{v-u(v,b_n)}}\\ &= c_n - \overline{F}'(u(v,b_n),b_n). \qquad (53)\end{aligned}$$

Now $\overline{F}'(w,b_n)$ is decreasing in w and if we may solve $\overline{F}'(\widehat{u},b_n) = b_n/R_n = c_n$ then $\Delta_3'(v) > 0$ for v with $u(v,b_n) > \widehat{u}$. Finally, if $w = w(b_n,R_n)$ is defined so that $u(w,b_n) = \widehat{u}$ we have

$$\Delta_3'(w) = 0$$

and so for $v > w$ we have $\Delta_3'(v) > 0$.

Now, since $u(w,b_n) = \widehat{u}$, and

$$\frac{1}{\sqrt{w-u(w,b_n)}} = \overline{F}'(\widehat{u},b_n) = c_n$$

we have

$$w = \widehat{u} + \frac{R_n^2}{b_n} = \frac{R_n}{b_n}\left(\widehat{u}(R_n)^2+1\right).$$

Furthermore

$$\begin{aligned}\Delta_3(w) &= \left(c_n(w-\widehat{u}) + \frac{1}{c_n} + \overline{F}(\widehat{u},b_n)\right)\\ &\quad - \left(2\sqrt{w-\widehat{u}} + \overline{F}(\widehat{u},b_n)\right)\\ &= c_n(\frac{1}{c_n}) + \frac{1}{c_n} - \frac{2}{c_n} = 0.\end{aligned}$$

Thus $\Delta_3(v) > 0$ for $v > w$. Of course the inequality is reversed for $v < w$ (for then $\Delta_3'(v) < 0$ by (53).

Remark. A similar argument may be applied to the function

$$\Delta(v) = \left(b(v-\widehat{u}) + \frac{1}{b} + \overline{F}(\widehat{u},b_n)\right) - \left(2\sqrt{v-u(v,b)} + \overline{F}(u(v,b),b)\right)$$

We note that letting

$$v(b) = \frac{1}{b^2} + \widehat{u}$$

we have

$$\frac{1}{\sqrt{v(b)-\widehat{u}}} = b = \overline{F}'(\widehat{u},b_n),$$

so $u(v(b),b)=\widehat{u}$. Also

$$\Delta(v(b))=0.$$

Appealing to the Monotonicity Lemma, we have if $v<v(b)$ that $u(v,b)<u(v(b),b)=\widehat{u}$. Thus

$$\Delta'(v)=b-\overline{F}'(u(v,b),b)<0.$$

so for $v<v(b)$ we have $\Delta(v)>0$, i.e. topping up is prefered. Of course the carry-forward option is unavailable as soon as $u(v,b)=0$ i.e. when $v=\overline{v}$. (see Sublemma 1). We note that the monotonicity property implies that

$$\overline{v}<v(b),$$

i.e. that

$$\frac{1}{\gamma^2}<1+\widehat{u}(1),$$

providing the lower bound

$$\widehat{u}(1)>\frac{1-\gamma^2}{\gamma^2}.$$

7.6 Form of F'_{n-1} and the infimum formula

The dominance lemma and the concluding remark together show which is the optimal policy in the three ranges $[0,v(b,1))$, $[v(b,1),v(b,R))$, $[v(b,R),\infty)$ when $R\in\Gamma_n$. If $R\notin\Gamma_n$ we are left with just the two ranges $[0,v(b,1))$, $[v(b,1),\infty)$.

Suppose the previous period buying-price was b_{n-1} and that a quantity v of input was carried forward into the current period. Suppose the current resale rate is R_n, given these three parameters, we may now compute the expected value over all prices b_n of the production from the current time onwwards, which we denote by $F_{n-1}(v,R_n,b_{-1})$. There are two cases to consider. First, when $R_n\in\Gamma_n$ we have

$$\begin{aligned}F_{n-1}(v,R_n,b_{n-1}) &= \int_0^{b(v,1)}\{b_n(v-\hat{u}_n(1,b_n))+\frac{1}{b_n}+\bar{F}_n(\hat{u}(1,b_n),b_n)\}q_n(b_n)db_n\\ &+\int_{b(v,1)}^{b(v,R_n)}\{2\sqrt{v-u_n(v,b_n)}+\bar{F}_n(u_n(v,b_n),b_n)\}q_n(b_n)db_n\\ &+\int_{b(v,R_n)}^{\infty}\{\frac{b_n}{R_n}(v-\hat{u}_n(R_n,b_n))+\frac{R_n}{b_n}+\bar{F}_n(\hat{u}_n(R_n,b_n),b_n)\}q_n(b_n)db_n.\end{aligned}$$

