An Alternative to the Feltham-Ohlson Valuation Framework: Using q-Theoretic Income to Predict Firm Value

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Abstract

In this model we provide a theoretical justi¯cation for why the functional relationship between earnings and value will be non linear. Moreover in our stylized model we derive a closed form for the relationship and show why earnings response coe±cients are lower for ¯rms that are contracting or expanding relative to those ¯rms that are maintaining a steady investment strategy. We extend earlier research which posits a simple convex relationship based upon ¯xed abandonment values and also generalize research which uses real options valuation models based upon the assumption that ¯rms only ever exercise one real investment option and then are committed to that strategy ad in¯nitum. In particular, since in some empirical settings the special case of `¯xed' abandonment will not apply, we show how the form of convexity changes. Secondly, in our model ¯rms are allowed to dynamically change investment strategies, for instance expanding in one period followed by contraction in the subsequent period. Given an objective of deriving comparative statics results for earnings response coe±cients, our dynamic model is able to capture more accurately real investment behavior than a model in which ¯rms only ever decide to expand or contract once. Our model provides both an alternative rationale for accounting measures having information content and an alternative framework for the empirical speci¯cation of tests of `accounting value relevance' based upon ¯nite mixture (regime-switching) distributions. Our model shows how one can view equity value as comprising opening cash, q-revalued opening stock, current q-income and future q-income.

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1 INTRODUCTION

In this section we discuss the established Feltham-Ohlson (FO) valuation model and brie°y review the main ¯ndings and some related research. We argue that, since the FO approach has no transparent role for management, the approach excludes consideration of important real options that typically arise empirically when investment decisions are undertaken. In addition, we present a simple example that shows that the traditional residual income number is not the only accounting measure which admits valuation equivalence to the discounted dividend stream. Another well-known criticism, following Peasnell (1982), of the clean surplus class of models, such as FO, is that the models do not give rise to any structural implications for the application of accounting rules. That is, it may be hard to argue that the models present a justi¯cation for accrual accounting when there is little evidence of the need for accrual adjustments. Exploiting this equivalence type result we show that a di®erent form of residual income valuation does give rise to a reasonably tractable method for analyzing optimal investment decisions and develops an approach to go beyond the general equivalence result and identify a restricted set of accounting measures that meet a certain `axiomatic' property, as follows. When considering candidate earnings numbers with the intention of predicting ¯rm value, we require that as the chosen earnings number increases, this rationally results in higher estimates of future ¯rm value. We thus propose that this simple monotonicity property should be satis¯ed by candidate earnings measures, on the grounds that investors will question any measurement methods of an earnings number for which current higher earnings, can mean lower ¯rm value in the future.

Initially, one may suspect that satisfaction of this seemingly quite mild axiomatic condition will not be particularly discriminating and that many earnings measures will satisfy the axiom. However, interestingly, we ¯nd that the established earning measure used in the literature (residual income) fails to satisfy the axiom and show how an alternative income measure based upon the established q-theory of income does satisfy the axiom. Clearly, with one simple axiom we cannot provide a way to discriminate between all possible earnings measures. However, we suggest that unlike the FO approach which provides no discrimination, our analytical approach is amenable to testing the satisfaction of additional well-speci¯ed axiomatic requirements and so o®ers the ability to re¯ne the number of candidate earnings numbers that satisfy a chosen set of axioms[1].

In addition to providing a theoretical means to discriminate between alternative earnings measures, our approach also contributes to empirical issues. In particular, since our approach is based upon multi-period optimization, we are able to derive comparative statics results which explain in a constructive way, why for instance, earnings response is non linear. In particular we show why for ¯rms that are expanding aggressively, the earnings response coe±cient may be quite low. Since our model is based upon optimizing behaviour we believe we may o®er a superior explanation for the role of earnings in estimating future value in such settings than those researchers who simply conclude that a low earnings response coe±cient may be interpreted

[1] We believe this to be of some importance at this time, given the active debate concerning the overall desirability of comprehensive and other earnings measures.

as evidence of the lack of usefulness of earnings numbers and rush to explore the explanatory power of non ¯nancial performance measures.

1.1 Real Options and the Feltham-Ohlson Model

In our model management need to evaluate real options embedded within typical investment decisions. We review an established model in section two which derives the q-theory of investment in such a setting. In section three we introduce a new investment model in which real options naturally arise and can be solved for optimally. This analysis allows us to make precise statements about expected ¯rm future value and leads us naturally to think about an alternative measure of income based upon q-theory. In section four we then consider how an investor could utilize alternative income measures to forecast future ¯rm value. We show that estimates based on residual income are subject to `hysteresis e®ects', and expected future ¯rm value can take multiple values for a given reported residual income number. We then show that our proposed income measure, q-theory pro¯t, is not subject to this same problem. We subsequently show how residual income can be shown to be equivalent to q-theory pro¯t in a restrictive setting. We present concluding remarks in section ¯ve.

We also note that there exists a number of review papers of the FO (Ohlson (1995) and Feltham and Ohlson (1995)) approach, such as Lo and Lys (2000) and Walker (1997), which thoroughly review the model and provide critiques of the approach. However, having subjected the model to a critique, those papers do not provide constructive alternative valuation approaches. In contrast we try to mount a constructive response to the identi¯ed limitations of the FO approach by developing a new model designed to overcome the lack of a well-de¯ned function for management with respect to project selection in FO. In the following sections we derive a valuation model in which management has a role to play via real options in project selection[2].

The FO model is normally developed by ¯rst recalling a well-known transformation of the traditional discounted future dividend valuation model:

$$\sum_{\tau=1}^{\infty} \gamma^{\tau} E_t(d_{t+\tau}). \tag{1}$$

at date t, where d_t = dividends paid at the end of each period t, $\gamma = (1 + r)^{-1}$ the discount rate and E_t = the expectations operator. Before considering the transformation, there are two natural interpretations of (1). The ¯rst has expectations computed using an equivalent martingale measure for the equity price (a modelling assumption is that such exists on the

[2] To the best of our knowledge only two other authors consider a similar modelling approach. Yee (2000) also incorporates project selection but in a very di®erent way from our model. In Yee ¯rms facing poor returns can switch out of existing projects as other exogenous projects are available. By contrast, in our model we are concerned with the expansion and contraction path of an *investment in place*, that is, the ¯rm does not completely abandon a project when things are bad, they ¯rst need to manage a contraction or later expansion on an ongoing basis. The other paper, much closer to ours in spirit, is Zhang (2000) which is discussed at the end of the subsection.

grounds of no arbitrage opportunities), and then the discount rate r is interpreted as the riskless rate. Alternatively, if the returns on equity W_t are modelled as independently and identically distributed (i.i.d.; assuming such a belief on the part of investors), then the physical probability for the distribution of equity price may be used as an equivalent procedure, in which case the discount rate becomes the constant expected rate of return, and that of necessity is set equal to the `required rate of return' for the given class of risk. Our model is based on the latter premise; that is to say, the model assumes that management control economic activities so that expected return is set equal to the `required rate of return'. The precise significance of this rule is studied in later sections, and involves recognition of elements of irreversibility. The study of such settings through identification of embedded investment call and put options is standard in the real-options approach to investment.

Equation (1) requires a technical assumption[3]. From this equation, and also subject to a similar kind of technicality[4], appealing to the clean surplus identity

$$B_t = B_{t-1} + y_t - d_t \tag{2}$$

(where B_t = book value of equity at t, y_t = earnings at the end of period t) leads to the residual income according to the identity:

$$S_t = \text{ Equity value at time } t = B_t + \sum_{\tau=1}^{\infty} \gamma^\tau E_t(x^a_{t+\tau}), \tag{3}$$

where residual income, or `abnormal earnings' as it is alternatively called, is defined by

$$x^a_t \equiv y_t - rB_{t-1}.$$

The most attractive feature of this approach is that it links valuation to observable accounting data. The ability to re-express (1) in a way that gives accounting centre stage via (3) has been well-known for a considerable time. Ohlson's particular contribution was to set out a specific proposal for how $x^a_{t+\tau}$ evolves. In particular he posited that

$$x^a_{t+1} = \omega x^a_t + x_t + \varepsilon_{t+1}, \tag{4}$$

where $0 \leq \omega$, x_t = value relevant information not yet captured by accounting and ε_{t+1} is a zero-mean disturbance term. In turn he assumed

$$x_{t+1} = gx_t + \eta_{t+1}, \tag{5}$$

where $g < 1$ and η_{t+1} is a zero-mean disturbance term. Together (4) and (5) imply that abnormal earnings follow an AR(1) process. It is apparent immediately that the Ohlson approach

[3] The `no bursting bubble' assumption $\gamma^\tau E[W_\tau] \to 0$ as $\tau \to \infty$ is required here.

[4] Namely: $\gamma^\tau B_\tau \to 0$ as $\tau \to \infty$, i.e. book value does not grow faster than the riskless or required rate of return (whichever is appropriate).

presents an opaque model of management, since nowhere does the Ohlson model consider managerial project selection or opportunities. Similarly the Feltham-Ohlson (FO) extension, which allows for conservative accruals, is silent with respect to project opportunities and the real options that these create. Thus, while the FO approach does establish a dependence of abnormal earnings on book value, it does so via a simple (decision opaque) mechanistic formulation. Lo and Lys (2000) pick up this point and comment in detail on links with the Gordon dividend growth model, pointing out that the assumption of an AR(1) process, although perhaps viewed initially as quite benign, implies very real restrictions on the economic settings in which the FO model can justi¯ably be applied.

Remark 1: The Feltham - Ohlson model is not well suited to applications where ¯rms adopt °exible investment strategies. One of our principal objectives is to derive an alternative model framework which puts at center stage a valuation model based upon ¯rm's period-by-period observed decision on whether or not to expand, contract or maintain investment.

That is, a signi¯cant limitation of the FO approach is that it is essentially a static strategic theory of investment in which once management make an investment they implicitly ignore the type of strategic new investments and divestments opportunities that typically characterize the rich empirical setting in which investment decisions are taken in practice. A central part of our model will be to identify a ¯rm's optimal dynamic investment strategy. That is, in our model we will consider how management dynamically adjust their investment strategy in response to time-varying stochastic conditions. We suggest that our model provides a more natural bridge upon which to structure empirical observations of ¯rms that routinely switch from contracting, shutting down, maintaining or expanding investment projects.

Remark 2: We show that an alternative accounting measure also provides an equivalence to valuation resulting from discounting dividend streams via (1), and furthermore that this alternative measure has a `desirable' feature.

Furthermore, we shall later argue that because of the decision-opaque nature of the FO approach, it is under-speci¯ed in terms of what role-informational asymmetries are being assumed, if any. When the possibility for asymmetries is allowed for, we then suggest one imposes[5] a regularity requirement which provides a simple test for what seems to be a `reasonable property' for an accruals system, namely, that when using an income measure to predict future ¯rm value there exist a functional relationship between the two. We show the FO residual income model may fail this test, and so fails to be a `satisfactory measure' upon which to condition forecasts of future ¯rm value. Again anticipating an argument that will be made more formally

[5] In later sections we shall provide a preliminary consideration of the issue of what constitutes a \good" accounting accruals measurement system. At this early stage we are just highlighting that our methodology can at least lead to some discrimination between alternative accruals processes unlike FO. We stress that at this early stage we are not claiming to be in a position to identify optimality of accruals measurement, simply that we can provide a partial ranking unlike the total inability to provide rankings under the FO approach.

in subsequent sections, this arises because we can show how the FO measure is subject to \hysteresis e®ects". Speci¯cally, we show that given the same level of FO residual income $\tilde{\mathcal{Y}}_t$ for two ¯rms, the prediction of optimal future ¯rm value must be conditioned upon whether the ¯rm is expanding or contracting its investment set. That is, if one ¯rm is expanding while the other is contracting, even though the residual income ¯gures are identical[6], our theory predicts that di®erent valuations be attached to the respective ¯rms. Put di®erently, simple linear extrapolation of future ¯rm value based upon current residual income omits important features central to characterizing the empirical nature of ¯rms' investment settings.

The approach of Zhang (2000) also considers how to revise the FO approach to include real option e®ects. In that respect the initial starting point of his approach and ours is identical. However, the Zhang model is essentially a one shot model in which ¯rms only ever once decide whether to expand, maintain or contract investment[7]. That is, after the one time decision they are locked into that decision ad in¯nitum. In contrast, our model is dynamic in the sense that for instance in three successive periods a ¯rm may expand, contract and then maintain investment. On the surface one may at ¯rst believe that the Zhang approach, although o®ering a simpli¯cation, may be able to capture most of the essential pertinent features of investment behaviour. However, since the model is essentially one shot, empirical issues of coping with over- or under-investment in the previous periods are not captured, that is, the Zhang model is not history dependent. We develop a model that is history dependent in the sense that we introduce an additional variable, opening capital stock, use of which management need to optimise given stochastic input prices. In contrast the Zhang approach depends only upon a stochastic e±ciency factor (which partly mirrors our price variable) while capital stock levels change according to a simple exogenous assumption. Thus at its simplest our model is a two variable investment model (a stochastic price or e±ciency parameter, and a history dependent opening investment stock parameter) whereas the Zhang model considers only the ¯rst variable. In terms of empirical implications our model potentially provides an explanation for why two ¯rms which, according to the Zhang model, would both expand investment may be seen to adopt di®ering maintenance and expansion strategies respectively given that one of them had \over-invested" in the previous period. That is, our approach allows a richer empirical model to be ¯tted to data[8] in which capital stocks, as well as e±ciency (or price variability), have an important explanatory e®ect.

In order to give an initial °avour[9] of our approach, we will introduce a simple two-period

[6]The informal intuition is as follows. Two ¯rms could have the same residual income, with one ¯rm making high revenues and expanding and purchasing signi¯cant additional amounts of capital, while the other ¯rm has only intermediate level revenues but can achieve the same overall pro¯t ¯gure by contracting and running down capital stocks.

[7]Zhang (2000) makes this point clearly in the text arguing that the assumptions are made to insure tractability. Hence one of our contributions is to maintain tractability for a more realistic investment setting in which ¯rms vary their investment startegies through time.

[8]Another important di®erence between our approaches is that rather than our focus upon dynamic optimization, Zhang's focus is upon the links bewteen valuation and `arbitrarily' biased accounting numbers.

[9]Although the di®erence presented in the subsection below may be considered by some readers as small, we actually introduce a far more signi¯cant change in emphasis on income measures away from the traditional

model which illustrates how we choose to account for values in our general model setting.

1.2 An Example of Equivalence with an Alternative Measure of Residual Income

We motivate our discussion by a simple two period model[10]. The returns technology is assumed to follow a simple square-root formulation so that period profit from applying x units of capital into production gives the firm a return of $2\sqrt{x}$. From this the purchase cost of the capital px needs to be deducted in order to determine profit. We shall assume that the firm expects the input price of capital to rise before the next period in which another production decision is taken and the firm actually chooses[11] to commence with $x+u$ units of capital at $t = 0$ purchased at p_0 a unit[12]. The firm plans to use x of the units in the first period and u of the units in the second period with the square-root returns function operating in both periods. Thus:

$$\text{opening net assets } B_0 = p_0(u + x).$$

We compute the two periods' respective earnings and residual incomes under the historic cost convention as:

$$\begin{array}{ll}
B_1 = 2\sqrt{x} + p_0 u & B_2 = (1+r)2\sqrt{x} + 2\sqrt{u} \\
B_1 - B_0 = y_1 & B_2 - B_1 = y_2 \\
y_1 = 2\sqrt{x} - p_0 x & y_2 = 2\sqrt{u} - p_0 u + 2\sqrt{x} r \\
\tilde{y}_1 = y_1 - r p_0(u + x) & \tilde{y}_2 = y_2 - r(2\sqrt{x} + p_0 u) \\
\tilde{y}_1 = 2\sqrt{x} - (1+r)p_0 x - r p_0 u & \tilde{y}_2 = 2\sqrt{u} - (1+r)p_0 u.
\end{array}$$

Note that the revenue $2\sqrt{x}$ included in B_1 is assumed to arise at the end of the first period (i.e. time $t = 1$) for discounting purposes. Since we will want to show valuation equivalence with another method of calculating residual income, we note that under the above historic cost assumptions the value of the firm at time $t = 0$ is given by opening book value plus the sum of discounted (historical) residual incomes:

$$B_0 + \frac{\tilde{y}_1}{1+r} + \frac{\tilde{y}_2}{(1+r)^2}$$

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residual income focus in sections three and four.

[10] This initial model is presented for paedagogic purposes. Many of the most interesting dynamic features are absent so as to first alert the reader's attention to pure accounting valuation issues before formally considering the investment optimality dynamics, which complicate the analysis, but adds important empirical richness to the setting.

