Statics and asymptotics of a price control limit: an optimal timing inventory problem

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Abstract

Suppose that the price of a specialised input to be used in a future production enterprise will in expectation rise, even though it might possibly fall. Given the opportunity to make an advance purchase at a fixed price what rule should be followed in selecting the amount if one is to profit from the possible price fall? The optimality condition associated with hedging against the price rise when the input is non-resellable (in its 'raw' pre-production state) requires selection of a 'censor' (or cap) X defined by the 'censor equation', namely $\int_0^X bq(b)db + \int_X^\infty Xq(b)db = 1$, where $q(b)$ is the probability density function for the future price of the input. If, in the time intervening between contracting the advance purchase and the initiation of production when further inputs may next be purchased, the price b_t of the input follows a geometric Brownian motion, then the distribution for b_t at the time of the next purchase is log-normal and so the quest for a censor transforms to finding the solution W of the equation

$$e^{-\mu} = \Phi(W - \sigma) + e^{\sigma W - \frac{1}{2}\sigma^2}\Phi(-W),$$

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where Φ denotes the standard cumulative normal distribution function, μ is the drift and σ^2 the variance per unit time of the geometric brownian motion. We study the comparative statics of the censor and show that $W = W(\mu,\sigma)$ is increasing with variance and decreasing with drift; we also study the monotonicity of $\bar{W}(t) = W(\bar{\mu}t,\bar{\sigma}\sqrt{t})$ and derive a number of asymptotic formulas for fixed μ and for σ small or large, e.g. $W(\mu,\sigma) = -\frac{\mu}{\sigma} + \frac{1}{2}\sigma + o(\sigma)$ as $\sigma \to 0+$ and $W(\mu,\sigma) = \sigma - \hat{\mu} - \frac{1}{\sigma - \hat{\mu}}\{1 + O(1)\}$ as $\sigma \to \infty$, where $\hat{\mu} = \Phi^{-1}(1 - e^{-\mu})$. These formulas are used to derive the dependence of the expected profit on the waiting period, drift and volatility.

Key words: asymptotic behaviour

1. Introduction: motivation, assumptions and key issues

We begin by describing two inter-related inventory optimization problems, both of which identify a critical price level which we call a 'censor' (to be derived in section 2). We then summarize our assumptions, spell out the key issue: to provide a description of the behaviour of the censor (achieved in part by asymptotic analysis), and finally outline the structure of the paper.

Our two inventory problems both have an embedded commitment to receive a fixed amount u of inventory (inputs for a production process), and an option to expand inventory by z at a later date. The expected price of the input at future dates rises over and above the interest rate, but the realised price may rise or fall; thus to determine the optimal level of the forward commitment component of inventory requires valuation of the other component: the option to expand by z.

The first of the two problems is a basic model of production. It is a simplified production process which consumes just one input at a known future date and so the model provides a 'one-shot' paradigm from which to build repetitions at consecutive dates by allowing at each date: an expansion of inventory (through an additional purchase), a consumption of some part of the inventory in the process of production, with the remainder being carried forward to the next date. This multi-period analysis has been done in Gietzmann and Ostaszewski (2004), where additional features allow for costly liquidation of inventory, but the the purpose there was to study in abstract the valuation of a firm in terms of accounting data. The discrete time model scenario is based on a real-options approach and is very close in spirit to the recent paper of Eberley and Van Mieghem (1997). By contrast with them, however, it is costs (rather like their returns) that evolve stochastically, and, by transposition, returns here are deterministic. Our returns are of the Cobb-Douglas form, and it is this that enables a tractable determination of the corresponding 'optimal ISD control limit' (acronym for Invest/Stay put/Disinvest) of Eberley and Van Mieghem. This control limit is referred to herein as a 'censor'; its value directly determines the optimal choice of u. However, we depreciate inputs over time entirely through consumption (in keeping with the inventory setting), rather than through an exogenous fixed rate no matter what the production, as in Eberley and Van Mieghem; in light of this, the 'optimal ISD control limit' of the current paper, though similar, is indeed different, and needs to be rederived ab initio.

The alternative model which we consider here is a more realistic re-formulation of the basic one. In our second model the production process consumes inputs continuously over a fixed time interval from an initial inventory and we allow a single additional purchase of inventory at some preselected date within the interval of activity, that date having been optimally selected at the beginning of the interval. The expected profit function of the basic model is a key ingredient of this optimal timing problem.

Apart from offering a real-options aproach, contrasting to the classical inventory literature (see for instance Bensousssan et al (1983), or Scarf (1960)), an additional contribution of the current paper is to provide information about the sensitivity of the 'control limit' (comparaitive statics and asymptotics) in regard to model parameters, an issue which is not considered in Eberley and Van Mieghem (1997).

1.1. The basic optimisation problem.

Today (at time $t = 0$) a manager knows that tomorrow (at time $t = 1$) he will be running, for once only, a production process; the revenue he will generate is given by a (non-linear) production function of the quantity input x and we assume that the quantity x is consumed in its entirety in the process. The input level x remains to be chosen at time $t = 1$, but, since the raw material may be bought in arbitrary amounts both at time $t = 0$ and at time $t = 1$, the manager may buy a quantity u today at today's price and retain the option to purchase an additional quantity $z \geq 0$ at date $t = 1$; no resale of inputs is allowed, so $x = u + z$. The manager's profit is the revenue less expenditure on inputs.

The present value of the input price at time $t = 1$ is in expectation higher than that at time $t = 0$, so given the opportunity to purchase a quantity u at today's price, what rule should be followed in selecting the amount u, if the manager is to take advantage from a possible price fall?

Assuming that the manager wishes to maximise expected profit, we wish to solve the optimisation problem: maximise for the two variables $z, u \geq 0$ the expression

$$\int_0^\infty \{f(z+u) - bz\}q(b)db - u, \tag{1.1}$$

where the function $f(x)$ models the revenue obtained from a quantity x of a raw material committed as input into a production process. We have in mind a Cobb-Douglas function like $f(x) = 2\sqrt{x}$, but the more general situation would have $f(x)$ a twice differentiable, increasing, strictly concave function. The variable b

denotes the price at time $t = 1$ (with the time $t = 0$ price taken as unity) and $q(b)$ denotes the observed probability density of the price b.

This is a classical inventory problem but amended by the inclusion of an option to expand the inventory and, in effect, by the inclusion of an optimally selected 'forward' contract (the choice of u). We evaluate the embedded option in a framework reminiscent of the Black-Scholes approach to valuing options: we let the price b have exponential growth and assume b has a log-normal distribution. The 'forward contract' is construed here as a contract signed at the earlier date $t = 0$ with an agreed specified delivered quantity, namely u, a specified delivery date $t = 1$, and a price standardized here to *unity* per unit delivered. The latter standardization fixes the unit of money, since, in the absence of arbitrage and storage costs, as is well-known, the forward price equals the price of inputs at the initial time of contracting, but compounded up to term-value at the required rate of interest. This forward contract leaves open a possible purchase of an additional quantity z to be made at the time of production at the 'spot' price (the price then prevailing). We identify in section 2 an optimality condition to be satisified by u.

1.2. Summary of assumptions

We emphasize here a number of points already raised. (i) The later 'spot' price is a random variable whose value b may be either below or above the forward price, even though the price *is expected to rise* (relative to its earlier value). (ii) The revenue is $f(z + u)$ and is generated from the initial stock u supplemented (if at all) by an amount z (acquired when the spot price is revealed to be b). (iii) It is assumed that the optimising agent is risk-neutral. (iv) The advance purchase of u has nil resale value on delivery. This makes the delivered asset a 'non-tradable', so that a martingale valuation approach is not necessarily appropriate; hence our analysis is of a dynamical programming character as in Eberley and Van Mieghem (1997).

We do note, however, that the optimality condition shown below in (1.2) may be re-interpreted as saying that q is a martingale measure for a problem in which only two future discounted prices arise: a price $\tilde{b}$ and a q-average price for the range $[0, \tilde{b}]$. The optimality condition equation then asserts that the opening price is indeed the expected future price.

1.3. Key issues and Structure of the paper

The structure of the paper is as follows. In the following section we derive a natural optimality condition associated with our optimisation problem. This refers to the solution for $\tilde{b}$ of the equation

$$\int_0^{\tilde{b}} bq(b)db + \tilde{b}\int_{\tilde{b}}^{\infty} q(b)db = 1, \tag{1.2}$$

where $q(b) = q(b, \mu, \sigma)$ is a two-parameter density function such that $E[b] < \infty$. In view of its character and central importance we will call the solution of (1.2) the **censor** $\tilde{b} = \tilde{b}(\mu, \sigma)$. Evidently

$$E[b \wedge \tilde{b}] = 1,$$

and so $\tilde{b} > 1$. This control limit, though similar to that studied by Eberley and Van Mieghem (1997) is distinct in respect of the treatment of capital depreciation, and thus follows the definition introduced in Gietzmann and Ostaszewski (1999) (in the course of studying profit in an optimally hedged future production, albeit there the future price is binary, i.e. may take one of two values). We review in the next section just enough of this idea as is required for the current application.

The more realistic version of our inventory problem is formulated in section 3 and we describe its solution in that section by drawing on the qualitative behaviour of the expected optimal profit function; that, in turn, is determined by the dependence of the censor $\tilde{b}(\mu, \sigma)$ on the two parameters μ and σ, where these describe the drift and volatility of an assumed geometric brownian motion model for the price b. The focus of the paper is indeed the censor since it identifies the optimal level for u , via equation (3.4), as a decreasing function of the censor. Definitions and principal findings are reported in 4; this includes comparative statics of the censor and asymptotics of the expected profit function. Technical calculations are then relegated to later section: censor statics are derived in section 5 and asymptotic analyses are carried out in sections 6 and 7.The final section 8 is dedicated to the consequences for the expected profit function.

2. Optimal forward purchase condition and the optimality of waiting

In this section we prove that the optimal level of advance purchase u for (1.1) satisfies

$$f'(u) = \tilde{b}, \tag{2.1}$$

where $\tilde{b}$ is the solution to (1.2) and also show that hedging offers greater value to the manager.

For given u the quantity $z = z(b)$ which maximizes $f(u+z) - bz$ when b is known, is either zero, or satisfies the first-order condition

$$f'(z+u) = b.$$

We now show why (2.1) follows from this condition. Begin by letting u be arbitrary; we put $\tilde{b} = f'(u)$. Let $G(b)$ denote the inverse function to f', so that G is a decreasing function of b. We have $z(b) = 0$, unless $b \leq \tilde{b}$, i.e. unless $G(b) \geq u = G(\tilde{b})$, in which case $z = G(b) - G(\tilde{b})$. The expected profit, given u has been purchased at unity, is thus

$$\begin{aligned}\Pi(u) &= \int_0^{\tilde{b}} \{f(G(b)) - b[G(b) - u]\}q(b)db + f(u)\int_{\tilde{b}}^{\infty} q(b)db - u \\ &= \int_0^{\tilde{b}} \{f(G(b)) - bG(b)\}q(b)db + f(u)\int_{\tilde{b}}^{\infty} q(b)db - u\{1 - \int_0^{\tilde{b}} bq(b)db\}.\end{aligned}$$

Differentiating $\Pi(u)$ according to the Leibniz rule we obtain, after some cancellations, the equation (1.2) and hence as $b = f'(u)$, also (2.1). Finally, with u selected optimally, we see that the expected optimal profit as a function of the parameters μ, σ is

$$g(\mu,\sigma) = \int_0^{\tilde{b}} \{f(G(b)) - bG(b)\}q(b)db + \{f(u) - u\tilde{b}\}\int_{\tilde{b}}^{\infty} q(b)db. \qquad (2.2)$$

Proposition (Value of Waiting)

The expected profit obtained by hedging optimally is no worse than the optimal profit obtained using only purchases at initial prices, namely

$$f(G(1)) - G(1) < g(\mu,\sigma).$$

Proof. We note that for $b > 0$ the function $h(b) = \max_{x>0}[f(x) - bx] = f(G(b)) - bG(b)$, the Fenchel conjugate of f, represents optimal profit, when input is acquired at a price of b; the dual is strictly convex in b (see Rockafellar (1970), or note that $h''(b) = -1/f''(G(b)))$ and $h(\tilde{b}) = f(u) - \tilde{b}u$. By Jensen's inequality for convex functions the expression for $g(\mu,\sigma)$ is greater than

$$h(\int_0^{\tilde{b}} bq(b)db + \tilde{b}\int_{\tilde{b}}^{\infty} q(b)db) = h(1).$$

3. Continuous production: optimal timing of one intermediate re-stocking

Here we introduce continuous time into consideration. We quote values in discounted terms, i.e. present-value terms relative to time $t = 0$. (We side-step a discussion of the relevant discount factor. In brief, discounting would be done relative to the required rate of return on capital given the risk-class of the investment project; see Dixit and Pindyck (1994).)