Here $b=b(v,R)$ solves the equation

$$v=\frac{R^2}{b^2}+\frac{\hat{u}_n(R)}{b^2},$$

i.e.

$$b(v,R)=\sqrt{R^2+\hat{u}_n(R)}/\sqrt{v}.$$

To obtain the formula for $F_{n-1}(v, R_n, b_{n-1})$ when $R_n \notin \Gamma_n$, we drop the third term and replace $b(v, R_n)$ by $+\infty$.

Using the fact that $\hat{u}(R, b_n) = u(w(R, b_n), b_n)$, it is routine to verify that when $R_n \in \Gamma_n$

$$\begin{aligned} F'_{n-1}(v, R_n, b_{n-1}) &= \int_0^{b(v,1)} b_n q_n(b_n) db_n \\ &\quad + \int_{b(v,1)}^{b(v,R_n)} \bar{F}'_n(u(v, b_n), b_n) q_n(b_n) db_n \\ &\quad + \int_{b(v,R_n)}^{\infty} \frac{b_n}{R_n} q_n(b_n) db_n, \end{aligned}$$

where the prime denotes differentiation with respect to v. The form of the middle term is derivable from the first-order condition (48). Evidently when $R_n \notin \Gamma_n$ we need to drop the third term and replace $b(v, R_n)$ by $+\infty$; in this connection note that as b_n tends to infinity

$$\bar{F}'_n(u(v, b_n), b_n) = b_n \bar{F}'_n(\tilde{u}(b_n^2 v) b_n^2, 1) \to b_n \bar{F}'_n(\infty, 1),$$

where we have use the homogeneity assumption (i). Suppose that the previous resale rate was R_{n-1}. Taking expectations over R_n it now follows subject to some mild technical assumptions connecting the inequality of growth rates (see the Appendix 2B) that

$$\bar{F}'_{n-1}(\infty, b_{n-1}, R_{n-1}) = E[b_n | b_{n-1}] \left\{ E[\frac{1}{R_n} | R_n \in \Gamma_n, R_{n-1}] + E[\bar{F}'_n(\infty, 1, R_n) | R_n \notin \Gamma_n, R_{n-1}] \right\}.$$

At this stage the formula is true subject to our inductive hypotheses. It will therefore be valid generally if the inductive hypothesis itself is validated. By the principle of induction all we need to do is now to verify the hypothesis at $n-1$. This we do by way of deductions this recursion formula.

First notice that $\bar{F}'_n(u(v, b(v, R), b(v, R)) = \bar{F}'_n(\hat{u}(R, b(v, R), b(v, R)) = b(v, R)$ so we may deduce that

$$F''_{n-1}(v, R_n, b_{n-1}) = \int_{b(v,1)}^{b(v,R_n)} \bar{F}''_n(u_n(v, b_n), b_n) u'_n(v, b_n) q_0(b_n) db_n,$$

where as usual the prime denotes differentiation with respect to the input, here v. By the Monotonicity Lemma $u' \geq 0$ (where defined) and by the inductive hypothesis $\bar{F}''_n \leq 0$. Thus $\bar{F}''_{n-1} \leq 0$. This verifies the concavity hypothesis (ii).

Next, since

$$b(b_{n-1}^2 v, R) = b(v, R)/b_{n-1}$$

we have

$$\bar{F}'_{n-1}(v, R_n, b_{n-1}) = b_{n-1} \bar{F}'_{n-1}(b_{n-1}^2 v, R_n, 1).$$

Thus the homogeneity hypothesis (i) is satisfied by $\bar{F}'_{n-1}$.

The inductive hypotheses have thus been validated for all n.

We note from the concavity property that

$$\bar{F}'_{n-1}(v, b_{n-1}|R_{n-1}) \geq \bar{F}'_{n-1}(\infty, b_{n-1}|R_{n-1}), \tag{54}$$

which we need elsewhere.