[11] Clearly one of the tasks of subsequent sections will be to show, when this is optimal and when it is not, to identify the optimal policy. The intuition here is that given the future value of the stock is expected to increase, the fact that the price is stochastic, means there is an economic value associated with not committing to purchase all resource needs in advance. That is the fact that prices could fall as well as rise leads to some value of waiting.

[12] Assume this is financed by the owners initial equity investment.

$$
\begin{aligned}
&= p_0(u+x) + \frac{2\sqrt{x} - (1+r)p_0 x - r p_0 u}{1+r} + \frac{2\sqrt{u} - (1+r)p_0 u}{(1+r)^2} \\
&= p_0 u + \frac{2\sqrt{x} - r p_0 u}{1+r} + \frac{2\sqrt{u} - (1+r)p_0 u}{(1+r)^2} \\
&= \frac{2\sqrt{x}}{1+r} + \frac{2\sqrt{u}}{(1+r)^2}.
\end{aligned}
$$

Finally the key thing to note from this simple example is that during intermediate periods (e.g. $t = 1$), calculating residual income requires one to keep track of both investment stock used up in the period (x) and investment stock carried forward (u) for future use in some other period, that is:

$$\tilde{y}_1 = 2\sqrt{x} - (1+r)p_0 x - r p_0 u, \qquad \tilde{y}_2 = 2\sqrt{u} - (1+r)p_0 u. \tag{6}$$

Now in contrast, rather than track historic-cost accounting income, as in the F-O framework, we shall instead track current-value accounting income adding an adjustment for per-period holding gains denoted HG (we thus include both realized and unrealized gains). That is, we shall assume that any physical stock valued at u which remains unused during a period is valued at $u(1+r)$ at the end, just as with any (banked) cash receipts generated in the previous period. Thus let us define current value accounting income that incorporates holding gains as:

$$
\begin{aligned}
y_t^{CV} &= (B_t + HG_t) - (B_{t-1} + HG_{t-1}) + d_t \qquad (7)\\
&= B_t^{CV} - B_{t-1}^{CV} + d_t \qquad (8)\\
&\text{where} \\
\tilde{y}_t^{CV} &= y_t^{CV} - rB_{t-1}^{CV} \text{ and } B_t^{CV} = B_t + HG_t.
\end{aligned}
$$

For our setting above, the current-value accounting values are given by:

$$
\begin{array}{ll}
HG_1 = r p_0 u & HG_2 = 2r\sqrt{x} \\
B_1^{CV} = 2\sqrt{x} + p_0 u(1+r) & B_2^{CV} = (1+r)2\sqrt{x} + 2\sqrt{u} \\
B_1^{CV} - B_0^{CV} = y_1^{CV} & B_2^{CV} - B_1 = y_2^{CV} \\
y_1^{CV} = 2\sqrt{x} - p_0 x + p_0 u r & y_2^{CV} = 2\sqrt{u} - p_0 u(1+r) + 2\sqrt{x} r \\
\tilde{y}_1^{CV} = y_1^{CV} - r p_0(u+x) & \tilde{y}_2^{CV} = y_2^{CV} - r(2\sqrt{x} + p_0 u(1+r)) \\
= 2\sqrt{x} - (1+r)p_0 x & = 2\sqrt{u} - (1+r)^2 p_0 u.
\end{array}
$$

Next we note that, under the above current value cost assumptions, the value of the firm at time $t = 0$ is given by opening book value plus the sum of discounted (current-value) residual incomes, which is identical to the above valuation with pure historic costs:

$$B_0 + \frac{\tilde{y}_1^{CV}}{1+r} + \frac{\tilde{y}_2^{CV}}{(1+r)^2}$$

$$
\begin{aligned}
&= p_0(u+x) + \frac{2\sqrt{x} - (1+r)p_0x}{1+r} + \frac{2\sqrt{u} - (1+r)^2p_0u}{(1+r)^2} \\
&= \frac{2\sqrt{x}}{1+r} + \frac{2\sqrt{u}}{(1+r)^2} \\
&= B_0 + \frac{\tilde{g}_1}{1+r} + \frac{\tilde{g}_2}{(1+r)^2},
\end{aligned}
$$

and thus from an investor-valuation perspective at $t = 0$ the two methods are equivalent. However, look at the two current-value residual incomes:

$$\tilde{g}_1^{CV} = 2\sqrt{x} - (1+r)p_0x, \qquad \tilde{g}_2^{CV} = 2\sqrt{u} - (1+r)^2p_0u.$$

Letting

$$b_t = (1+r)p_tx,$$

we see immediately that the current value residual incomes can simply be written as

$$\tilde{g}_1^{CV} = 2\sqrt{x} - b_0x, \qquad \tilde{g}_2^{CV} = 2\sqrt{u} - b_1u, \tag{9}$$

and hence unused stock in each period does not need to be included in the determination of current-value residual income as is the case in (6). It is important to recognize these expressions naturally lead to use of replacement-cost accounting. That is, given that we wish to consider whether intermediate-period residual income is useful for predicting future firm value, we shall find it simpler to characterize current value residual incomes as illustrated in (9).

Remark 3: Like the FO traditional historic cost residual income measure, our current-value residual income measure is equivalent to the discounted dividend stream.

Having shown an alternative decomposition of accounting income, we next return to the issue of the AR(1) process that FO employ. The reason why FO make this assumption in their model is because they need some method to predict how residual income is generated. In contrast to their mechanistic formalization, we assume that residual income results explicitly from firm-based microeconomic optimization. In the dynamic investment setting that we consider here, this corresponds to a requirement of solving for the optimal value function of the firm, which when added to book-value at any point in time, following a stochastic realization of a parameter, provides the appropriate valuation of the firm conditional upon optimal decision making[13]. Thus, provided we can solve for the optimal value function, we can critically appraise the question concerning how well an accounting measure, such as residual income, performs at predicting firm value. Indeed, one can directly refer to the relationship between the accounting-based measure and the optimal value function.

[13] As with the earlier discusion in this section we are trying to maintain an element of intuitive informality before subsequently introducing formal technical arguments.

Given that the identification of the optimal value function underpins our analysis, the following two sections are concerned with developing the optimization procedures required to determine the optimal value function. Section 2 presents a selected overview of a well-known general model which explains most succinctly why the implicit optimization of traditional static investment analyses, such as that of FO, is found to be deficient. The model shows that since the call and put options embedded in investment expansion and contraction options are omitted, these traditional approaches do not form the basis for identification of optimal investment decision making.

Remark 4: Attempting to show empirically how FO residual income relates to expected firm value can be misguided because if managers actually used FO residual income to rank projects, this would imply an element of sub-optimization on the part of managers.

We now turn to consider how to characterize optimal (dynamic) investment behavior.

2 The Real-Options Approach to Investment Valuation

We commence our discussion of the real options approach by briefly reviewing the work of Abel, Dixit, Eberley and Pindyck (1996) -hereinafter referred to as ADEP - which presents an easily accessible introduction to the literature and clearly demonstrates the above-outlined limitation with the FO model. After setting out the ADEP model we discuss various extensions which lead in a natural way to the specification of our alternative model.

In a simple two-period setting the model considers the problem of whether a firm should add to or reduce its opening (first-period) stock of capital K_0 which is purchased at a unit price of b_0. This is to be determined given the following three complications: the future (period one) purchase price of capital b_H may exceed its current price (costly expandability; $b_H > b_0$); the future resale price of capital b_L may be less than its current price (costly reversibility; $b_L < b_0$) and finally second-period revenues from employing capital are stochastic. The stochastic element is introduced as follows[14]. In the first period total revenue from installed capital is $r(K_0)$; in the second period the revenue, denoted $R(K, a)$, has a stochastic component determined by the realization of a. Subsequently in the second period after a has been revealed the firm adjusts the capital stock to a new optimal level denoted $K_1(a)$. Differentiating the revenue function with respect to K, the following two critical values of a are identified:

$$R_K(K_0, a_L) \equiv b_L \qquad \text{and} \qquad R_K(K_0, a_H) \equiv b_H.$$

That is, the optimal (marginal) decision rule is:

- when $a < a_L$ it is optimal to sell capital to the point that $R_K(K_1, a) = b_L$,
- when $a_L \leq a \leq a_H$ it is optimal to neither purchase nor sell capital, that is $K_1(a) = K_0$,

[14] For brevity we are not including details of all the regularity conditions since they can be found in the original text.

- when $a > a_H$ it is optimal to purchase capital until $R_K(K_1, a) = b_H$;

and so the present value of net cash flows $V(K_0)$ accruing to the firm commencing with capital stock K_0 in period zero with inter period discount rate γ, is given by

$$\begin{aligned} V(K_0) &= r(K_0) + \gamma \int_{-1}^{a_L} \{R(K_1(a), a) + b_L[K_0 - K_1(a)]\} dF(a) \\ &+ \gamma \int_{a_L}^{a_H} R(K_0, a) dF(a) + \gamma \int_{a_H}^{1} \{R(K_1(a), a) - b_H[K_1(a) - K_0]\} dF(a). \end{aligned} \tag{10}$$

Thus the period-one decision faced by the firm is

$$K_0 = \arg\max V(K_0) - b_0 K_0,$$

and the Net Present Value Rule can be interpreted from the first-order condition as requiring

$$\begin{aligned} V'(K_0) &\equiv r'(K_0) + \gamma b_L F(a_L) + \gamma \int_{a_L}^{a_H} R'(K_0, a) dF(a) + \gamma b_H [1 - F(a_H)] \\ &= b_0. \end{aligned} \tag{11}$$

This equates the period-one and onwards marginal return to capital to the initial marginal cost; note that the terms after $r'(K_0)$ which take into account the optimal change in capital stock in the following period. An alternative interpretation is also available. ADEP point out that equation (11) can be interpreted using Tobin's q-theory of the marginal value of capital. In this instance the marginal value of capital is

$$q \equiv V'(K_0),$$

and so the optimal investment rule can be identified by management if they determine q.

With respect to implementing this rule ADEP (p 761) comment that this (theoretically correct) rule can be difficult to apply in practice because "for a manager contemplating adding a unit of capital, it requires rational expectations of the path of the firm's marginal return to capital through the indefinite future" and thus in practice the most commonly used proxy for the correct NPV "treats the marginal unit of capital installed in period 1 as if the capital stock is not going to change again". In this case the marginal value of $V'(K_0)$ is approximated by:

$$\tilde{V}'(K_0) \equiv r'(K_0) + \gamma \int_{-1}^{1} R_K(K_0, a) dF(a), \tag{12}$$

and ADEP describe this replacement for the left-hand side of (11) as yielding the naive NPV rule.

At this point it is very helpful to note that the difference between $\tilde{V}'(K_0)$ and $V'(K_0)$ is given precisely by the embedded put and call options present in the problem. To see this we can rewrite (10) as

$$\begin{aligned} V(K_1) &= r(K_1) + \gamma \int_{-1}^{1} R(K_0, a) dF(a) \\ &+ \gamma \int_{-1}^{a_L} \{[R(K_1(a), a) - b_L K_1(a)] - [R(K_0, a) - b_L K_0]\} dF(a) \\ &+ \gamma \int_{a_H}^{1} \{[R(K_1(a), a) - b_H K_1(a)] - [R(K_0, a) - b_H K_0]\} dF(a), \end{aligned} \tag{13}$$

or more succinctly as

$$V(K_0) = \Pi(K_0) + \gamma P(K_0) - \gamma C(K_0), \tag{14}$$

where

$$\begin{aligned}
\Pi(K_0) &\equiv r(K_0) + \gamma \int_{-\infty}^{\infty} R(K_0, a)dF(a), \\
P(K_0) &\equiv \int_{-\infty}^{a_L} \{[R(K_1(a), a) - b_L K_1(a)] - [R(K_0, a) - b_L K_0]\} dF(a), \\
C(K_0) &\equiv \int_{a_H}^{\infty} \{-[R(K_1(a), a) - b_H K_1(a)] + [R(K_0, a) - b_H K_0]\} dF(a);
\end{aligned}$$

here $\Pi(K_0)$ is the expected present value over both periods keeping the capital stock fixed at K_0, i.e. not allowing expansion or contraction of the capital stock. Now

$$P'(K_0) = \int_{-\infty}^{a_L} \{b_L - R'(K_0, a)\} dF(a) = E[\max\{b_L - R'(K_0), 0\}].$$

is the value of a (marginal) put[15] on the marginal product of capital with exercise price b_L corresponding to selling back. Similarly $C'(K_0)$ is the value of a (marginal) call on the marginal product of capital with exercise price b_H:

$$C'(K_0) = \int_{-\infty}^{a_L} \{-b_H + R'(K_0, a)\} dF(a) = E[\max\{R'(K_0) - b_H, 0\}]$$

Thus, given (11), to capture the incentives to invest and divest we can decompose the marginal value into three components:

$$q = V'(K_0) = \Pi'(K_0) + \gamma P'(K_0) - \gamma C'(K_0).$$

Notice that the present value of expansion requires additional outlay (hence the negative term), whereas contraction generates additional income (hence the positive term).

To summarize, in the first period optimality requires management to choose K_0 so that

$$\Pi'(K_0) = b_0 - \gamma P'(K_0) + \gamma C'(K_0). \tag{15}$$

That is, under the naive rule in which management set $\Pi'(K_0) = b_0$, management are ignoring (strategic) option values to contract or expand in the second period and hence typically would choose K_1 suboptimally.

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[15] The put corresponds to the option to reduce the capital stock K_1 by selling k of the existing stock at b_L whenever $a < a_L$. Thus the realized value of the firm when the realization a is below a_L is to first order

$$\begin{aligned}
& r(K_1) + \gamma(R(K_1 - k) - R(K_1) + b_L k) \\
= \quad & r(K_1) + \gamma k(b_L - R'(K_1)).
\end{aligned}$$

Moreover it is straightforward to show[16] that the FO model is an implementation of the naive investment rule which ignores the options to expand and contract available in most real-options settings and hence accounting valuation theory based upon that approach is unlikely to be able to capture how accounting valuation impinges upon the ¯rm's actual dynamic investment strategy (including both expansion and contraction possibilities).

The objective of the next section is to develop a simple model which overcomes this de¯ciency in that management formally need to evaluate options to expand and contract each period and moreover it extends the two period ADEP model to more realistic investment horizons of $N > 2$ ¯nite periods[17]. After setting out the revised ¯nite-horizon investment model, we then return to consider accounting valuation issues in the following section.

3 Optimal Investment by Management: An Endogenous Regime-Switching Model of Investment

Our model speci¯cation is somewhat di®erent from that of ADEP. Before concentrating on the di®ering interpretation over speci¯c variables it is important to establish from the outset that our general methodological goal is also di®erent. Whereas ADEP were able to identify general statements concerning the conditions that optimal investment strategy should satisfy and how that leads one naturally to consider embedded put and call options, they did not actually characterize the functional form for the rewards from adopting an optimal investment strategy. That is, their analysis is not of direct use when trying to assess whether an accounting measure does, or does not, allow users to predict (optimal) future ¯rm value. We depart from their approach by introducing speci¯c functional forms to characterize the basic investment setting with the hope of being able to identify how optimal future ¯rm value depends parametrically upon decision variables that management face.

The following quite technical section shows that within our model speci¯cation we can in fact identify future ¯rm value as the optimal value function for the dynamic investment strategy adopted by management and that this takes a quite intuitive form[18].

Remark 5: In our model setting, future ¯rm value $V()$ is given by the sum of expected future period-by-period (optimized) indirect pro¯ts, plus the valuation of the existing stock of investment at its expected marginal value, which is the q-theoretic income.

Recalling the original Ohlson motivation for introducing an AR(1) process as a means for

[16] See Lo and Lys (2000). The FO approach simply assumes constant expansion (as in the Gordan growth model) rather than period-by-period expansion or contraction as will be allowed for in the model developed below.

[17] This is not the only di®erence between the two models. As we shall see in the following section there are a number of other di®erences, the most signi¯cant perhaps being that, in our model setting, depreciation occurs through use rather than at a constant rate, or alternatively not at all as in the ADEP model.

[18] The precise statement is given towards the end of this section by equation (31).

dealing with the need to model how expectations evolve, it may at ¯rst seem that we too are now in exactly the same situation - needing to impose a model of how expectations, albeit of future ¯rm pro¯tability rather than residual income, evolve over time. Appreciation of how we respond to this point provides the critical conceptual distinction between our approach and that of FO. In particular, working with the indirect pro¯t function[19] we are able to show in this section how the period t (indirect) pro¯t is functionally determined by the most recent observed investment input price b_t. That is, we show that when attempting to form expectations upon future values of the indirect pro¯t function, this requires expectations to be formed over how the stochastic input price b_t evolves. We state our assumption formally in equation (16) below. So have we simply replaced the FO, AR(1) assumption just with some other equally restrictive assumption? We would argue not, for the following reasons. Our distributional assumption is imposed upon an input price process which arises before any managerial action is taken. This is in contrast to Ohlson, who imposes a distributional assumption directly on the evolutionary path of residual income, and hence - as we have seen earlier - this imposes very real constraints upon the implied investment settings where this could logically be assumed to have followed from rational managerial behavior. Expressed alternatively, we would argue that it is less restrictive to impose a distributional assumption on an input than it is to impose one upon an output that results from managerial actions being applied to inputs. To summarize, it is our contention that the necessary distributional assumption that needs to be applied to compute expectations in any model of future ¯rm value, is applied at too late a stage in the model of managerial behavior in the Ohlson approach. Applying the distributional assumption to expected residual income necessarily restricts attention to only a subset of real-world decision scenarios that management may face in practice. For instance, as our earlier discussion makes clear, the FO model simply does not apply in a setting where a ¯rm has good and bad years. By contrast, in our model the `good' or `bad' realizations of the stochastic input price are at centre stage and the evaluation of the induced management's performance is e®ectively in terms of an assessment of their ability to exercise correctly the embedded growth, maintenance and or contraction options that come `into the money'.