3.1. Problem formulation

Suppose that at time 0 the price of inputs is $b_0 = 1$, and as time t progresses the present-value of the spot-price, b_t, follows the stochastic differential equation

$$\frac{db_t}{b_t} = \bar{\mu}dt + \bar{\sigma}dv_t, \tag{3.1}$$

where v_t is a standard Wiener process. Expressing the geometric Brownian process b_t explicitly:

$$b_t = \exp \bar{\mu}t \cdot \exp\{\bar{\sigma}v_t - \frac{1}{2}\bar{\sigma}^2 t\},$$

and assuming the constant growth rate $\bar{\mu}$ is positive, the expected (present-value/discounted) price at time t is $e^{\bar{\mu}t}$ and exceeds the initial price of unity. The price b_t is log-normally distributed with mean which we denote by $m = (\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)t$ and variance $\sigma^2 = \bar{\sigma}^2 t$. Thus $q(b) = q(b, \bar{\mu}t, \bar{\sigma}\sqrt{t})$ denotes the density that b_t takes the value b at the time t.

We assume that we are only permitted to acquire raw materials initially and at one other intermediate time $\theta \leq 1$, where θ is *selected optimally in advance* (and only b_θ will be observed). The inputs are utilized in a continuous production process running over $[0, 1]$ and the production creates instantaneous revenue at time t equal to $f(x_t)$ (again quoted in present-value terms), where we take the input rate instantaneously to be x_t. The revenue from any interval $[a, b]$ is

$$\int_a^b f(x_t)dt.$$

Thus if we use up $x\theta$ in the period $[0, \theta]$, then, with θ fixed, a constant instantaneous input rate x is optimal and we create revenue $f(x)\theta$ at a cost of $x\theta$ (assuming this stock was acquired at the initial price of unity).

We intend to acquire an optimal input $\hat{x}\theta$ for production in the interval [0, θ] and an optimal hedge $u \cdot (1-\theta)$ for the remaining time interval in which the revenue will be

$$\int_{\theta}^{1} f(x_t)dt = (1-\theta)f(u+z)$$

where $z \cdot (1-\theta) \geq 0$ denotes any additional purchase of inventory, again the optimal input rate is constant. Thus conditional on the initial choice of θ the expected future revenue is

$$\begin{aligned} & f(x)\theta + \int_0^\infty \{(1-\theta)f(z+u) - bz(1-\theta)\}q(b)db - (x\theta + (1-\theta))u \\ = \quad & (f(x)-x)\theta + (1-\theta)(\int_0^\infty \{f(z+u) - bz\}q(b)db - u). \end{aligned}$$

and is maximised in view of equation (2.1) by taking

$$u = G(\tilde{b}(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta})). \tag{3.2}$$

By equation (2.2) our optimal total expected revenue will thus be

$$R(\theta, \bar{\mu}, \bar{\sigma}) = \theta\{f(\hat{x}) - \hat{x}\} + (1-\theta)g(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta}).$$

Our optimiziation problem is to maximize over $\hat{x}$ and θ.

Evidently $\hat{x} = \arg\max_x[f(x) - x] = G(1)$. Now $h(1) = \max_x f(x) - x$, where h is the dual of f introduced at the end of the last section, and so since $g(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta}) \geq h(1)$ we have

$$h(1) < R(\theta, \bar{\mu}, \bar{\sigma}).$$

We are left with the problem of selecting a time $\theta \leq 1$ so as to maximise the quantity $R(\theta, \bar{\mu}, \bar{\sigma})$ with respect to θ.

To be specific we consider the optimisation problem when f is a Cobb-Douglas production function, i.e. for some $0 < \eta < 1$:

$$f(x) = \frac{x^\eta}{\eta}.$$

Thus $G(b) = f'^{-1}(b) = b^{1/(\eta-1)}$, and the Fenchel dual is

$$h(b) = (\frac{1-\eta}{\eta})b^{\eta/(\eta-1)} = \frac{b^{-\gamma}}{\gamma},$$

where $\gamma = \eta/(1-\eta)$. To simplify the presentation we take

$$f(x) = 2\sqrt{x},$$

and then $G(b) = 1/b^2$ and $h(b) = 1/b$. Thus $\bar{g}(0) = h(1) = 1$ and we need to optimise

$$R(\theta, \bar{\mu}, \bar{\sigma}) = \theta + (1-\theta) g(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta}),$$

so that R is a convex combination of unity and $g(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta})$; the form of the function g is calculated in section 4.1.

Remark. There is no real loss of generality in this switch to $2\sqrt{x}$, because in the presence of a log-normally distributed price the choice of any other value for γ is equivalent to a rescaling of $\bar{\mu}, \bar{\sigma}$ (since any power of a geometric Brownian motion is again a geometric Brownian motion, and the effect of the power is to transform the drift and variance coefficients.)

This rescaling feature leads us to a further consideration. Although it seems natural in the exposition above to let $t = 1$ be the date of termination of production, it is nevertheless advantageous to allow the termination date to be large and so to allow the variable t to be large at the expense of the parameters $\bar{\mu}, \bar{\sigma}$ now restricted to a small bounded set. This motivates our interest in an asymptotic analysis of the function

$$\bar{g}(t) = g(\bar{\mu}t, \bar{\sigma}\sqrt{t}).$$

Returning to our optimisation problem and assuming that the time θ may be selected freely in $[0, 1]$ (there being no associated management costs in choosing θ), the optimal choice of θ, assuming such exists, is evidently given by the first-order condition

$$\frac{\bar{g}(\theta) - \bar{g}(0)}{\bar{g}'(\theta)} = 1 - \theta, \tag{3.3}$$

with $\bar{g}(0) = 1$.

Observation. *The first-order condition is satisfied for some θ with $0 < \theta < 1$. The smallest solution is a local maximum of R. If $\bar{g}$ is concave then the solution is unique.*

Proof. In general, by the Proposition on the Value of Waiting, $\bar{g}(1) - \bar{g}(0) > 0$ and the first assertion is obvious since the right-hand side is zero at $\theta = 1$ and is positive at $\theta = 0$; moreover, by Theorem B below, the left-hand side has limiting value zero as $\theta \to 0+$. If, however, $\bar{g}(1) - \bar{g}(0) = 0$, then since the function $\bar{g}$ is initially increasing for $\theta > 0$, it has an internal local maximum at $\bar{\theta}$ for some $\bar{\theta}$

with $0 < \bar{\theta} < 1$. In this case the first-order condition is satisfied by some $\theta < \bar{\theta}$, since the left-hand side tends to $+\infty$ as $\theta \to \bar{\theta}$.

Any solution θ^* to the equation has $\bar{g}'(\theta^*) > 0$ and so the second assertion follows since $R'(\theta^*-) > 0$ and $R'(\theta^*+) < 0$.Observe that if $\bar{g}''(\theta) < 0$ we have

$$\frac{d}{d\theta}\left(\frac{\bar{g}(\theta) - \bar{g}(0)}{\bar{g}'(\theta)}\right) = 1 - \bar{g}''(\theta)\frac{\bar{g}(\theta) - \bar{g}(0)}{[\bar{g}'(\theta)]^2} > 0,$$

so the third assertion is clear since concavity ensures that the left-hand side is an increasing function of θ.

3.2. Description of the optimal solution and of a reduced problem

Thus far we have shown in (3.2) that the initial inventory is

$$\hat{x}\theta + u(1 - \theta)$$

where

$$\hat{x} = G(1), \qquad u = G(\tilde{b}(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta})) \tag{3.4}$$

and θ solves (3.3). Here G, the inverse of f', is a decreasing function of b.

One would wish to improve on the observation in the last paragraph to show in more general circumstances that (3.3) has a unique solution and to study dependence on the two parameters of the problem. This appears analytically intractable. For the purposes of gaining an insight we propose therefore to replace $\bar{g}(t)$ by a function related to it through asymptotic analysis (as t varies) on the grounds that from numeric observation the subsitute is qualitatively similar. The asymptotic analysis is summarised in the Theorems given in Section 3 and we are led to the considerably simpler problem obtained by making one of two substitutions for $\bar{g}(\theta)$, namely

$$1 + Ate^{-\alpha t}, \text{ or } \qquad e^{\alpha t},$$

according as $\bar{\sigma}^2 - \bar{\mu}$ is negative or positive and where $\alpha = |\bar{\sigma}^2 - \bar{\mu}| > 0$. In the first of the two situations this fits qualitatively with numeric observation of the *form* of $\bar{g}$; in the second situation it agrees with the general form observed and also of the asymptotic form as $t \to \infty$ (justified as earlier) derived in section 6.4.

3.2.1. Case (i): $\alpha = \bar{\mu} - \bar{\sigma}^2 > 0$

In this case the optimum time θ is the solution of

$$\frac{\theta}{1-\alpha\theta} = 1-\theta,$$

a quadratic relation. The left-hand side expression is increasing in θ for $0 < \theta < 1/\alpha$ since the derivative is $1/\ (1-\alpha\theta)^2$ and a unique solution for $0 < \theta < 1$ is guaranteed. Here we have explicitly

$$\theta = \theta(\alpha) = \frac{1}{2} - \frac{1}{\alpha}\left(-1 + \sqrt{1+\frac{\alpha^2}{4}}\right),$$

so that as α increases from zero the optimal time θ moves towards the origin (i.e. low volatilities bring the replenishment timing back). On the other hand, we have

3.2.2. Case (ii) $\alpha = \bar{\sigma}^2 - \bar{\mu} > 0$

The first-order condition reduces to

$$(1-e^{-\alpha\theta})/\alpha = 1-\theta,$$

with a unique solution in the unit interval. Here we can use a quadratic approximation for the exponential term and solve for θ to obtain the approximation

$$\theta(\alpha) = \frac{1}{1+\sqrt{1-\alpha/2}},$$

so that the optimal choice of θ is close to the midpoint $\theta = 1/2$, when α is small, but recedes, as α increases, towards unity (as a direct computation shows), i.e. high volatilities bring the replenishment position forward (meaning that waiting longer beyond the mid-term is optimal for higher volatilities).

4. Principal Findings

We begin by defining the normal censor and the corresponding expected profit function, as these are needed to state our results.

4.1. Definition of the normal censor

For the geometric brownian model adopted in (3.1) the censor equation (1.2) defining $\tilde{b} = \tilde{b}(\mu, \sigma)$ can be re-written thus:

$$1 = e^{\mu}\Phi(W-\sigma) + \tilde{b}\Phi(-W),$$

where $\Phi(x)$ denotes the standard normal cumulative distribution function and

$$W = w(\tilde{b}) = \frac{\ln \tilde{b} - m}{\sigma} \text{ and } m = \mu - \frac{1}{2}\sigma^2.$$

This formulation thus leads naturaly to the following.