8 Appendix 2B: Passage to the limit

We prove

Proposition 1. Provided $E[1/R] < \infty$ we have

$$\lim_{u\to\infty} \bar{F}'_1(u, b_n) = E[\frac{1}{R}]E[b_1|b_n].$$

This result follows easily from the following lemma and equation (heart) on writing $B = 1/\sqrt{u}$.

Lemma A.

$$\lim_{B\to 0} \int_1^\infty dR \int_0^{RB} bq(b)\frac{\rho(R)}{R} db = 0.$$

Proof. Changing the order of integration we need to consider two contributing integrals.

$$I_1(B) = \int_0^B db \int_1^\infty bq(b)\frac{\rho(R)}{R} dR = \int_0^B bq(b)db \int_1^\infty \frac{\rho(R)}{R} dR \leq BE[\frac{1}{R}]$$

which tends to zero with B. Also

$$\begin{aligned} I_2(B) &= \int_1^\infty dR \int_B^{RB} bq(b)\frac{\rho(R)}{R} db \leq \int_1^\infty \frac{\rho(R)}{R} \left\{\int_0^{RB} bq(b)db\right\} dR \\ &\leq \int_1^\infty \frac{\rho(R)}{R} \left\{\int_0^{RB} RBq(b)db\right\} dR \\ &\leq \int_1^\infty \frac{\rho(R)}{R} \left\{\int_0^{\infty} RBq(b)db\right\} dR = B. \end{aligned}$$

Place Diagram here.

Before stating and proving a fundamental technical result we amplify notation by giving a time index to some already defined quantities. We thus recall.

Definition. At period i the optimal hedge quantity $\hat{u}(R) = \hat{u}_i(R)$ is defined for $R \in \Gamma_i$ by

$$\bar{F}_i(\hat{u}(R), 1, R) = \frac{1}{R}.$$

We let

$$b_i(v, R) = \sqrt{1 + \hat{u}_i(R)}/\sqrt{v}.$$

We now prove three propositions which together justify certain manipulations in section 4. We write

$$E_i[\frac{1}{R'}|R] = \int_{\Gamma_i} \frac{\rho_i(R',R)}{R'} dR',$$

where $\rho_i(R',R)$ is the resale rate density for period i when the previous resale rate was R.

Proposition 2. Suppose that $E_i[1/R] < \infty$ and that the function $\hat{u}_i(R)/R^2$ remains bounded for $i = 1, ..., N-1$. Then we have

$$\lim_{v\to\infty} \int_{\Gamma_i} \frac{\rho(R',R)}{R'} \int_{b_i(v,R')}^{\infty} q_i(b)dbdR' = \gamma_i E[\frac{1}{R'}|R' \in \Gamma_i, R].$$

Proof. We consider Γ_i unbounded and follow Lemma A but this time write $B = b_i(v,1) = \sqrt{1+\hat{u}_i(1)}/\sqrt{v}$, then

$$b(v,R) = B\sqrt{\frac{R^2+\hat{u}_i(R)}{1+\hat{u}_i(1)}} = Bf_i(R).$$

Thus

$$\frac{f_i(R)}{R} = \sqrt{\frac{1+\hat{u}_i(R)/R^2}{1+\hat{u}_i(1)}}$$

Since $\hat{u}_i(R)/R^2$ remains bounded let the bound be M. The proof of Lemma A now gives

$$I_2(B) \leq \int_1^{\infty} \frac{\rho(R)}{R} \left\{ \int_0^{f(R)B} f(R)Bq(b)db \right\} dR \leq B \cdot E[f(R)/R] \leq B \cdot M.$$

We now prove inductively that $\hat{u}_i(R)/R^2$ remains bounded for each i. A technical assumption will be needed. At each stage of the induction we shall invoke Proposition 2; we begin with $i = N-1$ and then embelish the argument.

Proposition 3. Suppose that $\lim_{R\to\infty} E[\frac{R}{R'}|R] \neq 1/\gamma_{N-1}$ then $\sqrt{\hat{u}_{N-1}(R)}/R$ remains bounded.