Having outlined methodologically what we want to achieve in general terms, let us now turn to the detailed speci¯cation. However, just before doing so, we draw the reader's attention to the fact that there exists a di®erence in our model and that of ADEP in the way in which capital is utilized. In particular we develop a model of (installed) capital in which capital depreciates through use (as directed by management), rather than at a constant rate, or not at all, as in the ADEP model. We make this assumption to allow for the possibility that the net book value of an investment asset after subtracting accumulated depreciation could in principle be equal to the economic value of the asset to the organization. In contrast in the ADEP framework, the asset is assumed never to depreciate. In addition, we extend the investment planning horizon beyond a simple two-period framework to a general ¯nite-horizon setting. In order to introduce the di®erence in speci¯cation as transparently as possible, we ¯rst consider a two-period model variant of the ADEP model.

[19]See for instance Varian (1992) for a discussion of the use of the indirect pro¯t function.

3.1 The two-period model

In reality, firm investment is subject to multiple sources of uncertainty. In the ADEP model, the source of uncertainty is the price of finished output. By contrast, in our model we focus upon the input price of capital as the principle source of uncertainty[20]. Our objective here will be to characterize $V(K_0)$, the optimal value function for capital usage. As we shall see, by making certain functional assumptions for the operating environment, we will be able to go further than ADEP, since not only can we identify equivalent optimality conditions to (14), but moreover we can solve for the conditions once we have derived the functional form for the optimal value function $V(K_0)$.

We now develop our model via direct comparison to the ADEP approach. Simplifying the output-return side[21] we take the time $t = 0$ revenue to be $r(K) = 2\sqrt{K}$ and the time $t = 1$ revenue to be $R(K, a_1) = 2a_1\sqrt{K}$, where $a_1 > 0$ represents the unit sale price of the output at time $t = 1$. To further simplify the analysis, since in our model the input price is the prime source of uncertainty, we shall take $a_1 = 1$. Concentrating upon the source of uncertainty, we shall allow b_1, the input purchase price of capital at time $t = 1$ (corresponding to the constant b_H considered by ADEP), to be stochastic. In addition, we assume that the resale price of the input is $b_1\phi_1$ at time $t = 1$ (instead of b_L in ADEP notation), where the discount factor $\phi_1 < 1$ reflects the partial irreversibility of earlier investment. For clarity of exposition, ϕ_1 is deterministic in this model, but the model can be adapted to allow ϕ_1 to be stochastic. The fractional value of ϕ_1 is assumed to result from the input not being freely tradeable, and this creates a fundamental incompleteness in the specialist capital-input market. This has important implications for the valuation of the firm; the assumptions of the standard martingale approach in real-option theory posit the existence of a `traded twin security' perfectly correlated with the real asset. In our case the real asset is the additional capital, for which the purchase and sale prices diverge at time $t = 1$ by the factor ϕ_1, so that it is no longer possible to hold long and short positions at one price. Furthermore, the `input asset' most definitely has a `convenience yield' on account of its productive value - it is not held purely for trade. We therefore abandon the simple martingale approach[22], and instead adopt the standard `private values' dynamic programming approach[23] for valuation using the physical distribution of the input price b_1.

[20]Our focus here is with capital input hedging possibilities that may exist. For instance see Hopp and Nair (1991). A generalised version of our model in which both the input price and the output price are stochastic is available from the authors. The two sources of uncertainty complicate the analysis by requiring consideration be centered around the ratio of output to the input price without changing the general nature of results substantively.

[21]In general, we need not restrict attention to a square-root formulation: all we need is concavity. The role of the square root specification is to maximize the simplicity of the presentation. The reader should be warned, however, that the Cobb-Douglas revenue function can generate an arbitrarily large return, albeit only for small enough inputs. In general, a revenue function would exhibit a bounded return as input vanishes.

[22]A related situation is that of a four-state model in which prices of a traded asset move up or down and an investor receives a partially correlated preference shock to buy, sell (or even hold). This single-risky-asset model is evidently incomplete but presents two obvious martingales, one for `expansion' and one for `contraction' corresponding to an interpretation of the appropriate buy:sell ratio of the four-state model as a resale discount.

[23]See Dixit and Pindyck (1994) for an extended discusssion of this point.

This is the approach also taken by Abel and Eberley (1995) in their continuous-time infinite horizon model.

Commencing at time t_0, we assume that a firm has $u_0 = u_{t_0}$ ($u_{t_0} \geq 0$) units of capital in stock[24]. Given that the firm can purchase some more capital in the next period, the decision of how to allocate capital stock optimally between the current and latter period will ceteris paribus be driven by the capital input price process. We shall denote the 'one-period-appreciated' [25] price of capital by b_t.

3.1.1 Price of inputs

Although in general we use a sequence of times and corresponding prices that evolve geometrically, the price is nevertheless presented as though it evolves continuously as a geometric Brownian motion. Such an approach is dictated purely by mathematical convenience; the mathematics of optimization is much streamlined by the assumption that at each time, price is distributed continuously rather than multinomially; the presence of interperiod prices is not referred to in any way because we have periodic management decision making. The price b_t has positive drift (anticipated growth) $\mu_b > 0$, and is presented in the traditional stochastic differential form:

$$db_t = b_t(\mu_b dt + \sigma_b dW_b(t)), \tag{16}$$

where $W_b(t)$ is a standard Wiener process. It is assumed that $\gamma\phi_1 e^{\mu_b} < 1$ and that $\gamma e^{\mu_b} > 1$, i.e.

$$\mu_b + \ln\gamma > 0 > \mu_b + \ln\gamma + \ln\phi_1,$$

so that in particular per-period the expected rise in input price rises above the required return on capital and the resale price drops below it. For $t > s$, we let $Q(b_t|b_s)$ denote the (log-normal) cumulative distribution of b_t given b_s and we also let $Q_n(b) = Q(b_{t_n}|b_{t_{n-1}} = 1)$ denote the (log-normal) cumulative distribution of $b_n = b_{t_n}$ given that $b_{t_{n-1}} = 1$. When the context permits, we drop the subscript n.The development of the model depends on the multiplicative nature prices - the distribution of the ratio b_{t+1}/b_t is independent of b_t.

3.1.2 Optimal investment

In the simplest model the manager observes the price at discrete times, in this case at times t_0 and t_1, and can purchase/resell capital at these discrete moments in amounts which we shall denote $z_0 = z_{t_0}$ and $z_1 = z_{t_1}$. In order to track the stock of capital carried forward between periods we shall denote the period t_0 opening capital stock as v_{t_0}, or just v_0, and closing stock

[24]Note u_0 in our notation corresponds to K_0 in ADEP notation. We do not adopt their notation because of the different way in which capital is "consumed" in the two models.

[25]By 'one-period-appreciated', we mean that if the asset is purchased for $p_n = p_{t_n}$ at the commencement of the time interval $[t_n, t_{n+1})$, then the unit opportunity cost of funds tied up in the asset are $p_n(1+r) = b_n$, where b_n stands for $b_{t_n}$and r is the one-period interest rate. Alternatively, one can regard the supplier as rationally recognising that if payment for delivery from stock is to be delayed a period, then the price payable at the end of the period needs to include the cost of funds tied up in inventory.

as u_{t_0}, or just u_0. Let us now consider how to determine the optimal amount of capital u_{t_0} to carry forward to the next period given the amount purchased in the period is unrestricted, so that in this case $z_{t_0} \geq 0$.

The manager now needs to maximize over both z_0 (≥ 0) and x_0 the profit[26]

$$2\sqrt{x_0} - b_0 z_0 + \gamma V_0(v_0 + z_0 - x_0, b_0).$$

Here $V_0(u_0, b_0)$ denotes the optimal future expected value given the current price b_0 and the capital stock carried forward u_0 paid for in a previous period. (Thus V_0 is an increasing concave function). Equivalently, letting $u_0 = v_0 + z_0 - x_0$ we maximize over x_0 and u_0

$$2\sqrt{x_0} - b_0(u_0 + x_0 - v_0) + \gamma V_0(u_0, b_0). \tag{17}$$

Then when choosing optimally the closing stock of capital u_0 the first-order condition (if $u_0 > 0$) from (17) gives:

$$\gamma V_0'(u_0, b_0) = b_0 \tag{18}$$

and[27]

$$x_0 = \frac{1}{b_0^2}, \tag{19}$$

where the prime denotes the derivative $\partial V(u, b_0)/\partial u$. Note that (18) implies that for investment u_0 to be chosen optimally, the unit marginal return needs to be equated to the constant return $1 + r,$ i.e.

$$\frac{V_0'(u_0, b_0)}{b_0} = \gamma^{-1} = (1 + r),$$

and so the return on u_0, namely $[V_0(u_0, b_0) - b_0 u]/b_0 u,$ is greater[28] than $r.$

3.1.3 Optimal divestment

A firm planning to divest, i.e. taking $z_0 < 0,$ faces a similar problem. If the resale discount is ϕ_0 the firm considers the corresponding problem: maximize over both z_0 (< 0) and x_0 the profit[29]

$$2\sqrt{x_0} - \phi_0 b_0 z_0 + \gamma V_0(v_0 + z_0 - x_0, b_0),$$

[26] That is choice of the variables to maximise the sum of current profit plus the optimal value function $V(.)$ reflecting future optimal period payoffs.

[27] This very simple nature of this result is why we utilise the square-root specification.

[28] To see this fix $b > 0$ to be any price at time $t = 0$ and r an interest rate. Let the non-negative, concave, differentiable function $g(u)$ represent a deterministic value receivable at time $t = 1$ and assume that $\lim_{u \to \infty} g'(u) < b(1 + r)$. Let u^* maximise the profit $g(u) - b(1 + r)u$. Define the rate of return on $g(u)$ to be $R(u)$, where $1 + R(u) = g(u)/(bu)$. Evidently if $\hat{u} > 0$ satisfies $g(\hat{u}) = (1 + r)b\hat{u}$ then $u^* < \hat{u}$ and $R(\hat{u}) = r$. Now the rate is decreasing for $u < \hat{u}$; indeed $bR'(u) = -g^{\#}(u)/u^2$ and $g^{\#}(u)$ is an increasing function (since $D_u g^{\#}(u) = -g''(u) \geq 0$), but $g^{\#}(\hat{u}) = g(\hat{u}) - \hat{u}g'(\hat{u}) = \hat{u}[b(1 + r) - g'(\hat{u})] > 0$. Hence $R(u^*) > r$. Notice that $R(0+)$ is either unbounded (if $g(0) > 0$) or $g'(0+)/p$.

[29] That $\phi < 1$ is standard in the literature, otherwise if $\phi = 1$ we would have the possibility of simple portfolio rebalancing.

or equivalently, letting $u_0 = v_0 + z_0 - x_0$,

$$2\sqrt{x_0} - \phi_0 b_0(u_0 + x_0 - v_0) + \gamma V_0(u_0, b_0).$$

Thus the first order condition for u_0 (again assuming $u_0 > 0$) is

$$\gamma V_0'(u, b_0) = \phi_0 b_0, \tag{20}$$

and for x_0 is

$$x_0 = \frac{1}{\phi_0^2 b_0^2}. \tag{21}$$

3.1.4 Tobin's q and normalized inputs

We return to the investment version and let $u = \widehat{u}(b_0)$ denote the solution to equation (18).

Remark 6: Formal identification of the two-period optimal value function shows it is made up of three components conditioned upon whether the firm is expanding, maintaining or contracting investment.

We note that in our two-period model we have

$$\begin{aligned} V_0(u|\phi_1, b_0) \quad = \quad & \int_0^{\widetilde{b}_L} (\frac{1}{b_1} + b_1 u) dQ(b_1|b_0) + 2\sqrt{u} \int_{\widetilde{b}_L}^{\widetilde{b}_L/\phi_1} dQ(b_1|b_0) \\ & + \int_{\widetilde{b}_L/\phi_1}^{\infty} (\frac{1}{\phi_1 b_1} + \phi_1 b_1 u) dQ(b_1|b_0), \end{aligned} \tag{22}$$

where ϕ_1 is the resale rate for the second period and $\widetilde{b}_L = 1/\sqrt{u}$. The three integrals classify investment by the corresponding three input price policy ranges[30] according to the ranges of integration, as follows:

(U) The under-invested range ($0 \leq b_1 \leq \widetilde{b}_1$), in which additional investment in capital is made. Here as in (17) one maximizes over $x_1 \geq 0$ the second-period profit

$$2\sqrt{x_1} - b_1 x_1$$

with required input $x_1 = 1/b_1^2$ made available through the purchase of $x_1 - u$ at a price b_1 and net revenue $2/b_1 - b_1(x_1 - u) = 1/b_1 + b_1 u$. Clearly the extreme case is zero purchase when $u = 1/b_1^2$, whence the limit of integration $b_1 = \widetilde{b}_L$.

[30] Equivalent to the three output price ranges in the ADEP model.

(IO) The (endogenously) irreversible[31] over-invested range ($\tilde{b}_L \le b_1 \le \tilde{b}_L/\phi_1$), where all remaining capital (excess from period 0) is optimally applied between current and future production.
(RO) The reversible over-investment range ($\tilde{b}_1/\phi_1 \le b_1 \le \infty$), where some excess capital is resold. Here one maximizes over $x_1 \le u$ the second-period profit

$$2\sqrt{x_1} + \phi_1 b_1 (u - x_1)$$

obtained by reselling an amount $u - x_1$ of the capital stock. The required input is $x_1 = 1/(\phi_1 b_1)^2$, yielding net revenue $2/(\phi_1 b_1) + \phi_1 b_1 (u - x_1) = 1/(\phi_1 b_1) + \phi_1 b_1 u$. The extreme case is $u = 1/(\phi_1 b_1)^2$, giving the limit of integration $b_1 = 1/(\sqrt{u}\phi_1)$. Thus

$$V_0'(u, b_0, \phi_1) = \int_0^{\tilde{b}_L} b_1 dQ(b_1|b_0) + \tilde{b}_L \int_{\tilde{b}_L}^{\tilde{b}_L/\phi_1} dQ(b_1|b_0) + \int_{\tilde{b}_L/\phi_1}^{\infty} \phi_1 b_1 dQ(b_1|b_0).$$

In general, the resale factor ϕ_1 will not be known at time t_0 and so one should take expectations over $\phi_1$leading to an average version $\bar{V}_0'(u, b_0)$ of $V_0'(u, b_0, \phi_1)$. For presentational purposes we will usually avoid this additional expectation and pretend ϕ_1 is deterministic.

A critical interpretation of the marginal value V_0' of capital is now possible with reference to Tobin's q. Consider a policy of investment triggered by input prices below a threshold level of B. The average marginal benefit of such a strategy corresponds to the value of Tobin's marginal q. This we may compute from the last formula by writing B in place of $\tilde{b}_L$ (so by implication we are setting $B = 1/\sqrt{u}$), obtaining the function:

$$q(B, b_0) = \int_0^{B} b_1 dQ(b_1|b_0) + B \int_{B}^{B/\phi_1} dQ(b_1|b_0) + \int_{B/\phi_1}^{\infty} \phi_1 b_1 dQ(b_1|b_0).$$

At this point it is important to note that our assumption of a Cobb-Douglas type technology gives rise to the following **homogeneity property**:

$$q(B, b_0) = b_0 q(B/b_0, 1),$$

which we will wish to apply. An inductive argument shows that this homogeneity property extends to all periods in the context of a Cobb-Douglas production function (see Appendix D).

The function $q_0(B) =_{def} q(B, 1)$ is of course **Tobin's marginal quotient**, q, namely, the expected future return on an additional unit of capital measured in ratio to the market value (replacement cost) of the additional capital. This motivates our notation. Indeed we have

$$\lim_{h \to 0} \frac{V_0(u + h, b_0) - V_0(u, b_0)}{h b_0} = \frac{1}{b_0} V'(u, b_0) = \frac{q(B, b_0)}{b_0} = q_0(1/(b_0 \sqrt{u})).$$

[31]Endogenous in the sense that though reversal is possible, it is never optimal in this setting to choose it and hence the firm acts as if the situation was irreversible.