Definition. The **normal censor** is the function $W(\mu, \sigma)$ defined implicitly by the equation

$$e^{-\mu} = \Phi(W - \sigma) + e^{\sigma W - \frac{1}{2}\sigma^2}\Phi(-W). \tag{4.1}$$

We note that W is well defined since

$$\begin{aligned} F(W, \sigma) &= \Phi(W - \sigma) + e^{\sigma W - \frac{1}{2}\sigma^2}\Phi(-W) \\ &= \int_{-W+\sigma}^{\infty} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} + e^{\sigma W - \frac{1}{2}\sigma^2} \int_{W}^{\infty} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} \end{aligned}$$

is an increasing function of W.

Remark on notation. Employing the standard notation d_+ and d_- of financial theory, see for example Musiela- Rutkowski (1997), where:

$$d_\pm = d_\pm(r, \sigma, b_0, b) = [\ln(b_0/b) + (r \pm 0\frac{1}{2}\sigma^2)]/\sigma.$$

we have evidently,

$$W(\mu, \sigma) = w(\tilde{b}) = -[\ln(1/\tilde{b}) + (\mu - \frac{1}{2}\sigma^2)]/\sigma = -d_-(\mu, \sigma, 1, \tilde{b}).$$

Since $\tilde{b} > 1$ and large σ are commonly experienced, W is liable to be positive. We thus prefer W over d_-.

4.2. The expected profit formula

Assuming for the the revenue function, as earlier, the form $f(x) = 2\sqrt{x}$ with dual $h(b) = 1/b$, substitution into (2.2) yields the following formula for g:

$$g = e^{(\sigma^2 - \mu)}\Phi(W + \sigma) + \frac{1}{\tilde{b}}\Phi(-W).$$

Since

$$\tilde{b} = \exp(\sigma W + \mu - \frac{1}{2}\sigma^2) \tag{4.2}$$

the expected profit may also be expressed in the alternative form

$$g = e^{(\sigma^2-\mu)}\Phi(W+\sigma) + e^{-\mu-\sigma W+\frac{1}{2}\sigma^2}\Phi(-W). \tag{4.3}$$

To study this function we are led to characterise the behaviour of both $W(\mu, \sigma)$ and $\bar{W}(t) =_{def} W(\bar{\mu}t, \bar{\sigma}\sqrt{t})$.(Evidently $d_-(\mu, \sigma)$ and $\bar{d}_-(t) = d_-(\bar{\mu}t, \bar{\sigma}\sqrt{t})$ are their negatives).

4.3. Principal Results

We shall prove the following **theorems on censor statics**

$$\frac{\partial \tilde{b}}{\partial \mu} < 0, \frac{\partial \tilde{b}}{\partial \sigma} > 0,$$

so it is not surprising that in general for $\bar{\mu}, \bar{\sigma}$ constant

$$\bar{b}(\theta) = \tilde{b}(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta})$$

increases from *unity* to a maximum and then decreases, that is, it is **unimodal**; for $\bar{\sigma} > \sqrt{2\bar{\mu}}$ it will be merely monotonically increasing - (i.e. the maximum occurs at infinity).

We also have

$$\frac{\partial W}{\partial \mu} < 0, \frac{\partial W}{\partial \sigma} > 0,$$

so one may expect similarly that

$$\bar{W}(\theta) = W(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta})$$

to be **either monotonic or unimodal** (i.e. with one change in sign of the derivative); we do indeed observe this numerically and can derive asymptotic formulas as $\theta \to \infty$ of form $\pm\sqrt{\theta}$, but whilst we are able to find one stationary point, we are unable to prove there is at most one.

In view of the occurrence of both W and $W - \sigma$ in the censor equation we show that

$$\frac{\partial}{\partial \sigma}\{W - \sigma\} > 0$$

and

$$\frac{\partial}{\partial \sigma}\{\sigma W - \frac{1}{2}\sigma^2\} > 0.$$

For the purpose of understanding the expected revenue we derive the following **asymptotic formulas** for W. For fixed μ we show that

$$W = -\frac{\mu}{\sigma} + \frac{1}{2}\sigma + o(\sigma), \text{ as } \sigma \to 0+$$

and

$$W(\mu, \sigma) = \sigma - \hat{\mu} - \frac{1}{\sigma - \hat{\mu}}\{1 + O(1)\}, \text{ as } \sigma \to \infty,$$

where $\hat{\mu}$ satisfies

$$e^{-\mu} = \int_{\hat{\mu}}^{\infty} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}}$$

i.e. $\hat{\mu} = -\Phi^{-1}(e^{-\mu})$, or $\mu = -\ln \Phi(-\hat{\mu})$. Both asymptotic formulas are required to analyze the behaviour of the censor. An immediate corollary of the second formula is that for fixed μ

$$\tilde{b} = \tilde{b}(\mu, \sigma) \to \infty \text{ as } \sigma \to \infty,$$

implying that the hedge quantity tends to zero; the convergence is quite rapid since we have approximately that

$$\tilde{b}(\mu, \sigma) = \exp(\frac{1}{2}\sigma^2 - \sigma m + \mu - 1).$$

We will show that as $\theta \to 0+$ we have

$$\bar{W}(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta}) \to +\infty$$

and in fact $\sqrt{\theta}\bar{W} \to 0$ as $\theta \to 0+$.

We further show that as $\theta \to \infty$ there are two scenarios. If $2\bar{\mu} > \bar{\sigma}^2$ we have the asymptotic formula as $\theta \to \infty$ that

$$\bar{W}(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta}) = -\frac{2\bar{\mu} - \bar{\sigma}^2}{2\bar{\sigma}}\sqrt{\theta} + o(\sqrt{\theta}). \tag{4.4}$$

On the other hand if $\bar{\sigma}^2 > 2\bar{\mu}$ we have the asymptotic formula as $\theta \to \infty$ that

$$\bar{W}(\bar{\mu}\theta, \bar{\sigma}\sqrt{\theta}) = (\bar{\sigma} - \sqrt{2\bar{\mu}})\sqrt{\theta} + o(\sqrt{\theta}). \tag{4.5}$$

It is interesting to note that when $\bar{\sigma}^2 > 2\bar{\mu}$ we have

$$\bar{\sigma}\bar{W} + (\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)\theta = (\frac{1}{2}\bar{\sigma}^2 + \bar{\mu} - \bar{\sigma}\sqrt{2\bar{\mu}})\theta + o(\theta)$$

so

$$\bar{b}(\theta) \to \infty \text{ as } \theta \to \infty,$$

which shows that in this case $\bar{b}(\theta)$ is monotone rather than unimodal. (Note that the function $\frac{1}{2}\bar{\sigma}^2 + \bar{\mu} - \bar{\sigma}\sqrt{2\bar{\mu}}$ is increasing in $\bar{\sigma}$ from zero for $\bar{\sigma} \geq \sqrt{2\bar{\mu}}$.)

Remark. The formulas suggest the two case: $\bar{W}$ decreasing with θ when $\bar{\sigma}^2 < 2\bar{\mu}$ (from $+\infty$ to -∞); $\bar{W}$ initially decreasing with θ and then increasing to $+\infty$ when $\bar{\sigma}^2 > 2\bar{\mu}$. We are able to support the unimodality or monotonicity (existence of at most one stationary point) with numeric evidence by reducing the problem to some natural conjectures (see section 5.2).

The formulas above enables us to derive the following.

Theorem A (Asymptotic behaviour of the profit $g(\mu, \sigma)$**).**
(i) $g = e^{\sigma^2 - \mu}$ as $\sigma \to \infty$;
(ii) $g = e^{-\mu} + (1 - e^{-\mu})\Phi(\mu/\sigma)$ as $\sigma \to 0+$.

Theorem B (Behaviour of the profit $\bar{g}(\theta)$ **at the origin)**
We have $\bar{g}'(0) = \bar{\sigma}^2$ so that

$$\bar{g}(\theta) = 1 + \bar{\sigma}^2\theta + o(\theta).$$

Theorem C (Asymptotic behaviour of the profit $\bar{g}(\theta)$ **at infinity)**
(i) If $\bar{\sigma}^2 \leq \bar{\mu}$ we have as $\theta \to \infty$

$$\bar{g}(\theta) = 1 + o(1/\sqrt{\theta}) \to 1+,$$

and so $\bar{g}(\theta)$ has a maximum.

(ii) If $\bar{\mu} < \bar{\sigma}^2 < 2\bar{\mu}$ we have as $\theta \to \infty$

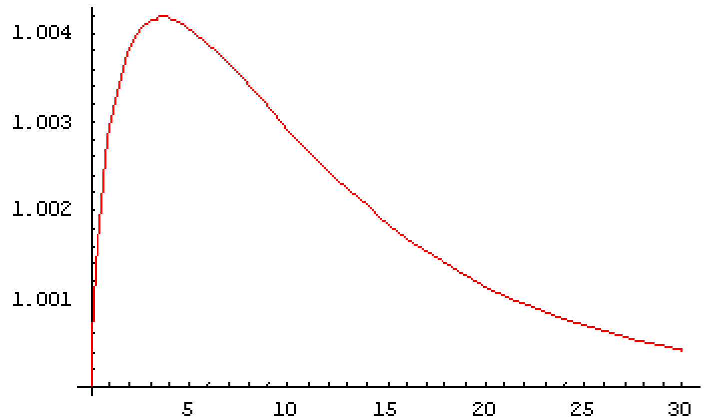
$$\bar{g}(\theta) = 1 + e^{(\bar{\sigma}^2 - \bar{\mu})\theta} + o(1/\sqrt{\theta}).$$

(iii) If $2\bar{\mu} < \bar{\sigma}^2$ we have as $\theta \to \infty$

$$\bar{g}(\theta) = e^{(\bar{\sigma}^2 - \bar{\mu})\theta} + o(1/\sqrt{\theta}).$$

(iv) If $\bar{\sigma}^2 = 2\bar{\mu}$ we have as $\theta \to \infty$

$$\bar{g}(\theta) = \frac{1}{4} + e^{\bar{\mu}\theta} + o(1/\sqrt{\theta}).$$



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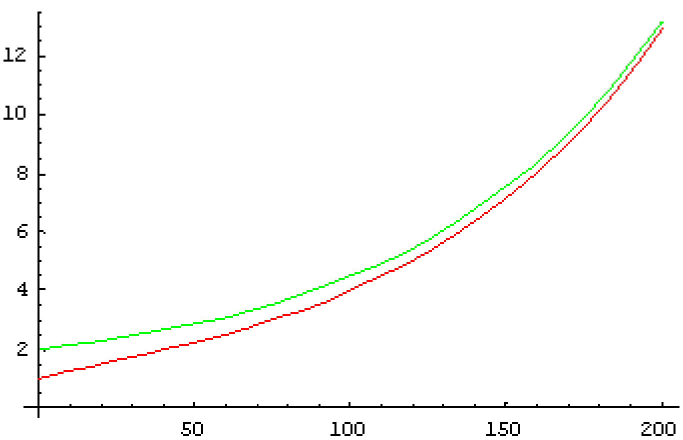

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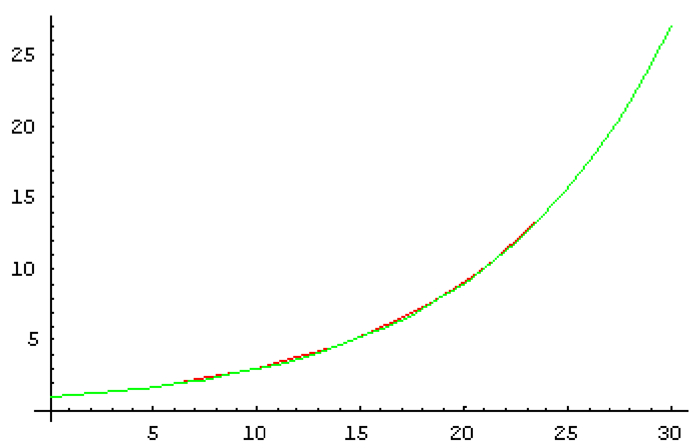

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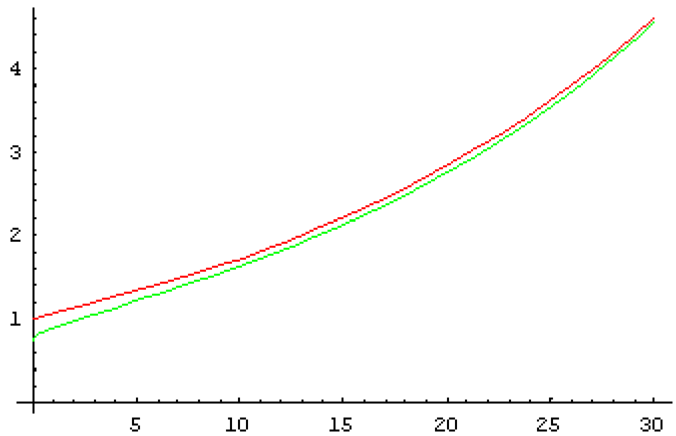

These cases are illustrated by computed graphs of $\bar{g}$ shown in Figure-1.