Proof. We re-express the defining equation for $\hat{u} = \hat{u}_{N-1}(R)$ in terms of the characterization of $\bar{F}'_{N-1}(u, b_{N-2}, R_{N-2})$. We will drop the tildes on $\tilde{q}_i(g)$ in the course of this proof and so write for $R \in \Gamma_{N-2}$ (which set we assume unbounded) as follows:

$$\int_0^{1/\sqrt{\hat{u}}} bq(b)db + E[\frac{1}{\sqrt{\hat{u}}} \int_{1/\sqrt{\hat{u}}}^{R'/\sqrt{\hat{u}}} q(b)db + \frac{1}{R'} \int_{R'/\sqrt{\hat{u}}}^{\infty} bq(b)db|R] = \frac{1}{R}.$$

Suppose, if possible, that $\sqrt{\hat{u}(R)}/R \to +\infty$ for some sequence of values of R tending to infinity, so that $R/\sqrt{\hat{u}(R)} \to 0$ and $1/\sqrt{\hat{u}(R)} \to 0$. Cross-multiplying by R and noting that

$$\int_0^{1/\sqrt{\hat{u}}} bq(b)db \leq \frac{1}{\sqrt{\hat{u}(R)}} \text{ and } E[\int_{1/\sqrt{\hat{u}}}^{R'/\sqrt{\hat{u}}} q(b)db] \leq 1,$$

we deduce from Lemma A that

$$\lim_{R\to\infty}\gamma_{N-1}E[\frac{R}{R'}|R]=1$$

and this is manifestly a contradiction if $\lim_{R\to\infty}E_{N-1}[\frac{R}{R'}|R]\neq 1/\gamma_{N-1}$.

Remark 1. The technical condition will be repeated each time, so we note now that appropriate perturbations in the choice of the resale rate parameters may easily be made to satisfy the omission of the stated values. Thus to all intents and purposes the function $f_i(R)/R$ will always be $O(1)$.

Remark 2. Evidently we have $\hat{u}_\infty=\lim_{R\to\infty}\hat{u}(R)=\infty$. For, if $0<\hat{u}_\infty<\infty$ we shall have in the limit that

$$0<\int_0^{1/\sqrt{\hat{u}_\infty}}bq(b)db\leq\int_0^{1/\sqrt{\hat{u}_\infty}}bq(b)db+\frac{1}{\sqrt{\hat{u}_\infty}}E[\int_{1/\sqrt{\hat{u}_\infty}}^{R'/\sqrt{\hat{u}_\infty}}q(b)db|R]=0,$$

a contradiction (and the same idea applies if $\hat{u}_\infty=0$). This observation enables us to see why $\sqrt{\hat{u}(R)}/R$ has a positive limit when the resale rate follows an asymptotically proportional process. Write $y=R'/R$ and suppose for large R that the distribution of R' has density $p(R'/R)$. Define

$$f(x)=\int_0^\infty\{p(y)\int_0^{xy}\tilde{q}(g)dg\}dy;$$

then $f'(x)=\int_0^\infty yp(y)\tilde{q}(xy)dy>0$ and $f(0)=0$, $f(\infty)=1$. Write $B=1/\sqrt{\hat{u}(R)}$. Letting λ be the limit of BR as $R\to\infty$, we deduce from

$$BR\frac{\int_0^B bq(b)db}{B}+BR\cdot E[\int_B^{BR'}q(b)db]+\frac{1}{R'}\int_{BR'}^\infty bq(b)db|R]=1.$$

that

$$\lambda f(\lambda)=1-e^\alpha$$

where $e^\alpha=\lim_{R\to\infty}E[\frac{R}{R'}|R]$. Hence if $\alpha<0$ we may solve for the unique λ such that

$$f(\lambda)=\frac{1-e^\alpha}{\lambda},$$

to obtain

$$R/\sqrt{\hat{u}(R)}\to\lambda.$$

In the log-normal case we have $R'=1+(R-1)e^{\sigma w+m}$, so we compute, changing the order of integration that

$$\begin{aligned}\int_1^\infty dR' \int_B^{BR'} q(b)\rho(R',R)db &= \int_B^\infty db \int_1^{b/B} q(b)\rho(R',R)dR' \\ &= \int_B^\infty db \int_{-\infty}^{W(b/B)} q(b)\varphi(w)dw \\ &= \int_B^\infty \Phi\left(\frac{\ln b - \ln B - (\ln(R-1)+m_R)}{\sigma_R}\right) q(b)db \\ &= \int_{W(B)}^\infty \Phi\left(\frac{\sigma_b w + m_b - \ln B - (\ln(R-1)+m_R)}{\sigma_R}\right)\varphi(w)dw \\ &\sim \int_{\ln(BR)/\sigma_b}^\infty \varphi(w)dw = \Phi(-\ln(BR)/\sigma_b).\end{aligned}$$