We note that the expression $1/(b_0\sqrt{u})$ is likewise a marginal quotient: $f'(u)/b_0$; is it is the marginal return of an investment u if it were currently consumed in production ('current' q rather than future q).

For a further insight into this equation, observe an important second homogeneity property (true here by inspection, but preserved also in a multi-period setting, as we show in Appendix E), namely that:

$$V_0(u, b_0) = \frac{1}{b_0} V_0(ub_0^2, 1).$$

A parallel derivation of the marginal return on investment starts from the remark that

$$\frac{V_0(u+h, b_0) - V_0(u, b_0)}{b_0 h} = \frac{V_0((u+h)b_0^2, 1) - V_0(ub_0^2, 1)}{b_0^2 h},$$

and so, if we put $\tilde{u} = ub_0^2$, we see that Tobin's marginal quotient is

$$\lim_{h \to 0} \frac{V_0(u+h, b_0) - V_0(u, b_0)}{b_0 h} = \lim_{\tilde{h} \to 0} \frac{V_0(\tilde{u}+\tilde{h}, 1) - V_0(\tilde{u}, 1)}{\tilde{h}} = \frac{\partial V_0(\tilde{u}, 1)}{\partial \tilde{u}}.$$

The transformation $B = 1/\sqrt{\tilde{u}}$ (noted earlier) shows the latter quotient to be $q_0(B) = q_0(1/\sqrt{\tilde{u}}) = q_0(1/(b_0\sqrt{u})$. The change of variable used here, $\tilde{u} = ub_0^2$, is natural, as it arises from solving the equation $f'(\tilde{u}) = f'(u)/b_0$ in which the input quantity u is scaled to $\tilde{u}$ - and has the equivalent marginal return corresponding to a unit input price. We will refer to $\tilde{u}$ as a normalized input quantity.[32]

The behavior of $q(B, 1)$ is indicated in Figure 1: a strictly increasing function (a property that is characteristic for multiple period models also)[33].

Place Figure 1 Here

3.1.5 Censor equation

The importance of the function q_0 stems from the induced decomposition of the solution of (18) into two steps. The first step is to solve for $\hat{\phi}_1$

$$\gamma q(\hat{\phi}_1, b_0) = b_0, \tag{23}$$

and the second is to solve $\hat{\phi}_1 = \hat{\phi}_L = 1/\sqrt{u}$ for u, to obtain

$$\hat{u}(b_0) = 1/(\hat{\phi}_1)^2.$$

[32] In the general Cobb-Douglas case the transformation of a quantity v to its normalization is given by $\tilde{v} = vb_0^{1/\alpha}$.

[33] We note that $q_0'(B) = \int_B^{B/\phi_1} dQ(b_1) > 0$. A similar calculation is shown later for the multiperiod q_n.

We call (23) the censor equation. The solution exists and is unique if and only if $\inf_{b_1} q(b_1, b_0) < (1 + r)b_0 < \sup_{b_1} q(b_1, b_0)$, that is[34],

$$E[\phi_1]E[b_1|b_0] < (1 + r)b_0 < E[b_1|b_0],$$

and, since we have assumed for simplicity that ϕ_1 is deterministic, this amounts to

$$\phi_1 E[b_1|b_0] < (1 + r)b_0 < E[b_1|b_0].$$

This is a proviso that while (discounted) prices are expected to rise the resale price is nevertheless expected to fall[35], a condition that is akin to absence of arbitrage opportunities. We assume this to hold. We call the value of the price b_1 given by $\hat{b}_1$, i.e. solving (23) above, the censor. Clearly $\hat{b}_1$ is a function of μ_b, σ, γ.

It may be shown that

$$\hat{b}_1(b_0) = b_0 \hat{g}_1(1 + r), \tag{24}$$

where $\hat{g}_1$ (the dynamic multiplier factor) is a function of $\mu = \mu_b + \ln\gamma$ and of σ. The intuition for this result may be traced to the fact that in our model the price b_1 is log-normally distributed, so that $\ln b_1$ has mean $\ln b_0 + (\mu_b - \frac{1}{2}\sigma^2)$; it thus makes sense to scale price b_1 not only by b_0 but also by the compounding factor $(1 + r)$.

To see why the result is valid note that, since $q(\hat{b}_1, b_0) = b_0 q(\hat{b}_1/b_0, 1)$, the censor equation may be written in equivalent form as

$$\gamma q(B, 1) = 1,$$

where $B = \hat{b}_1/b_0$. Shifting the drift from μ_b to $\mu = \mu_b + \ln\gamma$ (which is positive, by the assumption that $\gamma e^{\mu_b} > 1$) and letting the function corresponding to $q(B, 1)$ for this drift be denoted by $\tilde{q}(G, 1)$, we have[36]

$$q(B, 1) = (1 + r)\tilde{q}(\gamma B, 1).$$

Now we may simplify the equation to

$$1 = \gamma q(B, 1) = (1 + r)\gamma\tilde{q}(\gamma B, 1),$$

[34] If we assume the resale rate is independent of the sale price, then $\inf_B q(B, 1) = E[\phi_1]$.

[35] For simplicity we assume that ϕ_1 is independent of b_1; as the inter-period is assumed to be unity, we have $E[b_1|b_0] = e^{\mu}$ and the condition amounts to $\gamma e^{\mu} > 1 > \gamma E[\phi_1]$.

[36] Proof: Writing $b_1 = (1 + r)g_1$, $B = (1 + r)G$ and $\tilde{Q}(g_1) = Q((1 + r)g_1|1)$ we have

$$\begin{aligned} q(B, 1) &= q((1 + r)G, 1) = \int_0^G (1 + r)g_1 d\tilde{Q}(g_1) + (1 + r)G \int_B^{\psi B} d\tilde{Q}(g_1) + \int_{\psi B}^1 \phi_1(1 + r)g_1 d\tilde{Q}(g_1) \\ &= (1 + r)[\int_0^G g_1 d\tilde{Q}(g_1) + G \int_B^{\psi B} d\tilde{Q}(g_1) + \int_{\psi B}^1 \phi_1 g_1 d\tilde{Q}(g_1)] \\ &= (1 + r)\tilde{q}(G, 1) = (1 + r)\tilde{q}(\gamma B, 1). \end{aligned}$$

and so the final form of the equivalent censor equation reads

$$\tilde{q}(g, 1) = 1,$$

where $g = \gamma B = \gamma \hat{b}_1 / b_0$. If we denote the solution of this last equation by $\hat{g}_1$ then this quantity is evidently a function of μ and σ and we have as claimed

$$\hat{b}_1 = b_0 \hat{g}_1 (1 + r).$$

Thus, in particular

$$\hat{b}(b_0) = \frac{\gamma^2}{(\hat{g}_1 b_0)^2}. \tag{25}$$

It is of interest to point out that there is a critical value of $\phi_1 = \phi_{crit}$ for which it is the case that[37] $\hat{g}_1(1 + r) = 1$, and so $\hat{g}_1(1 + r) > 1$ iff $\phi_1 < \phi_{crit}$. In the case that $\hat{g}_1(1 + r) > 1$ the advance purchase is lower than the current demand.

The corresponding problem for divestment calls for the solution of

$$\gamma q(\hat{b}_{\phi_0}, b_0) = \phi_0 b_0,$$

$$\gamma q(\hat{b}_{\phi_0}/b_0, 1) = \phi_0, \tag{26}$$

and this will have a solution if and only if

$$\phi_0 > \inf_B \gamma q(B, 1) = \gamma E[\phi_1],$$

or, if the discount factor ϕ_1 is assumed deterministic, exactly when

$$\phi_0 > \inf_B \gamma q(B, 1) = \gamma \phi_1.$$

The intuition is simple: if there is no solution, then there is no resale possible in that period[38]. Here again we note that

$$\phi_0 = \gamma q(\hat{b}_{\phi_0}/b_0, 1) = (1 + r)\gamma \tilde{q}(\gamma \hat{b}_{\phi_0}/b_0, 1),$$

so that

$$\hat{b}_{\phi_0} = (1 + r) b_0 \hat{g}_1(\phi_0) \phi_0,$$

where $\tilde{q}(\phi_0 \hat{g}_1(\phi_0), 1) = \phi_0$.

[37] Regarding q as a fuction of ϕ_1 we see that for B fixed q or $\tilde{q}$ is increasing in ϕ_1 as e.g. $dq/d\phi_1 = E[b_1|1]$.Note that now for $\phi_1 = 1$ we have $q(B, 1) = E[b_1|1]$ and for $\phi_1 = 0$ we have the irreversible case for which evidently it is the case that $\hat{g}_1(1 + r) > 1$.

[38] If we assume the resale rate is independent of the sale price, then $\inf_B q(B, 1) = E[\phi_1]$.

3.1.6 The embedded options

Comparing (10) and (22), we can make the same re-arrangement as ADEP, to give

$$\begin{aligned} V_0(u,\phi_1,b_0) &= 2\sqrt{x(b_0)} + \gamma[2\sqrt{u}\int_0^\infty dQ(b_1|b_0) \\ &\quad + \int_0^{\tilde{b}_L}(\frac{1}{b_1} + b_1 u - 2\sqrt{u})dQ(b_1|b_0) + \int_{\tilde{b}_L/\phi_1}^\infty(\frac{1}{\phi_1 b_1} + \phi_1 b_1 u - 2\sqrt{u})dQ(b_1|b_0)]. \end{aligned}$$

Thus re-defining their notation - rather than introducing new notation (since we will not use their representation again) - we have similarly to (14)

$$V_0(u) = \tilde{V}_0(u|b_0) - \gamma P(u,\tilde{b}_1|b_0) + \gamma C(u,\tilde{b}_1|b_0),$$

where

$$\begin{aligned} \tilde{V}_0(u|b_0) &\equiv 2\sqrt{x(b_0)} + \gamma 2\sqrt{u}\int_0^\infty dQ(b_1|b_0), \\ P(u,\tilde{b}_1|b_0) &\equiv \int_0^{\tilde{b}_L} 2\sqrt{u} - (\frac{1}{b_1} + b_1 u)dQ(b_1|b_0), \\ C(u,\tilde{b}_1|b_0) &\equiv \int_{\tilde{b}_L/\phi_1}^\infty (\frac{1}{\phi_1 b_1} + \phi_1 b_1 u - 2\sqrt{u})dQ(b_1|b_0), \end{aligned}$$

with $\tilde{b}_L = 1/\sqrt{u}$, and where, just as before, $\tilde{V}_0(\tilde{b}_1)$ is the expected present value over both periods keeping the capital stock carried forward fixed at u. (Note that in view of the reciprocal relation between the a and b variables, the put and call have switched roles vis à vis ADEP.)

As before, looking at the first-order conditions[39] we have, now writing $\tilde{b}_1$ for $\tilde{b}_L$,

$$\begin{aligned} V_0'(u|\phi_1,b_0) &= \int_0^{\tilde{b}_1} b_1 dQ(b_1|b_0) + \tilde{b}_1\int_{\tilde{b}_1}^{\tilde{b}_1/\phi_1} dQ(b_1|b_0) + \\ &\quad + \phi_1\int_{\tilde{b}_1/\phi_1}^\infty b_1 dQ(b_1|b_0) \\ &= \tilde{b}_1\int_0^\infty dQ(b_1|b_0) - \int_0^{\tilde{b}_1}(\tilde{b}_1 - b_1)dQ(b_1|b_0) \\ &\quad + \phi_1\int_{\tilde{b}_1/\phi_1}^\infty (b_1 - \tilde{b}_1/\phi_1)dQ(b_1|b_0) \\ &= \tilde{b}_1 - E[\max(\tilde{b}_1 - b_1, 0)] + E[\max(\phi_1 b_1 - \tilde{b}_1, 0)] \qquad (27) \\ &= \tilde{V}_0'(\tilde{b}_1|b_0)/\gamma - P'(\tilde{b}_1|b_0) + C'(\tilde{b}_1|b_0) \\ &= q. \end{aligned}$$

39 With due consideration for the Leibniz Rule.

Comparison of (27) and (15) yields the key insight that the firm should evaluate the embedded investment call and put options with strike price given by the censor. In this respect the censor $\tilde{b}_1$ determines the effective `future' unit price (effective expected next-period price) of inputs, and thus delivery at that price requires the planner to: (i) receive compensation / revenue against that price for surrender of expansion potential, and (ii) pay additionally to that price a compensation / cost for the right of contraction potential[40].

Remark 7: The optimal investment rule is determined by evaluating the optimal investment or divestment such that the marginal benefit of capital (Tobin's q) is equal to the naive NPV together with the value of the marginal (short) put and (long) call options which have a strike price given by the optimally chosen censor.

3.2 Generalizing to $n > 2$ Periods and an Alternative to Applying Equivalence Between Residual Income and Discounted Dividends: q-theoretic Profit and Discounted Dividends

We now generalize the above simple two-period model for $n > 2$ and derive an alternative to the FO residual income valuation equation. The equivalence (3) between discounted dividend streams and residual income is only one of the possible equivalence relationships that could be used to demonstrate a role for accounting values in predicting future value. One of our contributions is to identify another equivalence relationship, namely (34) or (35), where residual income ceases to be the main focus for valuation. As we shall see when we derive the functional form for the optimal value function, it becomes natural to consider replacing residual income by a measure of `indirect profit', which can be interpreted as `optimal operating profit before

[40] Alternative interpretation: The naive non-linear view is that one unit of capital next period will be worth $\tilde{b}_1$ and leads to an inventory of $1/\tilde{b}_1^2$ but the marginal valuation ignores the present value of the option to expand when it is cheap to do so (i.e. $b_1 < \tilde{b}_1$) and this will call for extra outlay (hence the negative sign of this PV) and also ignores the option to contract when $b_1 > \tilde{b}_1/\phi$ so that it is worth selling for ϕb_1which brings in extra income. It is possible to use put-call symmetry (parity) to obtain

$$
\begin{aligned}
F_1^0(u,\phi,b_0) &= \int_0^{\tilde{b}_1} b_1 q(b_1|b_0)db_1 + \tilde{b}_1 \int_{\tilde{b}_1}^{\tilde{b}_1/\phi} q(b_1|b_0)db_1 + \\
&\quad + \phi \int_{\tilde{b}_1/\phi}^{1} b_1 q(b_1|b_0)db_1 \\
&= E[b_1] - \int_{\tilde{b}_1}^{\tilde{b}_1/\phi} (b_1 - \tilde{b}_1) q(b_1|b_0)db_1 + \\
&\quad - (1-\phi) \int_{\tilde{b}_1/\phi}^{1} b_1 q(b_1|b_0)db_1.
\end{aligned}
$$

This may be interpreted as comprising first the naive expected value of holding one unit of stock, secondly short one limited call (operable in a limited range), and finally $(1-\phi)$ units short of an asset-or-nothing option.

extraordinary items', and which we call `q-income' as defined below in this section. (See section 4.2 for its significance.) The future value is then a discounted sum of the future periods' `q-income'.

Remark 8: Our model of optimal investment choice by management requires consideration of the firm's indirect profit which within this setting we describe as `normal operating profit' regarded as optimal profit before extraordinary items[41].

In the next section we will compare the future-value prediction algorithms based upon our q-theoretic operating profit measure, to those based upon residual income. Let us now turn to introduce the new equivalence result.

We adopt the following notational assumption in order to minimize the use of subscripting. If at the end of period $t-1$ we have u_{t-1} capital stock left over for the commencement of production in period $t,$ we denote the capital stock at commencement of new production by

$$u_{t-1} = v_t.$$

When the period of analysis is unambiguous we shall drop the time subscript and simply refer to opening stock v and closing stock u for the period under consideration.