Place Figure 1 here Asymptotics when $\mu = 0.05\theta$; numerics for the four respective cases: (i) $\bar{\sigma}^2 < \bar{\mu}$; (ii) $\bar{\mu} < \bar{\sigma}^2 < 2\bar{\mu}$, (iii) $2\bar{\mu} < \bar{\sigma}^2$ (iv) $\bar{\sigma}^2 = 2\bar{\mu}$.(Feint graph is the approximation.)

Remarks. To see the formulas in context observe that if the production is run myopically over the unit interval and we wait a period $\theta < 1$ before replenishing, the expected profit is

$$\bar{g}(\theta) = E_0[1/b_\theta],$$

since $h(b) = 1/b$, and so as in (2.2)

$$E_0[1/b_\theta] = \int_0^\infty \frac{q_\theta(b)db}{b} = e^{(\bar{\sigma}^2 - \bar{\mu})\theta}.$$

5. Statics of the function $W(\mu, \sigma)$

Recall from (4.2) that

$$\widetilde{b}(\mu, \sigma) = e^{\sigma W(\mu,\sigma^2) - \frac{1}{2}\sigma^2 + \mu}$$

where $W(\mu, \sigma)$ is defined implicitly by (4.1) i.e.

$$e^{-\mu} \equiv \Phi(W(\mu, \sigma) - \sigma) + e^{\sigma W(\mu,\sigma) - \frac{1}{2}\sigma^2}\Phi(-W(\mu, \sigma)).$$

In this section we prove that $\widetilde{b}(\mu,\sigma)$ is a decreasing function of μ and an increasing function of σ. In the sections to follow we examine the asymptotic behaviour of $W(\mu,\sigma)$ for fixed μ and then of $\bar{W}(\theta) = W(\bar{\mu}\theta, \bar{\sigma}\theta)$ as a function of θ.

Theorem 1: The Censor $\widetilde{b}(\mu,\sigma)$ is Decreasing in the Drift

Proof. We wish to establish that:

$$\frac{\partial \widetilde{b}(\mu,\sigma)}{\partial \mu} < 0,$$

Differentiating (4.2) we see that for the Theorem to hold we require:

$$\frac{\partial \widetilde{b}(\mu,\sigma)}{\partial \mu} = \widetilde{b}(\mu,\sigma)\left\{\sigma \frac{\partial W(\mu,\sigma)}{\partial \mu} + 1\right\} < 0$$

$$\Leftrightarrow -\sigma \frac{\partial W(\mu,\sigma)}{\partial \mu} > 1$$

but differentiating the identity (4.1) we obtain:

$$\begin{aligned} -e^{-\mu} &= \varphi(W(\mu,\sigma)-\sigma)\frac{\partial W(\mu,\sigma)}{\partial \mu} + e^{\sigma W - \frac{1}{2}\sigma^2}\sigma\frac{\partial W}{\partial \mu}\Phi(-W(\mu,\sigma)) \\ &\quad - e^{\sigma W - \frac{1}{2}\sigma^2}\frac{\partial W}{\partial \mu}\varphi(-W(\mu,\sigma)) \\ &= e^{\sigma W - \frac{1}{2}\sigma^2}\sigma\frac{\partial W}{\partial \mu}\Phi(-W(\mu,\sigma)) \end{aligned}$$

- since $\varphi(W(\mu,\sigma)-\sigma) = e^{\sigma W - \frac{1}{2}\sigma^2}\varphi(W(\mu,\sigma))-$ yielding

$$-1 = \widetilde{b}\sigma\frac{\partial W}{\partial \mu}\Phi(-W(\mu,\sigma)).$$

So

$$\begin{aligned} -\sigma\frac{\partial W(\mu,\sigma)}{\partial \mu} &> 1 \quad \Leftrightarrow \quad 1 = -\widetilde{b}\sigma\frac{\partial W}{\partial \mu}\Phi(-W(\mu,\sigma)) > \widetilde{b}\Phi(-W(\mu,\sigma)) \\ &\Leftrightarrow \widetilde{b}\Phi(-W(\mu,\sigma)) < 1. \end{aligned}$$

But we may rewrite (4.1) as:

$$1 \equiv e^{\mu}\Phi(W(\mu,\sigma)-\sigma) + \widetilde{b}\Phi(-W(\mu,\sigma))$$

where clearly $e^{\mu}\Phi(W(\mu,\sigma)-\sigma)>0$,so

$$-\sigma\frac{\partial W(\mu,\sigma)}{\partial\mu}>1. \tag{5.1}$$

□

Theorem 2: The Censor $\widetilde{b}(\mu,\sigma)$ is Increasing in the Standard Deviation

Proof. We wish to establish that:

$$\frac{\partial\widetilde{b}(\mu,\sigma)}{\partial\sigma}>0.$$

As before

$$\widetilde{b}(\mu,\sigma)\equiv e^{\sigma W(\mu,\sigma)+\mu-\frac{1}{2}\sigma^2}$$

hence

$$\frac{\partial\widetilde{b}}{\partial\sigma}=\widetilde{b}(\mu,\sigma)\left\{\sigma\frac{\partial W}{\partial\sigma}+W(\mu,\sigma)-\sigma\right\},$$

and as before differentiating the censor equation (4.1) we obtain:

$$\begin{aligned}0 &= \varphi(W(\mu,\sigma)-\sigma)\{\frac{\partial W}{\partial\sigma}-1\}+e^{\sigma W(\mu,\sigma)-\frac{1}{2}\sigma^2}\{W(\mu,\sigma)+\sigma\frac{\partial W}{\partial\sigma}-\sigma\}\Phi(-W(\mu,\sigma))\\ &\quad -e^{\sigma W(\mu,\sigma)-\frac{1}{2}\sigma^2}\varphi(W(\mu,\sigma))\frac{\partial W}{\partial\sigma}\end{aligned}$$

so

$$\varphi(W(\mu,\sigma)-\sigma)=e^{\sigma W(\mu,\sigma)-\frac{1}{2}\sigma^2}\{W(\mu,\sigma)+\sigma\frac{\partial W}{\partial\sigma}-\sigma\}\Phi(-W(\mu,\sigma))$$

or

$$W(\mu,\sigma)+\sigma\frac{\partial W}{\partial\sigma}-\sigma=e^{\sigma^2}\frac{\varphi(W(\mu,\sigma))}{\Phi(-W(\mu,\sigma))}\equiv e^{\sigma^2}H(W(\mu,\sigma))>0 \tag{5.2}$$

where $H(.)$ denotes the hazard rate or Mills ratio $(\varphi(x)/\Phi(-x))$, so that:

$$\frac{\partial\widetilde{b}}{\partial\sigma}=\widetilde{b}(.)H(W(\mu,\sigma))e^{\sigma^2}>0 \quad \square$$

Since $e^{\sigma^2}>1$ and $H(x)>x$ for all x, the penultimate equation gives

$$\frac{\partial W}{\partial\sigma}>1$$

and

$$\frac{\partial}{\partial\sigma}\{\sigma W(\mu,\sigma)-\frac{1}{2}\sigma^2\} > \sigma + H(W(\mu,\sigma)).$$

We have just proved:

Theorem 3:

The two functions $\sigma W(\mu,\sigma)-\frac{1}{2}\sigma^2, \quad W(\mu,\sigma)-\sigma$ **are increasing in** σ.

We note that embedded in our proofs above are the following observations.

Theorem 4: The Normal Censor $W(\mu,\sigma)$ is increasing in Standard Deviation, decreasing with drift.

We have that:

$$\frac{\partial W(\mu,\sigma)}{\partial\sigma} > 0, \text{ and } \frac{\partial W(\mu,\sigma)}{\partial\mu} < 0.$$

Proof. This is immediate from (5.2) since

$$\frac{\partial W(\mu,\sigma)}{\partial\sigma}\sigma = \sigma + H(W) - W, \tag{5.3}$$

and the result follows since $H(w) > w$. Note that it is easy to show $H'(w) > 0$ whence the inequality holds via:

$$H'(w) = H(w)\{H(w) - w\}.$$

The second result is from inequality (5.1).

Theorem 5: The 'geometric Brownian' censor $\bar{b}(\theta)$ is unimodal or monotone.

Proof. Since

$$\bar{b}(\theta) = \widetilde{b}(\mu(\theta),\sigma(\theta))$$

we of course have

$$\frac{d\bar{b}(\theta)}{d\theta} = \frac{\partial\widetilde{b}}{\partial\mu}\frac{d\mu}{d\theta} + \frac{\partial\widetilde{b}}{\partial\sigma}\frac{d\sigma}{d\theta}$$

where we recall that

$$\mu = \overline{\mu}\theta \qquad \text{and} \qquad \sigma = \overline{\sigma}\sqrt{\theta}$$

which gives us:

$$\begin{aligned}\frac{d\bar{b}(\theta)}{d\theta} &= \overline{\mu}\frac{\partial\widetilde{b}}{\partial\mu} + \frac{1}{2}\overline{\sigma}\theta^{-\frac{1}{2}}\frac{\partial\widetilde{b}}{\partial\sigma} = \frac{1}{\theta}\{\mu\frac{\partial\widetilde{b}}{\partial\mu} + \frac{1}{2}\sigma\frac{\partial\widetilde{b}}{\partial\sigma}\} \\ &= \overline{\mu}\{\bar{b} - \frac{1}{\Phi(-W)}\} + \frac{1}{2}\overline{\sigma}\theta^{-\frac{1}{2}}\bar{b}\frac{\varphi(W)}{\Phi(-W)}\end{aligned}$$

so

$$\begin{aligned}\Phi(-W)\frac{d\bar{b}(\theta)}{d\theta} &= \overline{\mu}\{\bar{b}\Phi(-W)-1\}+\frac{1}{2}\overline{\sigma}\theta^{-\frac{1}{2}}\bar{b}\varphi(W) \\ &= \overline{\mu}\{-e^{\mu}\Phi(W-\sigma)\}+\frac{1}{2}\overline{\sigma}\theta^{-\frac{1}{2}}\bar{b}\varphi(W)\end{aligned}$$

and

$$\begin{aligned}\theta\Phi(-W)\frac{d\bar{b}(\theta)}{d\theta} &= -\mu\{e^{\mu}\Phi(W-\sigma)\}+\frac{1}{2}\sigma\bar{b}\varphi(W)=0 \\ &\Leftrightarrow \sigma H(-W+\sigma)-2\mu=0,\end{aligned}$$

since $\bar{b}\varphi(W)=\varphi(W)e^{\mu+\sigma W-\frac{1}{2}\sigma^2}=e^{\mu}\varphi(W-\sigma)$.