Here $W(B) = (\ln B - m_b)/\sigma_b \to -\infty$. The approximation to the integrand on the right-hand side is obtained by noting that it is above 0.95 as soon as $\sigma_b w + m_b - \ln B - (\ln(R-1)+m_R) \geq 3\sigma_R\sigma_b$, i.e. approximately as soon as $w \geq \ln(BR)/\sigma_b$. We may thus obtain an approximation of sorts ot λ by replacing $f(\lambda)$ by $\Phi(-(\ln\lambda)/\sigma_b)$ and solving the equation

$$\Phi(-x) = (1-e^\alpha)e^{-\sigma_b x}.$$

Proposition 4. Suppose that $\lim_{R\to\infty} E_i[\frac{R}{R'}|R] \neq 1/\gamma_i$ where γ_i is the finite inflation rate constant $E[b_i|b_{i-1}]/b_{i-1}$ (for $i = 1, ..., N-1$); then the functions $\hat{u}_i(R)/R^2$ remain bounded for $i = 1, ..., N-1$.

Proof. Again we will drop the tildes on $\tilde{q}_i(b)$ in the course of this proof. We repeat the argument of the last proposition inductively. Begin as before for $R \in \Gamma_i$ (assumed unbounded) with the definition of $\hat{u}_i = \hat{u}_i(R)$ (we drop the subscript i here whenever convenient).

$$R\int_0^{b(\hat{u},1)} bq(b)db + R\cdot E_{i-1}[\int_{b(\hat{u},1)}^{b(\hat{u},R')} \bar{F}'_{i-1}(u(\hat{u}(R),b),b,R')q_{i-1}(b)db + \frac{1}{R'}\int_{b_{i-1}(\hat{u},R')}^\infty bq(b)db] = 1.$$

Again suppose that $\sqrt{\hat{u}(R)}/R \to +\infty$, then we have $\hat{u}(R) \to \infty$ so

$$b_{i-1}(\hat{u},1) = \sqrt{\frac{1+\hat{u}_{i-1}(1)}{\hat{u}_i(R)}} \to 0$$

for $R \to +\infty$ and for fixed R'

$$b_{i-1}(\hat{u},R') = \sqrt{\frac{R'^2+\hat{u}_{i-1}(R')}{\hat{u}_i(R)}} \to 0.$$

Now $\bar{F}'_{i-1}(w, 1, R')$ is decreasing in w so we have

$$\int_{b(\hat{u},1)}^{b(\hat{u},R')} \bar{F}'_{i-1}(b^2\tilde{u}(\hat{u}(R)b^2), 1, R')bq(b)db \leq \bar{F}'_{i-1}(\infty, 1, R') \int_0^1 bq(b)db.$$

We now have

$$\lim_{R\to\infty} \frac{R}{\sqrt{\hat{u}(R)}} \cdot E_{i-1}[\bar{F}'_{i-1}(\infty, 1, R')|R] = 0,$$

since $\bar{F}'_{i-1}(\infty, 1, R') \leq \gamma_{i-1}$, by (43), and also

$$R\int_0^{b_{i-1}(\hat{u},1)} bq_{i-1}(b)db \leq R \cdot b_{i-1}(\hat{u}, 1) = \frac{R}{\sqrt{\hat{u}_i(R)}}\sqrt{1 + \hat{u}_{i-1}(1)} \to 0.$$

We deduce from Proposition 2 taking $v = \sqrt{\hat{u}_i(R)} \to \infty$ (since by inductive hypothesis $\sqrt{\hat{u}_{i-1}(R)}/R$ remains bounded) that

$$\lim_{R\to\infty} \gamma_{i-1}E_{i-1}[\frac{1}{R'}|R] = 1,$$

contradicting our assumption. Hence after all $\sqrt{\hat{u}_i(R)}/R$ remains bounded.