3.2.1 General optimal marginal value formula V_n'

Applying this simplified notation the following general characterization is then possible: for each n and corresponding time t_n there exists a `capital investment / carry-forward function'

$$u(v,b) = u_n(v,b), \tag{28}$$

which solves the equation

$$(v - u(v,b_{n+1}))^{-1/2} = \gamma V_{n+1}'(v,b_n,\phi_{n+1})$$

and an input price censor function $b(v) = b_n(v)$ and a constant $\psi = \psi_n,$ such that

$$\begin{aligned} V_n'(v,b_n,\phi_{n+1}) = & \int_0^{b(v)} b_{n+1}dQ(b_{n+1}|b_n) + \int_{b(v)}^{\psi b(v)} (v - u(v,b_{n+1}))^{-1/2}dQ(b_{n+1}|b_n) \\ & + \int_{\psi b(v)}^{\infty} \phi_{n+1}b_{n+1}dQ(b_{n+1}|b_n). \end{aligned}$$

Assuming a general concave revenue function $f(x)$ in place of the square-root form, the presence of an additional period of production, moves the exercise price (trigger) down. Here is the intuition: the provision for the future is the greater the further the horizon, but the trigger

[41] In our model setting the only extraordinary item is the opportunity gain or loss from purchasing the investment stock in advance.

varies inversely with quantity so the the trigger is smaller the further the horizon; at the same time the manager is less likely to sell stock back at a discount if he / she has the option to use that same stock at a later date. The general formula, though daunting, is not much different[42]. Assuming $u_{n-1} = v_n$ is carried into the future at time t_{n-1} we have:

$$
\begin{aligned}
V'_{n-1}(v_n|\phi_n, b_{n-1}) &= \bar{B}_n - E[\max(\bar{B}_n - b_n, 0)] + E[\max(\phi_n b_n - \bar{B}_n, 0)] \\
&\quad - \int_{\bar{B}_n}^{\bar{B}_n/\phi_n} (\bar{B}_n - f'(x_n(v_n, b_n)))dQ(b_n|b_{n-1}) \\
&\quad + \int_{\bar{B}_n/\phi_n}^{h_n(\bar{B}_n,\phi_n)} (f'(x_n(v_n, b_n)) - \bar{B}_n)dQ(b_n|b_{n-1}),
\end{aligned}
$$

where the first line refers to a strategy of not carrying forward capital (with put and call options referring to expansion and contraction), whereas the lines following refer to option values resulting from carrying stock forward. Here $\bar{B}_n = b_n(v_n, \phi_n)$ is the price at which management at tiem $t = t_n$ is indifferent between carrying-forward stock and selling stock off, while $x_n(v_n, b_n)$ is the optimal demand in period n for input, given a stock v_n of input and current input price of b_n. Here again for simplicity we have assumed ϕ_n is deterministic. The carrying-forward option in the displayed formula exists in a range from $\bar{B}_n$ to $h_n(\bar{B}_n, \phi_n)$ where the function $h_n(B, \phi)$ is the solution to the simultaneous equations

$$h_n(B, \phi) = b_n(v_n, \phi), \qquad B = b_n(v_n, 1), \tag{29}$$

and is further split into two intervals by reference to the point $\bar{B}_n/\phi_n$. In the Cobb-Douglas case the form of the function $h_n(\bar{B}_n, \phi_n)$ is determined in Appendix D.

3.2.2 General form of optimal future value V_n

Generalizing the two-period model we can then show[43] that the optimal future expected value is given by a formula incorporating three expected values according to which of its three options - investment, divestment or mere partitioning of its capital stock between current and future use - the firm uses. Next we need some notation to denote the choice of the optimal current-period production plan. Letting $G(b_n) = (f')^{-1}(b_n)$ represent the internal optimal demand for input that maximizes $f(x) - b_n x$ over x, then the exact form of the formula is (see Appendix A for more detail)

$$
\begin{aligned}
&V_{n-1}(v, b_{n-1}) = \\
&\int_0^{b_n(v,1)} [f(G(b_n)) - b_n G(b_n) + b_n(v - \bar{b}_n(1, b_n)) + V_n(\bar{b}_n(1, b_n), b_n)]dQ_n(b_n)
\end{aligned}
$$

[42] The form of the optimal solution changes as we change the number of periods. As we increase the number of periods, this increases the range of inactivity since, with more periods to follow (i.e. to act on the volatility), the chance of eventually experiencing sufficiently good demand conditions to use up existing "excess" stock increases; correspondingly the benefit of selling it at a discount is commensurately reduced.

[43] Technical details are available from the authors upon request.

$$+\int_{b_n(v,1)}^{b_n(v,\phi_n)} [f(v - u_n(v,b_n)) + V_n(u_n(v,b_n),b_n)]dQ_n(b_n)$$
$$+\int_{b_n(v,\phi_n)}^{1} [f(G(\phi_n b_n)) - \phi_n b_n G(\phi_n b_n) + \phi_n b_n(v - \hat{b}_n(\phi_n,b_n)) + V_n(\hat{b}_n(\phi_n,b_n),b_n)]dQ_n(b_n).$$

Here $b_n(v,1)$ replaces $b(v)$ while $b_n(v,\phi)$ replaces $\psi b(v)$, whereas v is the opening stock, ϕ_n the resale (discount) factor for the next period, $\hat{b}_n(1,b_n)$ is the optimal carry-forward into the following period when investing, $\hat{b}_n(\phi_n,b_n)$ is the optimal carry-forward when divesting, and $u_n(v,b_n)$ is the optimal carry-forward in the absence of investment or divestment. Under the integral signs we see period n production income, future costs of additional investment, or future income from divestments, given that prior period costs incurred purchasing stock are charged to the period in which the stock was acquired[44].

The first two terms on the right, namely $f(G(b_n)) - b_n G(b_n)$, merit particular attention. Here $G(b_n) = (f')^{-1}(b_n)$ is an internal optimal demand for input that maximizes $f(x) - b_n x$ over x; let us denote it temporarily by x_n. Since $b_n = f'(G(b_n)) = f'(x_n)$ we see that the indirect profit $f(G(b_n)) - b_n G(b_n)$ can also be written as $f^{\#}(x_n)$, where[45]

$$f^{\#}(x) := f(x) - xf'(x). \tag{30}$$

3.2.3 Future value as q-income stream

Since we will be comparing the ability of different income measures to forecast future firm value, we shall refer to our new indirect income measure $f^{\#}(x)$ with a y-variable notation. This is in order to follow traditional notation for income. Specifically, we set

$$Y^q(x^*(b)) =_{def} f^{\#}(x^*(b)).$$

An inductive application of the recurrence formula for $V(.)$ (shown earlier), coupled with some re-arrangements of the other terms, yields the following identity in terms of indirect profits for the undiscounted optimal future value of the project given a carried forward capital stock u_n. The details are given in Appendix C. That is, instead of working with the equivalence between (3) and (1) we consider the equivalence between (1) and :

$$V_n(u_n|b_n) = q_n u_n + E[\sum_{m=n+1} \gamma^{m-n-1} Y^q(x^*_m)], \tag{31}$$

On the right-hand side we sum the closing capital stock u_n evaluated at Tobin's q, plus the sum of all future indirect profits, where:

[44] Thus the total value of the firm in time t_n money must add to the given formula the cash position which includes past income and deductions of the historic cost of stock v suitably compounded.See later.

[45] Thus the function $\tilde{f}(b_n) = f^{\#}(G(b_n))$ is the Fenchel dual of f. However, we are also concerned with evaluating $f^{\#}$ at other points, eg at $G(\phi_n b_n)$.

i)$Y(x) = f^{\#}(x) = f(x) - xf'(x)$ denotes the indirect profit function associated with the production function $f(x)$;
ii) $u_{m+1} = u_{m+1}(u_m|b_n,...,b_m)$ is the optimal carry-forward from period m to period $m+1$ given the price history $b_n,...,b_m$;
iii) $x^*_m = x^*_m(u_{m-1}, b_m)$ is the general optimal demand for input at time m (so that when the firm expands $x^*_m = G(b_m)$);
iv) $q_m = q_m(u_m, b_m)$ is the period-m Tobin's marginal q, defined as the average marginal benefit of utilization of a unit of input in period m (given the current value of b_m and the closing stock u_m of the current period). When u_m is selected optimally (given opening stock v_m) the discounted value of q_m ranges between replacement cost b_m and resale cost $\phi_m b_m$. Indeed when u_m takes the value corresponding to optimal expansion, discounted q_m is the replacement cost and similarly when u_m takes the value corresponding to optimal contraction, discounted q_m takes the value $\phi_m b_m$.

Rewriting the identity thus

$$\gamma[V_n(u_n|b_n) - q_n u_n] = E[\sum_{m=n+1} \gamma^{m-n} Y(x^*_m)] = V_n^{\#}(u_n|b_n), \tag{32}$$

we see that the lefthand-side is the discounted future value less its marginal cost, and we denote this quantity by $\gamma V_n^{\#}(u_n|b_n)$, consistently so, since $q_n = V_n'$.

Our analysis of assessing future value shows the importance of Tobin's q, i.e. of marginal benefit, and we stress that this refers to replacement cost, as such, only in the expansion regime. It is natural therefore to measure current earnings as well by reference to Tobin's q, especially as both current demand and future demand have equal marginal value at an optimum (after taking due note of appropriate discounting).

Definition: The q-income at time t_n is the indirect profit, namely, the revenue less marginal cost of input, in symbols $f(x_n) - x_n f'(x_n)$, i.e. $f^{\#}(x_n)$, where x_n and u_n have been chosen to optimise the expression

$$f(x_n) + c_n(x_n + u_n - v_n) + \gamma V_n(u_n, b_n),$$

given v_n, and where $c_n = b_n$ for $x_n + u_n - v_n > 0$ and $c_n = \phi_n b_n$ for $x_n + u_n - v_n < 0$.

We note that q as introduced above is characterized along the lines of ADEP as being composed of:
- a certainty-equivalent price less the put option to expand plus the call option to contract plus the option to carry forward unused stock, i.e. typically it is of the form

$$\begin{aligned} q_0 \;=\; & \tilde{b}_1 - E[\max(\tilde{b}_1 - b_1, 0)] + E[\max(\phi_1 b_1 - \tilde{b}_1, 0)] \\ & + \int_{\tilde{b}_1}^{h(\tilde{b}_1,\phi)} (f'(x_1(u,b_1)) - \tilde{b}_1) dQ(b_1|b_0) \end{aligned} \tag{33}$$

for some function h and so includes the option to expand, to contract and to carry-forward optimally. Note that the future value of the firm, as measured in time t_{n+1} values, associated with the end of the production period $[t_n, t_{n+1}]$, is

$$Y(x_n^*) + \gamma V_n = Y(x_n^*) + \gamma q_n u_n + E[\sum_{m=n+1} \gamma^{m-n} Y(x_m^*)].$$

So recalling (1) and (3), we now see that we may write down the firm equity value S_n in terms of its book-value B_n at time t_n (i.e. the cash position k_n plus historic cost $h_n v_n$ of opening stock v_n), by means of the following identity:

$$S_n = B_n + Y(x_n^*) + v_n \cdot HG_n + E[\sum_{m=n+1} \gamma^{m-n} Y(x_m^*)], \tag{34}$$

where HG_n denotes the **holding gain** (per unit) on opening stock, and takes the following value, given a historic valuation of h_n per unit:
(U) $HG_n = b_n - h_n$, (IO) $HG_n = \gamma q_n b_n - h_n$, (RO) $HG_n = \phi_n b_n - h_n$.

Alternatively the equity value may be expressed in terms of the cash position k_n at time t_n and the corresponding opening stock position v_n as

$$S_n = k_n + \gamma q_n \cdot v_n + Y(x_n^*) + E[\sum_{m=n+1} \gamma^{m-n} Y(x_m^*)]. \tag{35}$$

In words: the equity value comprises opening cash, q-revalued opening stock, current q-income and future q-income $V^{\#}$. We recall that the q-revaluation price of stock is either b_n (i.e. replacement cost) in regime (U), or $\phi_n b_n$ (i.e. resale price) in regime (RO) or an intermediate value in regime (IO).

To summarize, our approach considers an alternative valuation identity and generalizes the earlier two-period model to multiple periods. After taking appropriate discounting, the form of the optimal value function $V(.)$ comprises:

-q adjusted value of the closing capital stock
- plus the expected q-income stream.

Moreover we have established the form of $Y(x_n^*(b_n))$ given a Cobb-Douglas technology (see Appendix B for the general details). It is satisfying that the q-income in this case is proportional to the revenue. For the square root function specifically:
- the period n indirect profit function $Y(x_n^*(b_n))$ takes a notionally simple form; it is $1/b_n$ when the project is under-invested, $1/(\phi_n b_n)$ when it is over-invested, and an intermediate value in the third regime.

Thus for our simple square-root returns model we can identify $V_n(u_n|b_n)$ by forming expectations over the input price process b_n. Furthermore forming this expectation simply requires looking at the appropriately censored integral of the next input price and the censoring value is the current input price b_n times a factor g_n, a generalized version[46] of (24).

Remark 9: (Existence of an Informational Asymmetry) We have shown how a manager determines the optimal expected future value of the firm by appropriate valuation of embedded put and call options. We must now ask why couldn't an external investor also directly identify the optimal value without the need to refer to other (subsidiary) information such as accounting income data.

3.2.4 Informational asymmetry

We shall henceforth assume that whereas the internal manager observes b_t, the investor does not. Thus at issue is whether some other accounting data may be helpful for the investor trying to form inferences on the value of $V_t()$. Before embarking on this route we note that it could be argued that an investor reading the annual financial accounts could be able to infer the input price of capital from the movements in capital items in the accounts. Our response here is as follows. Recall that in our introduction to the model we simplified the presentation by assuming the returns function faced by the firm was a simple square-root function and hence $f^{\#}(x_n) = \frac{1}{b}$, that is once the q-theoretic operating profit was reported an investor knows exactly the value of b and then can readily determine the value for $V()$. Note though that our initial working assumption of the square root function was simply to ease initial presentation. The important result we derive above (34) assumes only concavity of the returns function and in that case observing reported profit (i) does not allow the investor to infer what b was directly, and (ii) even if the investor had some other means of finding out the true value of b, that would still not be enough to infer the functional form of $V()$ since in the case of the general Cobb-Douglas returns function x^{θ} (for which $f^{\#}(x) = (1 - \theta)f(x)$), the function $V()$ cannot be recovered from knowledge of b without knowledge of the technology returns parameter θ.That is, if the reader feels uneasy about our modelling assumption that b is unobservable to an investor, then assuming that the general returns / technology factor θ is unobservable to the investor induces the same desired result that an investor cannot from an observed profit figure disentangle what the values for b and θ are - hence directly determine the optimal value function.

Fortunately since $f^{\#}(x)$ is directly proportional to revenue $f(x)$, the current marginal value $f'(x)$ is proportional to the ratio of current revenue over current consumption x.So despite the relevant q-theoretic variable being the current marginal profitability $f'(x_n)/b_n$, it is appropriate for an external investor to take an interest in $f^{\#}(x_n)$.

This identification of an asymmetry clearly raises the issue of optimal contract design within a principal-agent context[47]. We note that given our dynamic model setting, issues of dynamic

[46] We will give a specific example in the next section below.

[47] Our underlying model framework differs from that of Dutta and Reichelstein (1999) and Govindarajan

commitment and renegotiation will immediately arise and we leave to a following study the pursuit of these extensions. Our task here is to identify the ¯rst best. In contrast to most single period agency models where this is a trivial issue, in our setting it is not, evidenced perhaps by the fact that Feltham and Ohlson have been working with a model over the last decade which clearly has not been ¯rst best (since it ignores option values).

Having identi¯ed how managers can determine the optimal future value of the ¯rm, and on the assumption that an informational asymmetry exists between the manager and the investor, we now consider how the investor could use accounting measures to make inferences concerning the future value of the ¯rm.

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and Ramikrishnan (2001) because they assume that the change in cash °ow or earnings is linearly a®ected by exercise of e®ort on the part of the manager and do not recognise the option component of investment. As we have seen this options component does not necessarily add to cash °ow or earnings in a linear way.

4 Using an Earnings Measure to Infer Firm Value

Given Remark 9 let us now consider how a representative investor could use an earnings measure to value a firm. Recalling the fundamental FO result (4):

$$\tilde{\mathcal{G}}_{t+1} = \omega \tilde{\mathcal{G}}_t + x_t + \varepsilon_{t+1},$$

which has been used repeatedly by empiricists to test for the value relevance of accounting measures, at this stage we summarize our critique of this previous research via two remarks:

Remark 10: (Non-optimality of the FO Model)
Since the FO model does not recognize real options that arise in practice it is hard to know what (if anything) empirical tests[48] using the FO specification actually mean for the decision value significance of accounting measures. For instance, recalling our results on the underlying naivety of the investment model in the FO approach, we comment in subsection 4.3 that only in the restrictive case of non stochasticity in the underlying parameter is the FO residual income model consistent with our optimization model.

Remark 11: (Non-Linearity of the Firm Value)
By explicitly recognizing the real options omnipresent with investment in real assets, we have shown that the type of linear functional relationship embodied by (4), and used so frequently by empiricists[49], is inappropriate except in special cases. Instead our real-options analysis, which brings to the fore the three distinct regimes of optimal investment behavior (contraction, maintenance and expansion), suggests that, rather than testing for accounting value relevance with a single linear regression model, an approach using finite-mixture (distribution) models with three regime changes may be appropriate. We stress that the implied linearities are between our chosen earnings measure and future value[50].

At least two questions naturally arise from these remarks. What is the significance for earnings based valuation of the non optimality of FO residual income? If a simple linear regression is not representative of the underlying optimal investment environment what is an appropriate (perhaps approximate) empirical specification? We will address these questions in the following two subsections respectively.

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[48] Both Feltham and Ohlson have on occasion raised concerns about the empirical appllications of their model. Our critique here is with the claimed theoretical validity of empirical research claiming to apply the FO results.