We shall shortly show that for given μ there is a unique $\sigma=s_{\bar{b}}(\mu)$ for which the equation holds. Granted this define

$$S_{\bar{b}}(\mu)=s_{\bar{b}}(\mu)/\sqrt{\mu} \tag{5.4}$$

and we may solve the equation $\overline{\sigma}\theta=s_{\bar{b}}(\overline{\mu}\theta)$ by solving the two equations:

$$\frac{\overline{\sigma}}{\sqrt{\overline{\mu}}}=S_{\bar{b}}(\mu),\theta=\frac{\mu}{\overline{\mu}},\text{ provided }\frac{\overline{\sigma}}{\sqrt{\overline{\mu}}}\text{ is in the range of }S_{\bar{b}}.$$

As for the existence of the function $s_{\bar{b}}(\mu)$, first we show that for given μ there is σ for which the first-order condition holds. Indeed as $\sigma\to\infty$ we have $H(-W+\sigma)\to H(m)$ so $\sigma H(-W+\sigma)\to\infty$. On the other hand as $\sigma\to 0+$ we have $W=-\mu/\sigma+\frac{1}{2}\sigma+o(\sigma)$ and so, since $H(x)/x\to 1$ as $x\to\infty$ we have

$$\lim_{\sigma\to 0+}\sigma H(-W+\sigma)=\mu.$$

Next we show that the solution to the first-order condition is unique.

We compute that

$$\begin{aligned}&\frac{d}{d\sigma}\sigma H(-W+\sigma) \\ &= H(-W+\sigma)+H'(-W+\sigma)\sigma\left(1-\frac{\partial W}{\partial\sigma}\right) \\ &= H(-W+\sigma)+H(-W+\sigma)\left(H(-W+\sigma)-(-W+\sigma)\right)\left(\sigma-\sigma\frac{\partial W}{\partial\sigma}\right) \\ &= H(-W+\sigma)[1+\{H(-W+\sigma)-(-W+\sigma)\}\left(W-H(W)\right)]\end{aligned}$$

by (5.3). This is positive provided

$$\{H(-W+\sigma)-(-W+\sigma)\}\,(H(W)-W)<1. \tag{5.5}$$

We reduce the verification of this inequality to the observation that

$$\frac{2}{\pi}\leq (H(x)-x)(H(-x)+x)<1, \tag{5.6}$$

holds. [We have verified this expression numerically as increasing for $x>0$ and the bounds are readily identifiable analytically.] Now $H(-x)+x$ is increasing (as $1-H'(-x)>0$) hence we have $H(-x)+x>H(-x+\sigma)+x-\sigma$ which together imply (5.5).

The monotonicity of $\sigma H(-W+\sigma)$ allows us to see that for any σ there is a unique

$$\mu=\mu^*(\sigma)$$

such that

$$\mu=\frac{1}{2}\sigma H(-W(\mu,\sigma)+\sigma).$$

For given $\overline{\sigma}$ and $0<\bar{\theta}\leq 1$, reference to the equation which defines $\overline{\mu}^*$, namely

$$\overline{\mu}^*\bar{\theta}=\mu^*(\overline{\sigma}\bar{\theta}^{\frac{1}{2}}),$$

now permits us to assert that $\widetilde{b}(\overline{\mu}\theta,\overline{\sigma}^2\theta)$ as a function of θ has a maximum for some $\theta\leq\bar{\theta}$ provided $\overline{\mu}$ is large enough.

Conjecture. We find by numerical computation that $S_{\bar{b}}(\mu)$ is strictly increasing and concave in the range $0.001\leq\mu\leq 1$, with $S_{\bar{b}}(0.001)=0.07924$ and $S_{\bar{b}}(7.85)=1.3814$. This implies that $\bar{b}(\theta)$ is unimodal, at least for the range $0.07924\leq\overline{\sigma}/\sqrt{\overline{\mu}}\leq 1.3814$.

Conjecture. We conjecture that the function defined by (5.4) satisfies

$$\lim_{\mu\to 0}S_{\bar{b}}(\mu)=0,\quad \lim_{\mu\to\infty}S_{\bar{b}}(\mu)=\sqrt{2},$$

on the grounds that an over-approximation to $S_{\bar{b}}(\mu)$ is $2\sqrt{\mu}/H(m(\mu))$ and, after squaring, by l'Hôpital's Rule we have

$$\lim_{m\to\infty}\frac{-4\ln\Phi(-m)}{H(m)^2}=2. \tag{5.7}$$

Now for $\sigma<s_{\bar{b}}(\mu)$ we have $d\bar{b}/d\theta<0$ and for $\sigma>s_{\bar{b}}(\mu)$ we have $d\bar{b}/d\theta>0$. Evidently, directly from the definition $\bar{b}(0)=1$, and $\bar{b}(\theta)>1$ for $\theta>0$. Hence for fixed $\theta=\bar{\theta}$ and large enough $\bar{\sigma}$ the censor is increasing in $[0,\bar{\theta}]$.

6. Asymptotic Analysis of the function $W(\mu, \sigma)$

6.1. Asymptotics for fixed μ when $\sigma \to 0+$

Lemma 1 *We have for fixed* μ

$$\lim_{\sigma \to 0+} W(\mu, \sigma) = -\infty, \text{ and } \lim_{\sigma \to 0+} \sigma W(\mu, \sigma) = -\mu.$$

Proof. Let us write $V = W(\mu, \sigma) - \sigma$. Then the defining equation for $V = V(\sigma)$ is

$$e^{-\mu} = \Phi(V) + e^{\sigma V + \frac{1}{2}\sigma^2}\Phi(-\sigma - V)$$

or

$$e^{-\mu} - 1 = -\Phi(-V) + e^{\sigma V + \frac{1}{2}\sigma^2}\Phi(-\sigma - V)$$

so

$$(e^{-\mu} - 1) - \{\Phi(-\sigma - V) - \Phi(-V)\} = \{e^{\sigma V + \frac{1}{2}\sigma^2} - 1\}\Phi(-\sigma - V).$$

Thus, by continuity, if $\mu > 0$ the left-hand side is negative for small enough σ. Hence for such σ we have

$$\sigma V + \frac{1}{2}\sigma^2 < 0$$

and so $V < -\sigma/2$ and thus

$$\Phi(-\frac{1}{2}\sigma) < \Phi(-\sigma - V) < 1.$$

Letting $\bar{V} = \lim_{\sigma \to 0+} V(\mu, \sigma^2)$, we have $\bar{V} \leq 0$ and $\frac{1}{2} \leq \Phi(-\bar{V}) \leq 1$.So $\bar{V} \neq \infty$ and

$$(e^{-\mu} - 1) = \{\lim_{\sigma \to 0+} e^{\sigma V} - 1\}\Phi(-\bar{V}).$$

Thus if $-\bar{V} < \infty$ we would have $\sigma V \to 0$ and $\lim_{\sigma \to 0+} e^{\sigma V} = 1$, contradicting negativity. Thus

$$\bar{V} = -\infty.$$

Hence

$$(e^{-\mu} - 1) = \{\lim_{\sigma \to 0+} e^{\sigma V} - 1\}$$

i.e.

$$\lim_{\sigma \to 0+} \sigma V = -\mu.$$

We now prove:

Proposition 1.

$$W(\mu,\sigma) = -\frac{\mu}{\sigma} + \frac{1}{2}\sigma + o(\sigma)\textbf{as } \sigma \to 0+ \textbf{ for fixed } \mu.$$

Proof. To see why notice that for small enough σ we have

$$e^{-\mu} \simeq e^{\sigma W - \frac{1}{2}\sigma^2}$$

and so

$$W(\mu,\sigma) \approx -\frac{\mu}{\sigma} + \frac{1}{2}\sigma.$$

This argument can be embelished as follows. For any non-zero ε let

$$W(\varepsilon) = -\frac{\mu}{\sigma} + \frac{1}{2}\sigma + \sigma\varepsilon$$

and note that

$$\begin{aligned} \sigma W(\varepsilon) - \frac{1}{2}\sigma^2 &= -\mu + \sigma\varepsilon, \\ \sigma - W(\varepsilon) &= \frac{\mu}{\sigma} + \frac{1}{2}\sigma - \sigma\varepsilon \end{aligned}$$

We prove that for positive ε we have for small enough σ

$$W(-\varepsilon) < W(\mu,\sigma) < W(\varepsilon),$$

by showing that for small enough σ the following expression has the same sign as ε :

$$D(\sigma) = F(W(\varepsilon),\sigma) - F(W(\mu,\sigma),\sigma) = F(W(\varepsilon),\sigma) - e^{-\mu}.$$

This implies the Proposition. Now $D(0+) = 0$ and since

$$D(\sigma) = \int_{-W(\varepsilon)+\sigma}^{\infty} e^{-\frac{1}{2}x^2}\frac{dx}{\sqrt{2\pi}} + e^{-\mu+\sigma^2\varepsilon}\int_{W(\varepsilon)}^{\infty} e^{-\frac{1}{2}x^2}\frac{dx}{\sqrt{2\pi}} - e^{-\mu}$$

we have that

$$\begin{aligned} D'(\sigma) &= e^{-\frac{1}{2}(-W(\varepsilon)+\sigma)^2}\frac{1}{\sqrt{2\pi}}\{-\frac{\mu}{\sigma^2} + \frac{1}{2} - \varepsilon\} + e^{-\mu+\sigma^2\varepsilon}\{2\sigma\varepsilon\}(1+o(\sigma)) \\ &\quad + e^{-\mu+\sigma^2\varepsilon}e^{-\frac{1}{2}W(\varepsilon)^2}\{\frac{\mu}{\sigma^2} + \frac{1}{2} + \varepsilon\}. \end{aligned}$$

Note that the first and third terms contain a factor

$$\sigma \exp[-\frac{\mu^2}{\sigma^2}],$$

which is is small compared to, say, σ. So for small enough σ the derivative $D'(\sigma)$ has the same sign as ε. Thus for positive ε we have for small enough σ, the positive derivative implies $D(\sigma) > D(0+) = 0$; and for negative ε we have for small enough σ, the negative derivative implies $D(\sigma) < D(0+) = 0$, i.e. $D(\sigma)$ has the same sign as ε.

6.2. Asymptotics for fixed μ when $\sigma \to \infty$

For convenience define $R(W, \sigma)$ by

$$R(W, \sigma) = \sqrt{2\pi} F(W, \sigma) = \int_{-W+\sigma}^{\infty} e^{-\frac{1}{2}x^2} dx + e^{\sigma W - \frac{1}{2}\sigma^2} \int_{W}^{\infty} e^{-\frac{1}{2}x^2} dx.$$

It is easy to see that

$$\frac{\partial}{\partial W} R(W, \sigma) > 0$$

and

$$R(-\infty, \sigma) = 0, R(+\infty, \sigma) = \sqrt{2\pi}.$$

Let m be fixed; for the purposes of the current section only we re-define $W(m, \sigma)$ by

$$R(W(m, \sigma), \sigma) = \int_{m}^{\infty} e^{-\frac{1}{2}x^2} dx < \sqrt{2\pi}.$$

Claim.

$$\text{For } c \text{ any constant } \lim_{\sigma \to \infty} R(\sigma - c, \sigma) = \int_{c}^{\infty} e^{-\frac{1}{2}x^2} dx.$$

Conclusion from claim. Before proving the claim notice the consequences of the choices $c = (1 \pm \varepsilon)m$. Since

$$\lim_{\sigma \to \infty} R(\sigma - (1+\varepsilon)m, \sigma) = \int_{(1+\varepsilon)m}^{\infty} e^{-\frac{1}{2}x^2} dx > \int_{m}^{\infty} e^{-\frac{1}{2}x^2} dx$$

for large enough σ we have

$$R(\sigma - (1+\varepsilon)m, \sigma) > R(W, \sigma).$$

Hence for large enough σ we have

$$W < \sigma - (1+\varepsilon)m$$

Similarly taking $c = (1-\varepsilon)m$ we obtain

$$W > \sigma - (1-\varepsilon)m.$$

Thus

$$W(m,\sigma) = \sigma - m\{1 + o(1)\}.$$

Proof of claim. We have

$$\begin{aligned}
\lim_{\sigma\to\infty} R(\sigma - c, \sigma) &= \int_c^\infty e^{-\frac{1}{2}x^2}dx + \{\lim_{\sigma\to\infty} e^{\sigma(\sigma-c)-\frac{1}{2}\sigma^2}\int_{\sigma-c}^\infty e^{-\frac{1}{2}x^2}dx\} \\
&= \int_c^\infty e^{-\frac{1}{2}x^2}dx + \{\lim_{\sigma\to\infty} e^{-\sigma c+\frac{1}{2}\sigma^2}\cdot\frac{1}{\sigma - c}e^{-\frac{1}{2}(\sigma-c)^2}\} \\
&= \int_c^\infty e^{-\frac{1}{2}x^2}dx + \{\lim_{\sigma\to\infty}\frac{1}{\sigma-c}e^{-\frac{1}{2}c^2}\} = \int_c^\infty e^{-\frac{1}{2}x^2}dx
\end{aligned}$$

We complete the analysis and show the following.