Remark 3. Again we have $\hat{u}_\infty = \lim_{R\to\infty} \hat{u}(R) = \infty$, by the same argument as in Remark 2.

One step in the above proof is required elsewhere in section 4 so we isolate it here.

Lemma B . Provided $E[b_n] < \infty$ we have

$$\lim_{v\to\infty} \int_{b(v,1)}^{\infty} \bar{F}'_i(u(v, b_n), b_n)q_0(b_n)db_n = \bar{F}'_i(\infty, 1)E[b_n].$$

Proof. Dropping the subscript i we have for any constant K that

$$\begin{aligned} \lim_{v\to\infty} \int_{b(v,1)}^{K} \bar{F}'(u(v, b_n), b_n)q_0(b_n)db_n &= \lim_{v\to\infty} \int_{b(v,1)}^{K} \bar{F}'(b_n\tilde{u}(vb_n), 1)b_nq_0(b_n)db_n \\ &= \bar{F}'(\infty, 1) \int_0^K b_nq_0(b_n)db_n. \end{aligned}$$

Also, since $\bar{F}'' \leq 0$ we have $\bar{F}'(b_n\tilde{u}(vb_n), 1)b_n \leq \bar{F}'(0, 1)b_n$ so

$$\int_K^\infty \bar{F}'(u(v, b_n), b_n)q_0(b_n)db_n \leq \bar{F}'(0, 1) \int_K^\infty b_nq_0(b_n)db_n$$

9 Appendix 3: Derivation of the endogenous rate

We show that in the two period model when the production function is Cobb-Douglas of form x^θ/θ the endogenous depreciation rate is

$$\delta_0^*(\theta) = (1 + \widetilde{g}^{1/(\theta-1)})^{-1},$$

where $\widetilde{g}$ is the solution to equation (15), i.e.

$$1 = \Psi(\widetilde{g}, 1)$$

and depends only on μ_b, σ_b. We note that

$$\delta'(\theta) = \frac{\widetilde{g}^{1/(\theta-1)}(1 + \widetilde{g}^{1/(\theta-1)})^{-2} \ln \widetilde{g}}{(\theta-1)^2},$$

is increasing.

Indeed, since the maximisation of

$$\frac{x^\theta}{\theta} - bx$$

occurs with $x = b^{1/(\theta-1)}$ and has maximal value $\frac{1-\theta}{\theta} b^{\theta/(1-\theta)}$, it is easy to see that the same addition to investment rules apply so that in the two-period case we have

$$F(u, 1) = \int_0^{u^{\theta-1}} \left(\frac{1-\theta}{\theta} b^{\theta/(1-\theta)} + bu \right) q(b) db + \frac{u^\theta}{\theta} \int_{u^{\theta-1}}^{\infty} q(b) db.$$

Hence putting $B = u^{\theta-1}$ it is routine to verify that

$$F'(u, 1) = \int_0^{u^{\theta-1}} bq(b) db + u^{\theta-1} \int_{u^{\theta-1}}^{\infty} q(b) db = \Psi(B, 1).$$

Thus the equation $F'(u, 1) = 1$ has solution $u = \widetilde{g}^{1/(\theta-1)}$, where $\Psi(\widetilde{g}, 1) = 1$, as asserted.

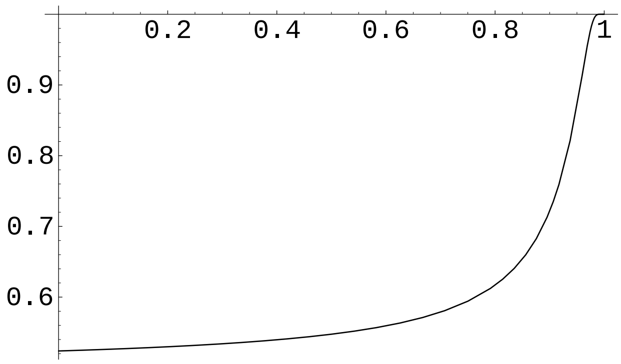

Figure 13: Graph of the endogeneous depreciation rate $\delta_0^*(\theta)$ against Cobb-Douglas index θ