[49] We shall discuss the extensions proposed by Burgsthaller and Dichev (1997) and others in subsection 4.2.

[50] Alternatively we could look at the relationship between our measure of current earnings and market value S_t by recovering the relationship from (34).

4.1 The significance of the non-optimality of FO residual income for earnings based valuation

A central feature that differentiates our approach from earlier studies of the use of earnings numbers to predict future firm value is that via (31) we can actually identify exactly the variable that is being estimated. Hence we can objectively appraise the ability of a chosen earnings method such as residual income to predict future firm value. Expressed precisely, if we let $\tilde{y}$ denote residual income, then we can consider analytically what is the relationship between the explanatory variable $\tilde{y}$ and the variable being predicted $V(.)$.

We have derived analytic expressions for the residual income and they are given below. However, the qualitative features driving the form of the dependence of $\tilde{y}$ on the unobservable b_i are pictured in the Figure below showing that the residual income $\tilde{y}_i$ at the end of the period $[t_i, t_{i+1}]$, as a function of the input price b_i, is asymptotically vertical as $b_i \to 0+$ and has a linear oblique asymptote with positive slope as $b_i \to +\infty$. Consequently, for each level of residual income in the range (apart from the minimum) there are at least two corresponding price levels b_i, making the future value of the project ambiguous.

Figure 2 : Graph of $\tilde{y}$ vs b

To see this we note that the residual income is defined by cases as follows. If we let h_i denote the historic unit cost of the investment asset holding of v_i at the beginning of the period $[t_i, t_{i+1}]$ it is straight-forward to show that:

$$\tilde{y}_{i+1} = \begin{cases} \frac{1}{b_i} + v_i[b_i - (1+r)h_i], & \text{for } b_i < 1/\sqrt{v_i}, \\ \frac{2}{b_i} - h_i\frac{1}{b_i^2} - rh_iv_i, & \text{for } 1/\sqrt{v_i} \le b_i \le b_i(v_i,1), \\ 2\sqrt{x_i(v_i,b_i)} - h_ix_i(v_i,b_i) - rh_iv_i, & \text{for } b_i(v_i,1) < b_i \le b_i(v_i,\phi), \\ \frac{1}{\phi b_i}(1 - u_i(1,\phi)) + \phi b_iv_i + \frac{u_i(1,\phi)}{\phi^2 b_i^2}h_i - (1+r)v_ih_i, & \text{for } b_i > b_i(v_i,\phi), \end{cases}$$

where in the first case the firm is expanding and the last case selling off some investment assets.

These formulae enable us to produce the required plot of future firm value less historic cost[51] of investment assets carried forward $(V_n(.) - h_nu_n)$ against residual income $\tilde{y}(.)$ as follows:

Figure 3: Hysteresis

This plot shows clearly why it could be misleading to condition expectations of future firm value solely on accounting residual income. The plot shows that for a given level of residual income a multiplicity of future firm values may be possible. That is, there does not exist a functional relationship between residual income and future firm value and hence there is no

[51] The same qualitative features arise if we simply plot $V(.)$ against $\tilde{y}(.)$. We have deducted the historic cost of the investment assets carried forward so as to capture the accounting convention of matching.

theoretical support for the empirical practice of linearly regressing future ¯rm value on residual income[52].

The intuition for this hysteresis e®ect arising is as follows. Compare two ¯rms with identical residual income, one expanding investment and the other contracting. The reason the two ¯rms with the same residual income have di®erent future values is that the expanding ¯rm faces a charge for the additional investment which reduces income whereas the contracting ¯rm is selling of assets which increases income. That is same the residual income number may result from two distinctly di®erent investment strategies which in turn imply di®erent future ¯rm value; ¯rms contracting now are not expected to have the same future value as ¯rms that are expanding now (holding currently observed residual income constant).

4.2 Considering q - income for earnings based valuation

Our earlier analysis has explained how upon observation of the current input price for the investment good, management face three investment-strategy regimes; expand investment when conditions are favorable (over the U under-invested range), neither add to nor sell any investment good (over the IO endogenously irreversible range) and sell some stock of the investment good (the RO reversibility range). Thus our ¯rst task is to derive the qualitative features for our prediction model of $V(.)$ based upon current q-income.

Regime (U): Under-invested in capital stock with v = 0 or v $<$ critical value $\hat{b}$. This case is where the stock of capital in place is insu±cient and it is optimal for management to increase the stock.The resulting q-income is $Y^q = 1/b$ and we consider the optimal multi-period behaviour in terms ¯rst of b and then, by substitution, in terms of Y^q.

In the multi-period setting, suppose ¯rst, for simplicity, that at the start of business there is no stock of capital. Via a generalization of (24) and (25) it can be shown that solving the ¯rst-order condition for the optimal value function to derive the optimal capital purchase corresponds to requiring that a capital stock be purchased equal to

$$\hat{b}_0 = \frac{1}{b_0^2} + \frac{\gamma^2}{(\hat{g}_{1,1} \ast b_0)^2} + ... + \frac{\gamma^{2n}}{(\hat{g}_{1,N} \ast b_0)^2}, \qquad (36)$$

where

$$\hat{g}_{n,m} = \hat{g}_n \cdot \hat{g}_{n+1} \cdot ... \cdot \hat{g}_m$$

and $\hat{g}_1, ..., \hat{g}_N$ are the period-by-period price input censor parameters of the model. A special case[53] perhaps illustrates best the e®ect of the input price persistence factors. Assuming in the

[52] In fact what has been done in the past is even more dubious as Lys and Lo (199*) point out since a truncated estimate for $V(.)$ is typically used.

[53] Note in our analysis the values for respective $\hat{g}_t$ are derived optimally, whereas the assumed values below are only for illustrative purposes. It need not in general be the case that $\hat{g}_t > 1$.See towards the end of section 3.1.

special case $b_0 = 1$, $\gamma = 1$ that the resale factors are such that $\hat{g}_t > 1$ for all t, and taking $\hat{g}_1 = \hat{g}_2 = \ldots \hat{g}_t = 1.5$ for all t, we have

$$\hat{\tau}_0 = 1 + \frac{1}{(1.5)^2} + \frac{1}{(2.25)^2} + \ldots.$$

that is, enough stock is purchased to meet demand for the current period, 44.4% of current demand for the following period and 19.75% of current demand for the period following that etc.

Intuition underlying choice of the optimal q-investment level when following the (U) expansion strategy

Given a low stock of capital, upon seeing an advantageous purchase price, the optimal strategy is to purchase additional stock for the current and future periods (expand investment) with the amount brought forward in this example being less for each period further into the future in order to guard against over-stocking before waiting to see how the input price evolves in the future.

To demonstrate how the total investment is planned to be applied across the sum of the periods we note that formula (36) may be derived as follows:

$$\begin{aligned}
\hat{\tau}_0 &= \frac{1}{b_0^2} + \hat{\tau}_1(b_0) \\
&= \frac{1}{b_0^2} + \hat{\tau}_1(b_0 \hat{g}_1(1+r)) \\
&= \frac{1}{b_0^2} + \frac{\gamma^2}{b_0^2 \hat{g}_1^2} + \hat{\tau}_2(b_0 \hat{g}_1 \hat{g}_2 (1+r)^2)\ldots \\
&= \frac{1}{b_0^2} + \frac{\gamma^2}{b_0^2 \hat{g}_1^2} + \frac{\gamma^4}{b_0^2 \hat{g}_1^2 \hat{g}_2^2} + \ldots
\end{aligned}$$

Thus the stock is built up "as if" the undiscounted prices in the future were known to be $b_m = \hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_m b_0$. A full derivation is given in Appendix B.

In general at time t_i the optimal opening investment stock to purchase is given by

$$\hat{\tau}_i(b_i) = \frac{1}{b_i^2} + \hat{\tau}_i(b_i) = \frac{1}{b_i^2} + \gamma^2 \left(\frac{1}{(\hat{g}_{i+1,i+1} \cdot b_i)^2} + \ldots.. + \frac{\gamma^{2(N-i)}}{(\hat{g}_{i+1,N} \cdot b_i)^2} \right). \tag{37}$$

Given the current optimal profit is given by $1/b_i$, the optimal expected future firm value is obtained by following a strategy of increasing the stock to $\hat{\tau}_i = \hat{\tau}(b_i)$, which results in[54]

$$\overline{V}_i(\hat{\tau}(b_i), b_i) = \frac{1}{b_i} \overline{V}_i(\hat{\tau}(1), 1),$$

[54] The following formula is derived in Appendix C

where the bar denotes expectation over the future resale factor ϕ_{i+1}. This homogeneity is derived in Appendix E. Notationally we may write this optimal expected value in the form

$$\hat{V}_i = \frac{C_{i;N}}{b_i}, \tag{38}$$

for some constant $C_{i;N} = \overline{V}_i(\hat{a}(1), 1)$. That is, qualitatively - the optimal future value of the ¯rm is given by the q-pro¯t $Y_i^q = 1/b_i$ multiplied by a certain constant[55] (which is dependent on volatility):

$$\hat{V}_i = Y_i^q \cdot C_{i;N}. \tag{39}$$

Regime (RO): Over-stocked in capital stock with some excess sold o® The case of a costly divestment is quite similar. For a given price b_n there are now two benchmark stock levels. The ¯rst and lower value is the optimum level $\hat{s}_n$ (computed as above) below which the stock level should not fall but there is now a second, larger, upper optimum level $\hat{s}_n(\phi_n, b_n)$, dependent also on the current resale rate, above which the stock should not rise. The ¯rst order condition (20) implies that current demand is as though the resale price was the purchase price so that the q-income is now $Y_n^q = 1/(\phi_n b_n)$. Again one considers optimal multi-period behaviour ¯rst in terms of b_n and then, by substitution, in terms of $Y_{q,n}$.

At time t_n the optimal highest stock level worth keeping exists and is given by

$$\hat{s}_n(\phi_n, b_n) = \frac{1}{(b_n\phi_n)^2} + \frac{\gamma^2}{(b_n\phi_n\hat{g}_n(\phi_n))^2}\left[1 + \frac{\gamma^2}{\tilde{g}^2_{n+2,n+2}} + \ldots + \frac{\gamma^{2(N-n)}}{\tilde{g}^2_{n+2,N}}\right],$$

i.e. as though the current price were $\phi_n b_n$ and discounted future prices were to be $\gamma b_{n+1} = \phi_n b_n \cdot \hat{g}_n(\phi_n)$, $\gamma^2 b_{n+2} = \phi_n b_n \hat{g}_n(\phi_n)\hat{g}_{n+2}, \ldots, \gamma^m b_{n+m} = \phi_n b_n \cdot \hat{g}_n(\phi_n)\tilde{g}_{n+2,n+m}, \ldots$. Corresponding to $\hat{s}_n(\phi_n, b_n)$ there is an optimal current revenue from production, namely $1/(\phi_n b_n)$, and an optimal carry-forward $\hat{a}_n(\phi_n, b_n) = \hat{s}_n(\phi_n, b_n) - 1/(b_n\phi_n)^2$, i.e. of the form $\hat{a}_n(\phi_n, b_n) = \hat{a}_n(\phi_n, 1)/b_n^2$. Note that in our notation $\hat{a}_n(\phi_n, 1) = \hat{a}_n(\phi_n)/\phi_n^2$ to ensure that $\hat{s}_n(\phi_n, b_n) = (1 + \hat{a}_n(\phi_n))/(\phi_n^2 b_n^2)$.

Intuition underlying choice of the optimal investment level when following the (RO) expansion strategy

Given a large stock of capital, upon seeing an advantageous resale rate, the optimal strategy is to sell some of the additional (investment) stock that was planned for use in this and future periods (contract investment), but the amount sold forward is less for each period further into the future that we consider; this is because the longer we wait the more chance there is that the ¯rm could move into an under-stocked position. Hence, analogously to the (U) regime, the future value from carrying-forward is:

$$\overline{V_n}(\hat{a}_n(\phi_n, b_n), b_n) = \frac{1}{b_n}\overline{V_n}(\hat{a}_n(\phi_n, 1), 1),$$

[55] We note that this last equation also holds in the general Cobb-Douglas case (for an appropriate rede¯ned constant).

(where the bar recalls the expectation over the future resale rate). We can again rewrite the displayed formula as: $\widehat{V}^{\phi}_n = \frac{C^{\phi}_{n,N}}{\phi_n b_n}$, where $C^{\phi}_{n,N} = \phi_n \overline{V_n}(\hat{b}_n(\phi_n, 1), 1)$. This, as before, follows from the general formula of Appendix E. Again we have a linear relationship:

$$\widehat{V}^{\phi}_n = Y^q_n \cdot C^{\phi}_{n,N}, \tag{40}$$

between future value[56] and accounting profit $Y^q_n = 1/(\phi_n b_n)$.

Regime (IO): Given $\phi_n < 1$ and the firm is neither over-invested nor under-invested Here we are concerned with the intermediate input price range:

$$\underline{b}_n = b_n(v, 1) < b_n < b_n(v, \phi_n) = \overline{b}_n. \tag{41}$$

The revenue is, as always,

$$2\sqrt{x_n(v_n, b_n)},$$

so we set the q-profit to be

$$Y^q_n = Y^q(x^*_n) = \sqrt{x_n(v_n, b_n)},$$

as required by our formula since $f^{\#}(x) = \sqrt{x}$. Thus we have, since $x_n(v_n, \underline{b}_n) = 1/\underline{b}_n^{\,2}$ and $x_n(v_n, \overline{b_n}) = 1/\overline{b_n}^{\,2}$, that

$$Y^q_n(\underline{b}_n) = \frac{1}{\underline{b}_n}, \quad Y^q_n(\overline{b}_n) = \frac{1}{\phi_n \overline{b}_n},$$

so the intermediate input price range corresponds to

$$\frac{1}{b_n} < Y^q_n < \frac{1}{\phi_n b_n},$$

as $1/\underline{b}_n > 1/b_n$ and $1/\overline{b}_n < 1/b_n$. In this range the firm neither invests nor divests. It partitions its stock v_n into current optimal consumption $x_n(v_n, b_n)$ and investment carried forward $u_n(v_n, b_n)$, and the cash income in this range is thus

$$2\sqrt{x_n(v_n, b_n)}.$$

The relation between the expected future firm value and q-income is then given[57] by:

$$V_n = V_n(v_n - (Y^q_n)^2, b_n(Y^q_n)),$$

where $b_n = b_n(Y^q_n)$ solves

$$Y^q_n = \sqrt{x_n(v_n, b_n)}.$$

Remark 12: (Monotonicity of $V^{\#}_n$ in Y^q_n)

[56] Again this last equation holds good in the general Cobb-Douglas case (for an appropriate constant).

[57] Observe that $\frac{dV_n}{dY_n} = \frac{\partial V_n}{\partial u}(-2Y_n) + \frac{\partial V_n}{\partial b}(\frac{dY_n}{db_n})^{-1} < 0$ if $\frac{\partial V_n}{\partial b} > 0$.

Provided either the volatility is large enough, i.e. $\sigma \geq \sigma^*(\phi_{n+1})$, or, equivalently, provided the forthcoming discount factor ϕ_{n+1} is close enough to unity, i.e. $\phi^*_{n+1}(\sigma) \leq \phi_{n+1} < 1$,the function $V^\#$ regarded as a function of Y^q_n is monotonic increasing. For instance, in a two-period model, a sufficient bound is provided by the inequality

$$\phi_1 > \exp(-1.65\sigma).$$

See Appendix H.

4.2.1 Convexity of future firm value in q-income

To compare our model predictions with thosed derived in the literature, we need to consider the equity value of the firm given its current q-income Y^q.A plot follows.

Figure 4: Graph of S vs Y^q

The graph has three sections correponding to the three regimes considered earlier. We comment on the qualitative features. For small enough value of Y^q (i.e. less than $\underline{Y}^q$) the equity value of the firm takes a convex (in fact hyperbolic) form[58]

$$\frac{v_n}{Y^q} - v_n h + o(Y_q)$$

in the square-root case (the first term generalizing to $v_n Y^{-\alpha/(1-\alpha)}$, in the case of a Cobb-Douglas index α). For large values of Y^q (i.e. greater than $\overline{Y}^q$) the equity value is asymptotically linear and takes the form

$$(2 + V(\hat{u}(1), 1))Y^q - v_n h + v_n (Y^q)^{-\alpha/(1-\alpha)}.$$

One may take the view that on our definition of income Y^q this quantity is unlikely to be very small and so the vertical asymptote is in itself is irrelevant, but the "convexity" it exhibits is not out of line with the cluster-plot given in Burgstahler and Dichev (B&D).