Proposition 2.

$$W(m,\sigma) = \sigma - m - \frac{1}{\sigma - m}\{1 + o(1)\}.$$

Proof. Consider an arbitrary non-zero ε; let $W_\varepsilon = \sigma - m - \delta$ and put

$$\delta = \frac{1-\varepsilon}{\sigma - m}.$$

Now consider

$$\begin{aligned}
D(\sigma) &= \left(\int_{-W+\sigma}^{\infty} e^{-\frac{1}{2}x^2}dx + e^{\sigma W-\frac{1}{2}\sigma^2}\int_{W}^{\infty} e^{-\frac{1}{2}x^2}dx\right) - \int_{m}^{\infty} e^{-\frac{1}{2}x^2}dx \\
&= \left(\int_{m+\delta}^{\infty} e^{-\frac{1}{2}x^2}dx - \int_{m}^{\infty} e^{-\frac{1}{2}x^2}dx\right) + e^{\sigma(\sigma-m-\delta)-\frac{1}{2}\sigma^2}\int_{\sigma-m-\delta}^{\infty} e^{-\frac{1}{2}x^2}dx \\
&= -\delta e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2) + e^{\sigma(\sigma-m-\delta)-\frac{1}{2}\sigma^2}\frac{1}{\sigma-m-\delta}e^{-\frac{1}{2}(m+\delta-\sigma)^2}\{1+O(\frac{1}{\sigma^2})\} \\
&= -\delta e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2) + \frac{1}{\sigma-m-\delta}e^{-\frac{1}{2}(m+\delta)^2}\{1+O(\frac{1}{\sigma^2})\} \\
&= \left(\frac{1}{\sigma-m-\delta}-\delta\right)e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2) + O(\frac{1}{\sigma^2}) \\
&= \left(\frac{1}{(\sigma-m)-\frac{1-\varepsilon}{\sigma-m}} - \frac{1-\varepsilon}{\sigma-m}\right)e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2) + O(\frac{1}{\sigma^2}) \\
&= \left(\frac{(\sigma-m)^2-(1-\varepsilon)\{(\sigma-m)^2-(1-\varepsilon)\}}{(\sigma-m)^3-(1-\varepsilon)(\sigma-m)}\right)e^{-\frac{1}{2}(m+\delta)^2} + O(\frac{1}{\sigma^2}) \\
&= \frac{\varepsilon(\sigma-m)^2+(1-\varepsilon)^2}{(\sigma-m)^3-(1-\varepsilon)(\sigma-m)}e^{-\frac{1}{2}(m+\delta)^2} + O(\frac{1}{\sigma^2}) \\
&= \frac{\varepsilon}{\sigma-m}e^{-\frac{1}{2}(m+\delta)^2} + O(\frac{1}{\sigma^2})
\end{aligned}$$

and this has the same sign as ε.

Thus for $\varepsilon > 0$ we have

$$R(W_{-\varepsilon},\sigma) < R(W(m,\sigma),\sigma) < R(W_{\varepsilon},\sigma)$$

and so

$$W_{-\varepsilon} < W(m,\sigma) < W_{\varepsilon}.$$

Conclusion. We are of course interested in the value $m = \hat{\mu}$ where

$$\sqrt{2\pi}e^{-\mu} = \int_{m}^{\infty} e^{-\frac{1}{2}x^2}dx,$$

as then $W(m,\sigma)$ coincides with the previously defined $W(\mu,\sigma)$. Evidently

$$e^{-\mu} = 1-\Phi(\hat{\mu}) = \Phi(-\hat{\mu}),$$

so

$$\mu = -\ln \Phi(-\hat{\mu}), \text{ or } \hat{\mu} = -\Phi^{-1}(e^{-\mu}).$$

Thus $\hat{\mu} > 0$ if and only if $\mu > \ln 2$; in particular for small μ we thus have $\hat{\mu} < 0$. Note for future use that

$$\frac{d\mu}{d\hat{\mu}} = \frac{\varphi(\hat{\mu})}{\Phi(-\hat{\mu})} = H(\hat{\mu}),$$

where $H(.)$ is the Mills ratio (or hazard rate).

7. Asymptotics of the function $\bar{W}(\theta) = W(\bar{\mu}\theta, \bar{\sigma}\theta)$

This section studies the behaviour of $\bar{W}(\theta)$ for values of θ which are small or large. We also consider numeric evidence for intermediate values of θ.

7.1. $\bar{W}(\theta)$ for θ near zero

Lemma 2.

$$\lim_{\theta\to 0+} \sqrt{\theta}\bar{W}(\theta) = 0 \text{ and } \lim_{\theta\to 0+} \bar{W}(\theta) = +\infty \textit{for fixed } \bar{\mu}, \bar{\sigma}.$$

Proof. As before write $V = V(\theta) = \bar{W}(\theta) - \bar{\sigma}\sqrt{\theta}$ then from (4.1)

$$(e^{-\mu} - 1) - \{\Phi(-\sigma - V) - \Phi(-V)\} = \{e^{\sigma V + \frac{1}{2}\sigma^2} - 1\}\Phi(-\sigma - V)$$

and we deduce that

$$(e^{-\mu} - 1) - \sigma\varphi(V^*) = \{e^{\sigma V + \frac{1}{2}\sigma^2} - 1\}\Phi(-\sigma - V)$$

for some V^* between V and $V + \sigma$ or approximately

$$-\bar{\mu}\theta + \sigma\varphi(V^*) = \{e^{\sigma V + \frac{1}{2}\sigma^2} - 1\}\Phi(-\sigma - V).$$

Let $\bar{V} = \lim_{\theta\to 0+} V(\theta)$. We now need to prove the following claim before returning to the lemma.

Claim.

$$L = \lim_{\theta\to 0+} \sigma V(\theta) = 0.$$

Proof of Claim. Suppose that $L = \lim_{\theta \to 0+} \sigma V(\theta) \neq 0$ along a sequence of values of θ,then

$$V(\theta) \approx \frac{L}{\bar{\sigma}\sqrt{\theta}}$$

hence

$$\sigma\varphi(V^*) \approx \bar{\sigma}\sqrt{\theta}\exp\{-\frac{L^2}{\bar{\sigma}^2\theta}\}/\sqrt{2\pi}$$

and so

$$-\bar{\mu}\theta\{1 - \frac{\bar{\sigma}}{\bar{\mu}\sqrt{\theta}}\exp\{-\frac{L^2}{\bar{\sigma}^2\theta}\}/\sqrt{2\pi}\} \approx -\bar{\mu}\theta.$$

Thus for small enough θ we have

$$\{e^{\sigma V + \frac{1}{2}\sigma^2} - 1\}\Phi(-\sigma - V) < 0,$$

so that $\bar{V} \leq 0$.

Suppose first that $\bar{V} = -\infty$, then $\Phi(\infty) = 1$ yields by way of

$$0 = (e^L - 1) = 1$$

that $L = 0$. This in turn implies that for small enough θ we have approximately

$$-\bar{\mu}\theta + \sigma\varphi(V^*) = \{e^{\sigma V + \frac{1}{2}\sigma^2} - 1\} = \sigma V + \frac{1}{2}\sigma^2$$

or

$$-\frac{\bar{\mu} + \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\sqrt{\theta} = V$$

and so contradicts $\bar{V} = -\infty$. But we conclude from the finiteness of $\bar{V}$ again that $L = 0$. This final contradiction proves the claim.

Proof of Lemma 2 continued. We have as above

$$(e^{-\mu} - 1) + \sigma\varphi(V^*) = \{e^{\sigma V + \frac{1}{2}\sigma^2} - 1\}\Phi(-\sigma - V).$$

By the Claim, σV is small, so we may expand the exponential and dividing by $\sigma = \bar{\sigma}\sqrt{\theta}$ obtain

$$-\frac{\bar{\mu}}{\bar{\sigma}}\sqrt{\theta} + \varphi(V^*) = (V + \frac{1}{2}\sigma)\Phi(-\sigma - V).$$

If $V \to \bar{V}$ a finite limit we would have the Mills ratio, or hazard rate $H(\bar{V})$

$$H(\bar{V}) \equiv \frac{\varphi(\bar{V})}{\Phi(-\bar{V})},$$

satisfy $H(\bar{V}) = \bar{V}$, a contradiction since the ratio is always greater than $\bar{V}$. Thus the limit $\bar{V}$ must be infinite, hence $\varphi(\bar{V}) = 0$ and so $\bar{V} = +\infty$, otherwise $\bar{V} = -\infty$ leads to the contradiction

$$0 = \bar{V}\Phi\left(-\bar{V}\right) = \bar{V} \cdot 1.$$

Comment. For small θ we may write

$$e^{-\bar{\mu}\theta} \approx \Phi(\bar{W} - \sigma)$$

and so we have

$$\bar{W}(\theta) \approx \Phi^{-1}(e^{-\bar{\mu}\theta})$$

or even

$$\bar{W}(\theta) \approx -\Phi^{-1}(\bar{\mu}\theta).$$

One might solve

$$e^{-\bar{\mu}\theta} = 1 - \bar{\mu}\theta = \Phi(\bar{W} - \sigma) = 1 - \frac{\exp -(\bar{W} - \sigma)^2/2}{(\bar{W} - \sigma)\sqrt{2\pi}}$$

i.e.

$$\bar{\mu}\theta = \frac{\exp -(\bar{W} - \sigma)^2/2}{(\bar{W} - \sigma)\sqrt{2\pi}}$$

and an overestimate of the true $\bar{W}$ is given by

$$\sigma + \sqrt{-2\log[\theta\bar{\mu}\sqrt{2\pi}]}$$

and so $\sigma\bar{W}$ is over-estimated by

$$\bar{\sigma}^2\theta + \bar{\sigma}\sqrt{-2\theta\log[\theta\bar{\mu}\sqrt{2\pi}]}$$

which tends to zero.

Proof of Theorem B. Differentiation gives

$$-\bar{\mu}e^{-\bar{\mu}\theta} = \varphi(W-\sigma)(W'-\sigma') + e^{\sigma W - \frac{1}{2}\sigma^2}\varphi(-W)(-W') + \Phi(-W)e^{\sigma W - \frac{1}{2}\sigma^2}[-\frac{1}{2}\bar{\sigma}^2 + (\sigma W)'].$$

Now as $\theta \to 0+$we have

$$\varphi(W-\sigma)\sigma' = e^{\sigma W - \frac{1}{2}\sigma^2}\varphi(W)\frac{\bar{\sigma}}{2\sqrt{\theta}} = e^{\sigma W - \frac{1}{2}\sigma^2}\frac{\varphi(W)}{W}\frac{1}{\theta}\frac{W\bar{\sigma}\sqrt{\theta}}{2} \to \bar{\mu}\cdot 0 = 0.$$

Thus

$$-\bar{\mu} = \lim_{\theta\to 0+}[\Phi(-W)(\sigma W)'].$$

Differentiation gives

$$\begin{aligned}\bar{g}' &= [\bar{\sigma}^2 - \bar{\mu}]e^{(\sigma^2-\mu)}\Phi(W+\sigma) + e^{(\sigma^2-\mu)}\varphi(W+\sigma)(W'+\sigma') \\ &\quad + e^{-\mu-\sigma W+\frac{1}{2}\sigma^2}\varphi(-W)(-W') + e^{-\mu-\sigma W+\frac{1}{2}\sigma^2}\Phi(-W)[\frac{1}{2}\bar{\sigma}^2 - \bar{\mu} - (\sigma W)'].\end{aligned}$$

Now as $\theta \to 0+$ we have

$$\bar{g}' = [\bar{\sigma}^2 - \bar{\mu}] - \lim_{\theta\to 0+}\Phi(-W)[(\sigma W)'] = \bar{\sigma}^2.$$

7.2. Intermediate behaviour of $\bar{W}(\theta)$: numeric evidence and a conjecture

Before we consider the behaviour of $\bar{W}(\theta)$ as $\theta \to \infty$, let us study the behaviour of the derivative $\bar{W}'(\theta)$ (for fixed $\bar{\mu}, \bar{\sigma}$). We find that just as with $\bar{b}(\theta)$ the derivative must change sign at least once. Uniqueness of this change appears, however, to be intractable to analysis. Numeric evidence is strong, as we indicate.

Routine implicit differentiation of the censor equation with respect to θ yields

$$\sigma\bar{W}'(\theta) = \sigma'[\sigma + H(\bar{W}(\theta)) - \bar{W}(\theta)] - \bar{\mu}[1 + \frac{H(\bar{W}(\theta))}{H(\sigma - \bar{W}(\theta))}].$$

The stationarity condition may thus be re-written

$$G(\mu,\sigma) \equiv \sigma\frac{(H(W(\mu,\sigma)) - W(\mu,\sigma) + \sigma)H(\sigma - W(\mu,\sigma))}{H(\sigma - W(\mu,\sigma)) + H(W(\mu,\sigma))} = 2\mu,$$

where $\mu = \bar{\mu}\theta, \sigma = \bar{\sigma}\theta$. Just as in the proof of Theorem 5 (section 3) one may show that $G(\mu, 0+) = \mu$, $\lim_{\sigma\to\infty} G(\mu,\sigma) = \lim_{\sigma\to\infty}\sigma H(m) = \infty$. Let $s_w(\mu)$ denote the least σ solving the equation $G(\mu,\sigma) = 2\mu$. Numeric investigation suggests that $G(\mu,\sigma)$ appears to be strictly increasing. Assuming this is the case $\bar{W}(\theta)$ is unimodal, since the equation $\bar{W}'(\theta) = 0$ has the unique solution given by

$$S_w(\mu) = \frac{\bar{\sigma}}{\sqrt{\bar{\mu}}} \text{ and } \theta = \frac{\mu}{\bar{\mu}}, \text{ provided } \frac{\bar{\sigma}}{\sqrt{\bar{\mu}}} \text{ is in the range of } S_w,$$

where

$$S_w(\mu) = s_w(\mu)/\sqrt{\mu}. \tag{7.1}$$

The graph of $S_w(\mu)$ is found by calculation to be convex, decreasing and we have $S_w(.001) = 11.79$, $S_w(7.85) = 1.464$.

Conjecture. We conjecture that

$$\lim_{\mu\to 0} S_w(\mu) = \infty, \lim_{\mu\to\infty} S_w(\mu) = \sqrt{2},$$

again on the grounds of (5.7). If true, the conjecture would imply that for $\bar{\mu} > \frac{1}{2}\bar{\sigma}^2$ there is no stationary point and so $\bar{W}(\theta)$ is monotonic decreasing. This would agree with numeric evidence.

Remark. One may also study the behaviour of the derivative $\bar{W}'(\theta)$ when θ is fixed, say at $\bar{\theta}$ and $\bar{\mu}$ is fixed but $\bar{\sigma}$ varies. We find again that the derivative must change sign at least once. Uniqueness of this change appears hopelessly intractable to analysis.

Armed with these observations we are not surprised by the findings in the next section.

7.3. Asymptotics of $\bar{W}(\theta)$ for $\theta \to \infty$

We begin with a preliminary analysis which enables us to derive asymptotic formulas by cases.

Lemma 3. If $\frac{1}{2}\bar{\sigma}^2 \neq \bar{\mu}$ then

$$\lim_{\theta\to\infty} \bar{W}(\theta) = \pm\infty.$$

Remark. We will later identify the appropriate sign which depends on the relative size of $\frac{1}{2}\bar{\sigma}^2$ and $\bar{\mu}$.

Proof. As usual

$$e^{-\bar{\mu}\theta} = \Phi(\bar{W}(\theta) - \sigma) + e^{\sigma\bar{W}(\theta)-\frac{1}{2}\sigma^2}\Phi(-\bar{W}(\theta))$$

or

$$e^{-\bar{\mu}\theta-\sigma\bar{W}(\theta)+\frac{1}{2}\sigma^2} = e^{-\sigma\bar{W}(\theta)+\frac{1}{2}\sigma^2}\Phi(\bar{W}(\theta) - \sigma) + \Phi(-\bar{W}(\theta)) \tag{7.2}$$

i.e.

$$e^{-\bar{\mu}\theta-\sigma\bar{W}(\theta)+\frac{1}{2}\sigma^2} = \Phi(-\bar{W}(\theta)) + \varphi(\bar{W}(\theta))/H(\sigma - \bar{W}(\theta)), \tag{7.3}$$

where, as above, $H(.)$ denotes the hazard rate. Assume that $\bar{W}(\theta) \to \bar{w}$. We are to prove that $\bar{w}$ is not finite. We argue by cases.

If $\frac{1}{2}\bar{\sigma}^2 > \bar{\mu}$, the left hand side is unbounded, whereas the right-hand side is bounded for large θ by

$$1 + \varphi(\bar{w})/(\bar{\sigma}\sqrt{\theta} - \bar{w}).$$

If $\frac{1}{2}\bar{\sigma}^2 < \bar{\mu}$ then letting $\theta \to \infty$ we obtain

$$0 = \Phi(-\bar{w}) + 0$$

a contradiction.

We now prove a related result:

Lemma 4.

$$\lim_{\theta\to\infty} \bar{W}(\theta) - \sigma = -\infty.$$

Proof. As before if $V = \bar{W}(\theta) - \sigma$ we have

$$e^{-\mu} = \Phi(V) + e^{\sigma V + \frac{1}{2}\sigma^2}\Phi(-\sigma - V).$$

Suppose that $V \to -\infty$ is false. Then either $V \to \infty$, or $V \to \bar{V}$, a finite limit. In either case we have

$$e^{\sigma V + \frac{1}{2}\sigma^2}\Phi(-\sigma - V) \leq e^{-\frac{1}{2}V^2}\varphi(V + \sigma)/(V + \sigma) \to 0$$

as $\theta \to \infty$, (since $\sigma \to \infty$). This implies that $0 = \Phi(\bar{V})$, a contradiction in either case. So $V \to -\infty$.

Lemma 5. If $\frac{1}{2}\bar{\sigma}^2 = \bar{\mu}$, then

$$\lim_{\theta\to\infty} \bar{\sigma}\sqrt{\theta}\bar{W}(\theta) = \log 2.$$

Proof. As before suppose $\bar{W}(\theta) \to \bar{w}$. If $\bar{w} < 0$ (possibly infinite), then we have in the limit $\infty = \Phi(-\bar{w})$, a contradiction. If $0 < \bar{w} < \infty$, then by (7.2) we have $0 = \Phi(-\bar{w})$,again a contradiction.This leaves two possibilities: either $\bar{w} = \infty$ or $\bar{w} = 0$.

Suppose the former case arises. Noting that since

$$1 = \lim_{\theta\to\infty}[e^{\bar{\mu}\theta}\Phi(\bar{W}(\theta) - \sigma) + e^{\sigma\bar{W}(\theta)}\Phi(-\bar{W}(\theta))],$$

then $e^{\sigma\bar{W}(\theta)}\Phi(-\bar{W}(\theta))$ is bounded. But

$$\lim_{\theta\to\infty} e^{\sigma\bar{W}(\theta)}\Phi(-\bar{W}(\theta)) = \lim_{\theta\to\infty}\frac{e^{\sigma\bar{W}(\theta)}e^{-\frac{1}{2}\bar{W}^2}}{\bar{W}(\theta)\sqrt{2\pi}} = \lim_{\theta\to\infty} e^{[\bar{W}(\theta)(\sigma - W)]}\frac{e^{+\frac{1}{2}\bar{W}^2}}{\bar{W}(\theta)\sqrt{2\pi}} = \infty$$

by Lemma 4 and by our assumption.

Thus $\bar{w} = 0$. We now obtain

$$\begin{aligned} 1 &= \lim_{\theta\to\infty}[e^{\bar{\mu}\theta}\Phi(\bar{W}(\theta)-\sigma)+e^{\sigma\bar{W}(\theta)}\Phi(0)] \\ &= \lim_{\theta\to\infty}\frac{e^{\frac{1}{2}\bar{\sigma}^2\theta}e^{-\frac{1}{2}\bar{W}^2-\frac{1}{2}\bar{\sigma}^2\theta+\sigma\bar{W}(\theta)}}{(\sigma-\bar{W}(\theta))\sqrt{2\pi}}+e^{\sigma\bar{W}(\theta)}\Phi(0)] \\ &= \lim_{\theta\to\infty}e^{\sigma\bar{W}(\theta)}[\frac{1}{2}-\frac{1}{\sigma\sqrt{2\pi}}]=\frac{1}{2}\lim_{\theta\to\infty}e^{\sigma\bar{W}(\theta)}, \end{aligned}$$

hence our result. The intermediate behaviour between the other two possibilities is not surprising.

Computations. Armed with these lemmas we can now compute $\bar{W}(\theta)$ for large θ. First assume that $\bar{W}(\theta)\to-\infty$. We write

$$W_\varepsilon(\theta)=\frac{\bar{\mu}-\frac{1}{2}\bar{\sigma}^2+\varepsilon}{\bar{\sigma}}\sqrt{\theta},$$

then

$$e^{-\bar{\mu}\theta+\sigma W_\varepsilon(\theta)+\frac{1}{2}\sigma^2}=e^{\varepsilon\sqrt{\theta}},$$

and hence if $\varepsilon>0$ we have for large enough θ that

$$\begin{aligned} e^{-\bar{\mu}\theta-\sigma\bar{W}(\theta)+\frac{1}{2}\sigma^2} &= \Phi(-\bar{W}(\theta))+\varphi(-\bar{W}(\theta))/H(\sigma-\bar{W}(\theta)) \\ &< e^{\varepsilon\sqrt{\theta}} \\ &= e^{-\bar{\mu}\theta+\sigma W_\varepsilon(\theta)+\frac{1}{2}\sigma^2}. \end{aligned}$$

So for $\varepsilon>0$ and large enough θ that

$$-\bar{W}(\theta)<W_\varepsilon(\theta).$$

Now if $\varepsilon<0$ we have for large enough θ that

$$\begin{aligned} e^{-\bar{\mu}\theta-\sigma\bar{W}(\theta)+\frac{1}{2}\sigma^2} &> \Phi(-\bar{W}(\theta)) \\ &> e^{\varepsilon\sqrt{\theta}} \\ &= e^{-\bar{\mu}\theta+\sigma W_\varepsilon(\theta)+\frac{1}{2}\sigma^2} \end{aligned}$$

and again we have for large enough θ that

$$-\bar{W}(\theta)>W_\varepsilon(\theta).$$

Conclusion 1:

$$\bar{W}(\theta) = -\frac{\bar{\mu} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\sqrt{\theta} + o(\sqrt{\theta}), \text{ for } \bar{\mu} > \frac{1}{2}\bar{\sigma}^2.$$

Next suppose that $\bar{W}(\theta) \to \infty$.