To see how this relates to the Burgstahler and Dichev (1997) findings, recall that in essence the B&D paper empirically tests the future value of a firm by a two-period model. In the later of the two periods the earnings E_1 predict a possible future earnings stream valued at $W^1_1 = cE_1$ (where c is the earnings capitalization factor). Management have the option to switch from this earnings stream to an alternative activity. That activity also generates a future earnings stream W^2_1, known as adaptation value, which is a constant independent of E_1 and assumed fixed a priori at a value A. The model's empirical proxy for A is the book value B_0. The firm currently has earnings E_0, so the current market value of the firm S_0 comprises book value B_0, the current earnings E_0, and the expected value V_0 of the claim: $\max\{W^2_1, W^2_1\} = \max\{cE_1, A\}$. In a log-normal setting for the distribution of E_1 given E_0, the value V_0 has the well-known convex shape of a call-value struck at $K = A/c$. Our model agrees with the linear valuation $W^1_1 = cE_1$ but only provided E_1 is large enough. On the other hand, our model does not give

[58] In fact, the exact form is $\phi_n b_n(v_n - \hat{u}(\phi, 1)/b^2) + \frac{1}{b_n} V(\hat{u}(\phi, 1), 1) + \frac{1}{\phi b_n} - h_n v_n$

management the option to receive a fixed income stream A for values of E_1 below some strike value K. Instead the adaptation value depends on the value of E_1 and is at best viewed as piecewise linear in E_1 with ranges of linearity endogenously defined by $\underline{Y}^q$ and $\overline{Y}^q$ as given by (??). Note that B&D also subdivide the earnings range into three intervals in order to verify convexity (by testing whether the slope of the respective best linear fit to the data is increasing). Another interestin point is that presumably in order to ensure large enough subsample sizes, B&D chose to have equal numbers of observations in each interval. Given the option-valuation basis for their model, it would have been economically more intuitive had they selected their middle interval centered on the implied option strike K.

To summarize our findings are in broad agreement with the stylized facts proposed by B&D in their option style valuation model. That is our model predicts asymptotic linearity for large values of current earnings and a convex valuation for low current earnings. However, in our model the market value can be negatively relatated to (low) earnings as was observed by B&D in their empirical study. Whereas they could provide no formal explanation our model shows it may be consistent firm optimization with non constant abandonment value.

Before moving to the concluding section we shall now present a subsection which shows that residual income is in fact a special case (under restrictive conditions) of q-income and hence in these special circumstances applying FO residual income is consistent with optimality of investment behaviour.

4.3 The Equivalence Between Residual Income and q-Theoretic Operating Profit: Intuition Underlying the Special Case

We start by recalling the example of subsection 2.1. Working with current value residual income we have from (9) that

$$\pi_1^{CV} = 2\sqrt{x} - b_0 x, \qquad \pi_2^{CV} = 2\sqrt{u} - b_1 u.$$

Noting that by suitable redefinition of notation and introducing a time script, if

$$b_0(1 + r) = b_1$$

then

$$\pi_1^{CV} = 2\sqrt{x_0} - b x_0 = Y(x_0^*), \qquad \pi_2^{CV} = 2\sqrt{x_1} - b x_1 = Y(x_1^*).$$

That is, provided $x_0 = x_0^* = (\frac{1}{b_0})^2$ and $x_1 = x_1^* = (\frac{1}{b_1})^2$, the current value residual income is identical to q-theoretic operating profit. This simple example shows how management focusing upon residual income is a special case of adopting a focus upon q-theoretic operating profit. Specifically the equivalence holds in the restricted case that the discounted input prices are constant through time. This result is just another recurrence of what we have established earlier: focusing upon residual income does not take into account the put and call, expansion and contraction options that arise with investment decisions taken in a stochastic environment. Only in the special case where those options have no value, because the input price is constant (non stochastic), will prediction relative to the two respective measures be equivalent. To see this recall (35) -

$$S_n = k_n + \gamma q_n \cdot v_n + Y(x_n^*) + E[\sum_{m=n+1} \gamma^{m-n} Y(x_m^*)],$$

- and that optimization is with respect to current production x_n and future stock carried forward u_n. What is appealing about (35) is that, since q_n includes the value[59] of embedded put and call options and of carrying forward stock, the optimization problem is essentially separable - that is, after identification of Tobin's q_n, management can think about current period optimization over x independently of expansion or contraction decisions for the overall level of investment stock to carry forward u_n. With reference to equation (34)

$$S_n = B_n + Y(x_n^*) + v_n \cdot HG_n + E[\sum_{m=n+1} \gamma^{m-n} Y(x_m^*)],$$

note that in the case when the input price b is fixed, the $\gamma q_n u_n$ term cancels against the market price paid for the stock (zero holding gains), and

$$Y_n(x_n^*(b_n)) = \tilde{y}_n^{CV}(x_n^*(b_n)).$$

However, note the converse, when $\gamma q_n u_n \neq b_n$, then $Y_n(x_n^*(b_n)) \neq \tilde{y}_n^{CV}(\tilde{x}_n(b_n))$.

5 Conclusion

For the FO model recall that FO superimpose (4) and (5) on (3). However, as has been argued extensively above, superimposing this simple $AR(1)$ process on the way residual income grows, considerably restricts the type of underlying investment behavior that could be consistent with the model. The objective of the paper has been to establish a more flexible model which facilitates an alternative representation of the expected income stream of terms $E_t(\tilde{y}_{t+\tau})$ based upon optimal managerial real-options evaluation. These findings are significant because the Feltham-Ohlson valuation framework has been used by empiricists to test the value relevance of accounting data. Some researchers have criticized how empiricists have used the model to try to specify appropriate empirical testing procedures for the value relevance of accounting information. We address both the underlying validity of the FO model and the implications for specification of empirical testing routines. With regard to validity, we show how, independently of specifications issues, the underlying constant growth assumption which is central to the Feltham-Ohlson framework removes the possibility for management to have a role in deciding whether or not to exercise expansion and contraction possibilities which do occur with most investment projects. Given this limitation we develop an alternative valuation framework which does not suffer from these limitations because the option to expand or contract optimally is given centre-stage in our model of managerial decision-making. This flexible model which puts

[59] Recall (33).

three investment regimes at center stage also shows that a single linear regression model of the link between firm value and accounting measures is inappropriate. Instead our model shows how a regime-shifting specification (giving rise to a tri-mixture of distributions) would more effectively capture the underlying statistical relationships that apply. In the previous section we have derived the basic regime functional forms needed to implement such testing procedures.

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6 Appendix A: NPV Rule

In this section we derive the NPV Rule. The notation (see section 3.2) is as follows: $F_n(v, b_0, \phi)$, or simply $F(v, b_0, \phi)$ with time t_n suppressed, denotes the discounted future maximum expected profit ignoring the historic cost of the carry–forward input v. The price at the time t_n is here denoted b_0 (sic!) and so that the price at time t_{n+1} is then b_1;the resale rate revealed at time t_{n+1} is ϕ. When the time t_n is suppressed $\overline{F}_+(v, b_1)$ denotes expectation over ϕ' of $F_{n+1}(v, b_1, \phi')$. Thus, for example $F_0(v, b_0, \phi) = \gamma V_0(v, b_0, \phi)$ and the corresponding value of the firm ignoring past costs and revenues is $V_0(v|b_0) = f(G(b_0)) - b_0 G(b_0) + F_1(v, b_0, \phi)$. Now we have

$$\begin{aligned}
\gamma^{-1}F(v,b_0,\phi) &= \int_0^{b(v,\infty)} \left[f^{\#}(G(b_1)) + b_1(v - \hat{b}(\infty,b_1)) + \overline{F_+}(\hat{b}(\infty,b_1),b_1) \right] dQ_1 \\
&+ \int_{b(v,\infty)}^{b(v,\phi)} \left[f(v - u(v,b_1)) + \overline{F_+}(u(v,b_1),b_1) \right] dQ_1 \\
&+ \int_{b(v,\phi)}^{\infty} \left[\phi b_1(v - \hat{b}(\phi,b_1)) + f^{\#}(G(\phi b_1)) + \overline{F_+}(\hat{b}(\phi,b_1),b_1) \right] dQ_1.
\end{aligned}$$

Hence proceeding formally and applying the Liebniz Rule[60]

$$\begin{aligned}
\gamma^{-1}F'(v,b_0,\phi) &= \int_0^{b(v,\infty)} b_1 dQ_1 \\
&+ \int_{b(v,\infty)}^{b(v,\phi)} \left[f'(v - u(v,b_1))(1 - u') + \overline{F_+}'(u(v,b_1),b_1)u' \right] dQ_1 \\
&+ \int_{b(v,\phi)}^{\infty} \phi b_1 dQ_1.
\end{aligned}$$

But $f'(v - u(v,b_1)) = \overline{F_+}'(u(v,b_1),b_1)$ by definition of $u(v,b_1)$. So

$$F'(v,b_0,\phi) = \int_0^{b(v,\infty)} b_1 dQ_1 + \int_{b(v,\infty)}^{b(v,\phi)} f'(v - u(v,b_1))dQ_1 + \int_{b(v,\phi)}^{\infty} \phi b_1 dQ_1.$$

Hence

$$\begin{aligned}
\gamma^{-1}F'(u,b_0,\phi) &= \int_0^{b(u,\infty)} b_1 dQ_1 + \int_{b(u,\infty)}^{b(u,\phi)} f'(x(u,b_1))dQ_1 + \int_{b(u,\phi)}^{\infty} \phi b_1 dQ_1, \\
&= \int_0^{\tilde{b}_1} b_1 dQ_1 + \int_{\tilde{b}_1}^{h_1(\tilde{b}_1,\phi)} f'(x(u,b_1))dQ_1 + \int_{h_1(\tilde{b}_1,\phi)}^{\infty} \phi b_1 dQ_1, \\
&= \tilde{b}_1 + \int_0^{\tilde{b}_1} (b_1 - \tilde{b}_1)dQ_1 + \int_{\tilde{b}_1}^{h_1(\tilde{b}_1,\phi)} (f'(x(u,b_1)) - \tilde{b}_1)dQ_1 \\
&+ \int_{h_1(\tilde{b}_1,\phi)}^{\infty} (\phi b_1 - \tilde{b}_1)dQ_1.
\end{aligned}$$

[60] We do not show cancelling terms.

7 Appendix B: The optimal replenishment policy.

We prove the following Recurrence Lemma

$$\hat{u}_n(b_n,\phi_{n+1}) = G(\hat{b}_{n+1}(b_n)) + \hat{u}_{n+1}(\hat{b}_{n+1}(b_n),\phi_{n+2}).$$

Proof. Recall that the function $u = \hat{u}_n(b,\phi)$ is defined by the equation[61]

$$F_n'(u,b) = \phi b.$$

We work inductively. To obtain the solution $v = \hat{u}_n(b_n,\phi_{n+1})$ of the first-order condition

$$F_n'(v,b_n) = \phi_{n+1} b_n$$

we begin by first solving the censor equation

$$q_n(B,\phi_{n+1},,b_n) = \phi b_n.$$

We denote the solution[62] by $\hat{b}_{n+1}(b_n,\phi)$. Recall that

$$\begin{aligned} q_m(B,\phi_{m+1},b_m) &= \int_0^B b_m\,dQ(b_m|b_{m-1}) \\ &+ \int_B^{h_m(B,\phi_{m+1})} f'(x_m(\hat{h}_m(B),b_m))\,dQ(b_m|b_{m-1}) + \phi_{m+1}\int_{h_m(B,\phi_{m+1})}^{\infty} b_m\,dQ(b_m|b_{m-1}). \end{aligned}$$

Here $\hat{h}_m(b) = G(b) + \hat{u}_m(b,\phi_{m+1})$, and $G(b) = \{f'\}^{-1}(b)$. Now for an appropriate function $B_{n+1}(v)$ we have

$$F_n'(v,b_n) = q_n(B_{n+1}(v),\phi_{n+1},b_n),$$

so we now need to solve

$$B_{n+1}(\mathbf{b}) = \hat{b}_{n+1}(b_n,\phi).$$

But recalling that in general $F(v,b) = f(v - u(v,b) + F_+(u,b)$, we have

$$\begin{aligned} F_n'(\hat{h}_n(b),b) &= f'(\hat{h}_n(b) - u(\hat{h}_n(b),b))(1-u') + F_{n+1}'(u,b)u' \\ &= f'(\hat{h}_n(b) - u(\hat{h}_n(b),b)) = b. \end{aligned}$$

Thus we have the identity

$$B_{n+1}(\hat{h}_n(B)) = F_n'(\hat{h}_n(B), B_{n+1}(\hat{h}_n(B))) = B.$$

---

[61] Recall the convention that $F = \gamma V$.

[62] In the Cobb-Douglas case

$$\hat{b}_{n+1}(b_n) = \phi b_n \mathbf{b}_n(\phi)$$

for some constant $\mathbf{b}_n(\phi)$.

Hence for $B = \hat{b}_{n+1}(b_n, \phi)$ we have identified that $\hat{b} = v_B^n$. In conclusion we have

$$\begin{aligned}\hat{u}_n(b_n, \phi) &= G(\hat{b}_{n+1}(b_n, \phi)) + \hat{u}_{n+1}(\hat{b}_{n+1}(b_n, \phi), 1) \\ &= G(\hat{b}_{n+1}(b_n, \phi)) + G(\hat{b}_{n+2}(\hat{b}_{n+1}(b_n, \phi), 1)) + \ldots\end{aligned}$$

Corollary. The analysis prescribes an aggregate demand of

$$D_n^{\phi}(b_n) = G(\hat{b}_{n+1}) + \ldots + G(\hat{b}_{n+i}) + \ldots,$$

where $G(b) = \{f'\}^{-1}(b)$ and the sequence $\hat{b}_{n+i}$ is given by the iteration

$$\begin{aligned}\hat{b}_{n+1} &= \hat{b}_{n+1}(b_n, \phi), \\ \hat{b}_{n+2} &= \hat{b}_{n+1}(\hat{b}_{n+1}, 1), \\ \hat{b}_{n+3} &= \hat{b}_{n+3}(\hat{b}_{n+2}, 1), \\ &\ldots\end{aligned}$$

It is now easy to describe the replenishment programme. Suppose we have n periods remaining and we have a stock v. The acquisition programme calls for an optimal aggregate demand to be purchased of

$$D_n^1(b_n) = G(\hat{b}_{n+1}) + \ldots + G(\hat{b}_{n+i}) + \ldots,$$

(i.e. with $\phi = 1$) and either we have v below this amount in which case we need to top up to this amount or else we are moderately over-stocked and must carry-forward $u_n^*(v, \phi_{n+1}, b_n)$ without selling, or else we must sell to the point where the stock is $D_n^{\phi_{n+1}}(b_n)$. Thus

$$u_n^*(v, b_n) = \begin{cases} \hat{u}_n(1, b_n) & b_n < b_n(v, 1), \\ u(v, b_n) & b_n(v, 1) < b_n < b_n(v, \phi_{n+1}), \\ \hat{u}_n(\phi, b_n) & b_n(v, \phi_{n+1}) < b_n. \end{cases}$$

8 Appendix C: Derivation of Valuation formula

We study first the general two-stage situation. The current price is b_0 the next period price is b_1 and the resale rate is ϕ. Our notation in this section for the maximum expected value given a stock v of inputs and given the knowledge of $\phi = \phi_1$ is $F(v, b_0, \phi)$; once b_1 is revealed and u is carried forward into the future, the maximum expected revenue from the period beyond is $\overline{F_+}(u, b_1)$, where the bar signifies expectation over ϕ_2.

We note that $F = \gamma V$.

8.1 Step 1. We prove a recurrence

$$\gamma^{-1} F(v, b_0, \phi) = E_{b_1}[f^{\#}(x^*(v, b_1)) + \overline{F_+}^{\#}(u^*(v, b_1), b_1)] + \gamma v q,$$

where the notation is as in section 3.2 above and is recalled below.

Proof. We have as in Appendix A that

$$
\begin{aligned}
\gamma^{-1}F(v,b_0,\phi) &= \int_0^{b(v,1)} \Big[f(G(b_1)) - b_1G(b_1) + b_1(v - \hat{b}(1,b_1)) + \overline{F_+}(\hat{b}(1,b_1),b_1) \Big] dQ_1 \\
&+ \int_{b(v,1)}^{b(v,\phi)} \Big[f(v - u(v,b_1)) + \overline{F_+}(u(v,b_1),b_1) \Big] dQ_1 \\
&+ \int_{b(v,\phi)}^{1} \Big[\phi b_1(v - \hat{b}(\phi,b_1)) + f^{\#}(G(\phi b_1)) + \overline{F_+}(\hat{b}(\phi,b_1),b_1) \Big] dQ_1.
\end{aligned}
$$

To understand the first integral (corresponding to the understocked situation), note that the additional purchase z is specified by $v + z = G(b_1) + \hat{b}(1,b_1)$ and so the revenue is $f(G(b_1)) - b_1(G(b_1) + \hat{b}(1,b_1) - v)$.