We begin by writing

$$e^{-\bar{\mu}\theta + \frac{1}{2}(\sigma - \bar{W}(\theta))^2} = e^{\frac{1}{2}\bar{W}(\theta)^2} e^{-\sigma \bar{W}(\theta) + \frac{1}{2}\sigma^2} \Phi(\bar{W}(\theta) - \sigma) + e^{\frac{1}{2}\bar{W}(\theta)^2} \Phi(-\bar{W}(\theta))$$

or

$$e^{-\bar{\mu}\theta} \varphi(\bar{W}(\theta) - \sigma) = \Phi(-\bar{W}(\theta))/\varphi(\bar{W}(\theta)) + \Phi(-\sigma + \bar{W}(\theta))/\varphi(-\bar{W}(\theta) + \sigma). \quad (7.4)$$

But with $y = \bar{W}(\theta)$, we have

$$\frac{\varphi(y)}{\Phi(-y)} = H(y) \geq y$$

and with $z = \sigma - \bar{W}(\theta)$ we have

$$\frac{\varphi(z)}{\Phi(-z)} \geq z.$$

We thus have for $\bar{W}(\theta) > 0$ and $\sigma - \bar{W}(\theta) > 0$ that

$$\frac{1}{\sqrt{2\pi}} e^{-\bar{\mu}\theta + \frac{1}{2}(\sigma - \bar{W}(\theta))^2} \leq \frac{1}{\bar{W}(\theta)} + \frac{1}{\sigma - \bar{W}(\theta)}. \quad (7.5)$$

But $\bar{W}(\theta) \to +\infty$, and likewise, since $z = \sigma - \bar{W}(\theta) \to +\infty$ (by Lemma 4), so the right-hand side is small.

We now guess the solution and write

$$\bar{W}(\theta) = M\sqrt{\theta}.$$

So provided M is bounded we have from (7.5) that

$$e^{-\bar{\mu}\theta + \frac{1}{2}(\sigma - \bar{W}(\theta))^2} = O(\frac{1}{\sqrt{\theta}}) \quad (7.6)$$

whence we may write

$$-\bar{\mu}\theta + \frac{1}{2}(\sigma - \bar{W}(\theta))^2 = -\ln(\rho\sqrt{\theta})$$

(with ρ bounded) and so

$$\sigma - \bar{W}(\theta) = \sqrt{2\bar{\mu}\theta - \ln \rho^2\theta} = \sqrt{2\bar{\mu}\theta}\{1 - \frac{1}{2}\frac{\ln \rho^2\theta}{\bar{\mu}\theta}\}^{1/2}$$

from which we have:

Conclusion 2:

$$\bar{W}(\theta) = (\bar{\sigma} - \sqrt{2\bar{\mu}})\sqrt{\theta} + O(1/\sqrt{\theta}),$$

for

$$2\bar{\mu} < \bar{\sigma}^2.$$

8. Consequences for the revenue function $\bar{g}(\theta)$

We argue by cases. Recall from (4.3) that

$$\bar{g}(\theta) = e^{(\bar{\sigma}^2-\bar{\mu})\theta}\Phi(\sigma + \bar{W}(\theta)) + e^{-\sigma\bar{W}+(\frac{1}{2}\bar{\sigma}^2-\bar{\mu})\theta}\Phi(-\bar{W}(\theta)).$$

(a) The case $2\bar{\mu} > \bar{\sigma}^2$. Here, for large θ we have

$$\bar{W}(\theta) = -(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)\sqrt{\theta} + o(\sqrt{\theta}).$$

It is immediate that

$$\begin{aligned} \sigma + \bar{W}(\theta) &= \frac{\frac{3}{2}\bar{\sigma}^2 - \bar{\mu}}{\bar{\sigma}}\sqrt{\theta} + o(\sqrt{\theta}), \\ \sigma - \bar{W}(\theta) &= \frac{\frac{1}{2}\bar{\sigma}^2 + \bar{\mu}}{\bar{\sigma}}\sqrt{\theta} + o(\sqrt{\theta}). \end{aligned}$$

Furthermore we have, just as with (7.4), that

$$e^{-\bar{\mu}\theta+\frac{1}{2}\sigma^2-\sigma\bar{W}(\theta)} = e^{-\frac{1}{2}\bar{W}(\theta)^2}\Phi(-\sigma + \bar{W}(\theta))/\varphi(-\bar{W}(\theta) + \sigma) + \Phi(-\bar{W}(\theta)).$$

So by Lemma 4 and since $\bar{W}(\theta) \to -\infty$ we have

$$\lim_{\theta\to\infty} e^{-\bar{\mu}\theta+\frac{1}{2}\sigma^2-\sigma\bar{W}(\theta)} = 1,$$

and in fact we conclude that $e^{-\bar{\mu}\theta+\frac{1}{2}\sigma^2-\sigma\bar{W}(\theta)} = 1 + o(1/\sqrt{\theta})$. Thus we have

$$\begin{aligned} g(\theta) &= e^{(\bar{\sigma}^2-\bar{\mu})\theta}\Phi(\sigma + \bar{W}(\theta)) + \Phi(-\bar{W}(\theta)) + o(1/\sqrt{\theta}) \\ &= e^{(\bar{\sigma}^2-\bar{\mu})\theta}\Phi(\frac{\frac{3}{2}\bar{\sigma}^2 - \bar{\mu}}{\bar{\sigma}}\sqrt{\theta}) + \Phi(\frac{\bar{\mu} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\sqrt{\theta}) + o(1/\sqrt{\theta}). \end{aligned} \tag{8.1}$$

We now use the following asymptotic expansion (see Abramowicz and Stegun (1972)) valid for $x \to +\infty$

$$\Phi(x) = 1 - \frac{e^{-x^2/2}}{x\sqrt{2\pi}}.$$

We note that since $2\bar{\mu} > \bar{\sigma}^2$ we have

$$(\bar{\sigma}^2 - \bar{\mu}) - \frac{1}{2}(\frac{3}{2}\bar{\sigma} - \frac{\bar{\mu}}{\bar{\sigma}})^2 = (\bar{\sigma}^2 - \bar{\mu}) - \frac{9}{8}\bar{\sigma}^2 - \frac{1}{2}\frac{\bar{\mu}^2}{\bar{\sigma}^2} + \frac{3}{2}\bar{\mu} = -\frac{1}{8}\bar{\sigma}^2 + \frac{1}{2}\bar{\mu} - \frac{1}{2}\frac{\bar{\mu}^2}{\bar{\sigma}^2} < 0$$

since

$$\bar{\mu}^2 - \bar{\mu}\bar{\sigma}^2 + \frac{1}{4}\bar{\sigma}^4 = (\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)^2 > 0,$$

so that

$$e^{(\bar{\sigma}^2-\bar{\mu})\theta}\Phi(\frac{\frac{3}{2}\bar{\sigma}^2 - \bar{\mu}}{\bar{\sigma}}\sqrt{\theta}) = e^{(\bar{\sigma}^2-\bar{\mu})\theta} + o(1/\sqrt{\theta}).$$

(b) The case $2\bar{\mu} < \bar{\sigma}^2$. This time

$$\bar{W}(\theta) = (\bar{\sigma} - \sqrt{2\bar{\mu}})\sqrt{\theta} + o(\sqrt{\theta})$$

We have that $\bar{W}(\theta) \to \infty$ and so

$$e^{-\bar{\mu}\theta+\frac{1}{2}(\sigma-\bar{W}(\theta))^2}\Phi(-\bar{W}(\theta))e^{-\frac{1}{2}\bar{W}(\theta)^2} = o(1/\sqrt{\theta})$$

since the first factor tends to zero by (7.6) and the final factor decays exponentially in θ. This leaves us to consider

$$e^{(\bar{\sigma}^2-\bar{\mu})\theta}\Phi((2\bar{\sigma} - \sqrt{2\bar{\mu}})\sqrt{\theta}). \tag{8.2}$$

As before we use the asymptotic expansion for $\Phi(x)$ noted above. We observe that since $\bar{\sigma}\sqrt{2\bar{\mu}} \leq \bar{\sigma}^2$ we have

$$(\bar{\sigma}^2 - \bar{\mu}) - \frac{1}{2}(2\bar{\sigma} - \sqrt{2\bar{\mu}})^2 = (\bar{\sigma}^2 - \bar{\mu}) - 2\bar{\sigma}^2 - \bar{\mu} + \bar{\sigma}\sqrt{2\bar{\mu}} = -\bar{\sigma}^2 - 2\bar{\mu} + \bar{\sigma}\sqrt{2\bar{\mu}} \leq -2\bar{\mu}$$

so that

$$e^{(\bar{\sigma}^2-\bar{\mu})\theta}\Phi((2\bar{\sigma}-\sqrt{2\bar{\mu}})\sqrt{\theta}) = e^{(\bar{\sigma}^2-\bar{\mu})\theta} + o(1/\sqrt{\theta})$$

and so

$$\bar{g}(\theta) = e^{(\bar{\sigma}^2-\bar{\mu})\theta} + o(1/\sqrt{\theta}).$$

We consider the volatility range subdivided into the three intervals $[0,\bar{\mu}]$, $(\bar{\mu}, 2\bar{\mu}), (2\bar{\mu}, \infty)$ and we have the following conclusions:

(i) If $\bar{\sigma}^2 \leq \bar{\mu}$, then by (8.1) we have

$$\bar{g}(\theta) = 1 + o(1/\sqrt{\theta}) \downarrow 1 \text{ as } \theta \to \infty.$$

Here we refer to the Proposition on the Value of Waiting, telling us that $g(\theta) \geq 1$. Note that when $\bar{\sigma}^2 = \bar{\mu}$ the first two terms in (8.1) sum to unity.

(ii) If $2\bar{\mu} > \bar{\sigma}^2 > \bar{\mu}$, then by (8.1) we have

$$\bar{g}(\theta) = 1 + e^{(\bar{\sigma}^2-\bar{\mu})\theta} + o(1/\sqrt{\theta}).$$

(iii) If $2\bar{\mu} < \bar{\sigma}^2$, then by (8.2)

$$\bar{g}(\theta) = e^{(\bar{\sigma}^2-\bar{\mu})\theta} + o(1/\sqrt{\theta}),$$

since $\sigma + \bar{W}(\theta) \to +\infty$ and $\bar{W}(\theta) \to +\infty$

(iv) If $\bar{\sigma}^2 = 2\bar{\mu}$, then $\sigma + \bar{W}(\theta) \to +\infty$. Now

$$\lim_{\theta\to\infty} e^{\sigma\bar{W}(\theta)} = 2$$

and we have

$$\begin{aligned}
\bar{g}(\theta) &= e^{(\bar{\sigma}^2-\bar{\mu})\theta}\Phi(\sigma+\bar{W}(\theta)) + e^{-\sigma\bar{W}+(\frac{1}{2}\bar{\sigma}^2-\bar{\mu})\theta}\Phi(-\bar{W}(\theta)) \\
&= e^{\bar{\mu}\theta}\Phi(\sigma+\bar{W}(\theta)) + e^{-\sigma\bar{W}}\Phi(-\bar{W}(\theta)) \\
&= e^{\bar{\mu}\theta} + \frac{1}{4} + o(1/\sqrt{\theta}).
\end{aligned}$$

Closing Remark. Taking logarithmic derivatives we obtain the following approximation

$$\frac{\bar{g}'(\theta)}{\bar{g}(\theta)} = -(\bar{\mu}-\bar{\sigma}^2) + H(-\bar{W}(\theta)-\sigma)\cdot\{\frac{1}{2\sqrt{\theta}}\bar{\sigma} + \bar{W}'(\theta)\}$$

so if $\bar{\sigma}^2 < \bar{\mu}$ we expect to see a maximal value for $\bar{g}(\theta)$.

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