Now we reorganize the expression on the right. First note that $f^{\#}(x) = f(x) - xf'(x)$, and since G is the inverse of f' we have

$$f^{\#}(G(b_1)) = f(G(b_1)) - b_1G(b_1).$$

Similarly $\overline{F_+}^{*}(x) = \overline{F_+}(x) - x\overline{F_+}'(x)$. But since $u = \hat{b}(1,b_1)$ solves

$$b_1 = \gamma V_+'(u,b_1) = \overline{F_+}'(u,b_1),$$

we have

$$\overline{F_+}^{\#}(\hat{b}(1,b_1),b_1) = \overline{F_+}(\hat{b}(1,b_1),b_1) - b_1\hat{b}(1,b_1).$$

Likewise

$$\overline{F_+}^{\#}(\hat{b}(\phi,b_1),b_1) = \overline{F_+}(\hat{b}(\phi,b_1),b_1) - \phi b_1\hat{b}(\phi,b_1).$$

Lastly $u = u(v,b_1)$ solves

$$f'(v - u) = \gamma\overline{V_+}'(u,b_1) = \overline{F_+}'(u,b_1),$$

hence

$$
\begin{aligned}
\overline{F_+}^{\#}(u(v,b_1),b_1) &= \overline{F_+}(u(v,b_1),b_1) - u(v,b_1)f'(v - u(v,b_1)) \\
&= \overline{F_+}(u(v,b_1),b_1) - u(v,b_1)f'(x(v,b_1)),
\end{aligned}
$$

where $x(v,b_1) = v - u(v,b_1)$. Of course

$$f^{\#}(x(v,b_1)) = f(x(v,b_1)) - x(v,b_1)f'(x(v,b_1))$$

We thus have, writing x for $x(v,b_1)$,

$$
\begin{aligned}
\gamma^{-1}F(v,b_0,\phi) &= \int_0^{b(v,1)} \Big[f^{\#}(G(b_1)) + \overline{F_+}^{\#}(\hat{b}(1,b_1),b_1) \Big] dQ_1 + v\left[\int_0^{b(v,1)} b_1 dQ_1\right] \\
&+ \int_{b(v,1)}^{b(v,\phi)} \Big[f^{\#}(x) + \overline{F_+}^{\#}(u(v,b_1),b_1) \Big] dQ_1 + \int_{b(v,1)}^{b(v,\phi)} [xf'(x) + u(v,b_1)f'(x)]\, dQ_1 \\
&+ \int_{b(v,\phi)}^{1} \Big[f^{\#}(G(\phi b_1)) + \overline{F_+}^{\#}(\hat{b}(\phi,b_1),b_1) \Big] dQ_1 + v\left[\int_{b(v,\phi)}^{1} \phi b_1 dQ_1\right],
\end{aligned}
$$

or just

$$
\begin{aligned}
\gamma^{-1}F(v,b_0,\phi) = & \int_0^{b(v,1)} \left[f^{\#}(G(b_1)) + \overline{F_+}^{\#}(\hat{u}(1,b_1),b_1) \right] dQ_1 \\
& + \int_{b(v,1)}^{b(v,\phi)} \left[f^{\#}(x) + \overline{F_+}^{\#}(u(v,b_1),b_1) \right] dQ_1 \\
& + \int_{b(v,\phi)}^{1} \left[f^{\#}(G(\phi b_1)) + \overline{F_+}^{\#}(\hat{u}(\phi,b_1),b_1) \right] dQ_1 \\
& + v\left[\int_0^{b(v,1)} b_1 dQ_1 + \int_{b(v,1)}^{b(v,\phi)} f'(x) dQ_1 + \int_{b(v,\phi)}^{1} \phi b_1 dQ_1 \right].
\end{aligned}
$$

This may be rendered in a more compact way as asserted above, namely

$$\gamma^{-1}F(v,b_0,\phi) = E_{b_1}[f^{\#}(x^*(v,b_1)) + \overline{F_+}^{\#}(u^*(v,b_1),b_1)] + vq,$$

provided we introduce the notation

$$x^*(v,b_1) = \begin{cases} G(b_1) & b_1 < b(v,1), \\ x(v,b_1) & b(v,1) < b_1 < b(v,\phi), \\ G(\phi b_1) & b(v,\phi) < b_1, \end{cases}$$

and

$$u^*(v,b_1) = \begin{cases} \hat{u}(1,b_1) & b_1 < b_1(v,1), \\ u(v,b_1) & b_1(v,1) < b_1 < b_1(v,\phi), \\ \hat{u}(\phi,b_1) & b_1(v,\phi) < b_1, \end{cases}$$

where

$$x(v,b_1) = v - u(v,b_1)$$

and

$$q = \int_0^{b_1(v,1)} b_1 dQ_1 + \int_{b_1(v,1)}^{b_1(v,\phi)} f'(x(v,b_1)) dQ_1 + \int_{b_1(v,\phi)}^{1} \phi b_1 dQ_1.$$

It is convenient to define a function h_1 by the simultaneous equations

$$\left. \begin{aligned} h_1(B,\phi) &= b_1(v,\phi), \\ B &= b_1(v,1), \end{aligned} \right\}$$

i.e. $h_1(B,\phi) = b_1(v_B^1,\phi)$, where $v = v_B^1$ solves $B = b_1(v,1)$. We identify these functions in the Cobb-Douglas case in a later section. In conclusion we may define an important function $q_0(B)$ as follows:

$$q_0(B) = \int_0^{B} b_1 dQ_1 + \int_{B}^{h_1(B,\phi)} f'(x(v_B,b_1)) dQ_1 + \int_{h_1(B,\phi)}^{1} \phi b_1 dQ_1.$$

The solution for B of $q_0(B) = b_0$ is the censor $\tilde{b}_1 = \tilde{b}_1(b_0)$.

8.2 Step 2. Deduction from the Recurrence

We prove

$$V_0(v, b_0, \phi) = E[\sum_{n=1} \gamma^{n-1} f^{\#}(x_n^*(b_n))] + vq_0(b_1(v, 1)),$$

or

$$F_0(v, b_0, \phi) = E[\sum_{n=1} \gamma^{n} f^{\#}(x_n^*(b_n))] + \gamma vq_0(b_1(v, 1)).$$

Proof. We already know that

$$\gamma^{-1}F'(v, b_0) = V' = q(b(v, 1)) = \int_0^{b(v,1)} bdQ(b) + \int_{b(v,1)}^{b(v,\phi)} f'(x(v, b))dQ(b) + \int_{b(v,\phi)}^{1} \phi bdQ(b)$$

(using generic notation), hence we may also write

$$\begin{aligned} V - vq &= \gamma^{-1}[F - vq\gamma] \\ &= \gamma^{-1}[F(v, b_0, \phi) - vF'(v, b_0)] \\ &= \gamma^{-1}F^{\#}(v, b_0, \phi) \\ &= E_{b_1}[f^{\#}(x^*(v, b_1)) + \overline{F_+}^*(u^*(v, b_1), b_1)]. \end{aligned}$$

Taking expectations over ϕ, we have

$$\gamma^{-1}\overline{F}^*(v, b_0) = E_{\phi,b_1}[f^{\#}(x^*(v, b_1)) + \overline{F_+}^*(u^*(v, b_1), b_1)].$$

We may now apply this result inductively, the first steps being

$$\begin{aligned} \gamma^{-1}F_0^{\#}(v, b_0, \phi) &= E_{b_1}[f^{\#}(x_1^*(v, b_1)) + \overline{F_1}^{\#}(u_1^*(v, b_1), b_1)] \\ &= E_{b_1}[f^{\#}(x_1^*(v, b_1)) + \gamma E_{b_2}[f^{\#}(x_2^*(u_1^*, b_2)) + \overline{F_2}^{\#}(u_2^*(u_1^*, b_2), b_2)]] \\ &= E_{b_1}[f^{\#}(x_1^*(v, b_1)) + \gamma E_{b_2}[f^{\#}(x_2^*(u_1^*, b_2)) + \gamma E_{b_3}[f^{\#}(x_3^*(u_2^*, b_3)) + \overline{F_3}^{\#}(u_3^*(u_2^*, b_3), b_2) \\ &= E_{b_1}[f^{\#}(x_1^*(v, b_1)) + E_{b_2}[\gamma f^{\#}(x_2^*(u_1^*, b_2)) + E_{b_3}[\gamma^2 f^{\#}(x_3^*(u_2^*, b_3)) + \gamma^2\overline{F_3}^{\#}(u_3^*(u_2^*, b_3), \end{aligned}$$

Assuming N steps, so that $F_{N+1} = 0$, we obtain on suppressing some notation that

$$\gamma^{-1}F_0^{\#}(v, b_0, \phi_1) = E[\sum_{n=1} \gamma^{n-1} f^{\#}(x_n^*(b_n))].$$

So

$$V_0(v, b_0, \phi_1) - vq_1 = E[\sum_{n=1} \gamma^{n-1} f^{\#}(x_n^*(b_n))].$$

Rewriting, we obtain the valuation

$$\begin{aligned} & f^{\#}(x_0^*(v,b_0)) + \gamma V_0(v,b_0,\phi_1) \\ = \; & f^{\#}(x_0^*(v,b_0)) + E[\sum_{n=1}^{\infty} \gamma^n f^{\#}(x_n^*(b_n))] + v\gamma q_0(b_1(v,1)), \end{aligned}$$

where

$$q_0(B) = \int_0^B b_1 dQ_1 + \int_B^{h_1(B,\phi)} f'(x(v_B^1, b_1)) dQ_1 + \int_{h_1(B,\phi)}^1 \phi b_1 dQ_1.$$

In the next section we identify the functional form for the Cobb-Douglas case.

9 Appendix D: The Cobb-Douglas case

In this section we give the explicit form of all relevant auxiliary functions needed to compute the terms in the accounting identity. In particular we recall relevant formulas and results derived in our CDAM paper apropriate to the case $f(x) = x^{1-\alpha}/(1-\alpha)$ when $0 < \alpha < 1$. Note that $f' = x^{-\alpha}$ has inverse $G(b) = b^{-1/\alpha}$, and so

$$\begin{aligned} f^{\#}(x) &= \frac{\alpha}{1-\alpha} x^{1-\alpha}, \\ f^{\#}(G(b)) &= \frac{\alpha}{1-\alpha} b^{-(1-\alpha)/\alpha}. \end{aligned}$$

The homogeneity property is given by

$$\overline{V}'_n(u,b_n) = b_n \overline{V}'_n(u b_n^{1/\alpha}, 1),$$

and more interestingly by

$$\overline{V}'_n(u,b_n) = B\overline{V}'_n(uB^{1/\alpha}, b_n/B).$$

Thus for $\phi = \phi_n$ or $\phi = 1$ the solution $u = \hat{u}(b_n,\phi)$ to $\overline{V}'_n(u) = \overline{V}'_n(u,b_n) = b_n\phi$ is of the form $ub_n^{1/\alpha} = \overline{u}$ (i.e. $u = \overline{u}b_n^{-1/\alpha}$), where $\overline{u} = \overline{u}_n(\phi)$ solves

$$\overline{V}'_n(\overline{u}_n, 1) = \phi,$$

assuming

$$\phi > \inf_u \overline{V}'_n(u,1),$$

otherwise there is no solution (and therefore no need to sell-back stock).

It may be shown quite generally (see Appendix B) that

$$\begin{aligned} \hat{u}_{n+1}(b_n,\phi) &= G(\hat{b}_{n+1}(b_n,\phi)) + \hat{u}_{n+2}(\hat{b}_{n+1}(b_n,\phi),1) \\ &= (\hat{b}_{n+1}(b_n))^{-1/\alpha} + \hat{u}_{n+2}(\hat{b}_{n+1}(b_n),1), \end{aligned}$$

just as in (36).

Now the equation

$$\overline{V}^0_n(\overline{u}_n, 1) = \phi,$$

is equivalent (see Appendix B) under a transformation of variables to

$$q_n(B, b_n) = \phi b_n,$$

and this in turn to

$$q_n(B/b_n, 1) = \phi.$$

So we let $g = g_n(\phi)$ solve

$$q_n(g\phi, 1) = \phi,$$

with the convention[63] that $g_{n+1}(\phi) = 0$ when

$$\phi < \inf_g q_n(g, 1).$$

Thus the original equation for B is solved by setting

$$B/b_n = g_n(\phi)\phi,$$

i.e. $B = g_{n+1}(\phi)(\phi b_n)$. We thus have

$$\begin{aligned} \hat{u}_n(b_n, \phi) &= (g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi)\phi b_n)^{-1/\alpha} \\ &\quad + (g_{n+3}(1)g_{n+2}(1)g_{n+1}(\phi)\phi b_n)^{-1/\alpha} \end{aligned}$$

and

$$\begin{aligned} \widehat{v}_n(b_n, \phi) &= (\phi b_n)^{-1/\alpha} + (g_n(\phi)\phi b_n)^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + \ldots \\ &= (\phi b_n)^{-1/\alpha}\left[1 + (g_{n+1}(\phi))^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi))^{-1/\alpha} + \ldots\right] \end{aligned} \tag{42}$$

By (42) we have

$$\widehat{v}_n(b_n, \phi) = (\kappa_n(\phi b))^{-1/\alpha},$$

where

$$\kappa_n(\phi)^{-1/\alpha} = 1 + (g_{n+1}(\phi))^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi))^{-1/\alpha} + \ldots \quad . \tag{43}$$

Note that $\kappa_N \equiv 1$.Thus the solution $b = b_n(v, \phi)$ to $v = v^n_{\phi b}$ is

$$b_n(v, \phi) = \phi^{-1}\kappa_n(\phi)^{-1}v^{-\alpha}.$$

Note the identity

$$b_{n+1}(\hat{u}_n(b_n, \phi), 1) = \phi g_{n+1}(\phi) b_n. \tag{44}$$

[63] This ensures that the bench-mark stock, above which all is to be sold is infinity, in keeping with the idea that there should be no resale.

This is evident if we notice that we have to solve

$$
\begin{aligned}
\hat{u}_n(b_n,\phi) &= \hat{b}_{n+1}(b_{n+1},\phi) \\
&= (b_{n+1})^{-1/\alpha}\left[1 + (g_{n+2})^{-1/\alpha} + (g_{n+3}g_{n+2})^{-1/\alpha} + \dots\right] \\
&= (g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + \dots \quad .
\end{aligned}
$$

If the project is overstocked, the carry-forward equation

$$
f'(v - u) = \overline{V}'_n(u, b_n) \tag{45}
$$

may be re-written using homogeneity as

$$
f'(\tilde{v} - \tilde{u}(\tilde{v})) = \overline{V}'_n(u b_n^{1/\alpha}, 1),
$$

where $\tilde{u} = u b_n^{1/\alpha}$, $\tilde{v} = v b_n^{1/\alpha}$, or in standardised form as

$$
f'(\tilde{v} - \tilde{u}(\tilde{v})) = \overline{V}'_n(\tilde{u}, 1),
$$

with solution $\tilde{u}_n(\tilde{v})$. The solution of (45) is then $u_n(v, b_n) = \tilde{u}(v b_n^{1/\alpha}) b_n^{-1/\alpha}$. Note also

$$
f'([v/u] - 1) = \overline{V}'_n(1, b_n u^{\alpha})
$$

so the utilization ratio

$$
\frac{v}{u} = 1 + G(\overline{V}'_n(1, \lambda_n b_n / b_{n+1}(u)))
$$

is a function of the ratio of the current price and the top-up limit. Here λ_n is a constant.

Evidently the special functions $\tilde{u}_n(\tilde{v})$ need numeric evaluation. They are defined inductively as follows. The base of the induction is

$$
\begin{aligned}
x_N(v, b_N) &= v, \\
u_{N+1}(v) &= 0, \\
q_{N-1}(B) &= \int_0^B b_N\, dQ(b_N) + \int_B^{B\psi_N} f'(x_N(v_B, b_N))\, dQ(b_N) + \phi_N \int_{B\psi_N}^1 b_N\, dQ(b_N), \\
\psi_N &= 1/\phi_N, \\
v_B &= 1/B^2, \\
b_N(u, 1) &= 1/\sqrt{u}, \\
W_{N-1}(u) &= u + [q_{N-1}(b_N(u))]^{-2}, \\
\tilde{u}_N(v) &= W_{N-1}^{-1}(v) \\
u_N(v, b_N) &= \tilde{u}_N(v b_N^{1/\alpha}) b_N^{-1/\alpha}.
\end{aligned}
$$

The inductive step is very similar:

$$
\begin{aligned}
x_n(v, b_n) &= v - e_n(v), \\
q_{n-1}(B) &= \int_0^B b_n dQ(b_n) + \int_B^{B\psi_n} f'(x_n(v_B^n, b_n)) dQ(b_n) + \phi_n \int_{B\psi_n}^1 b_n dQ(b_n), \\
\psi_n &= \phi_n^{-1} \kappa_n(\phi_n)^{-1} \kappa_n(1), \\
v_B^n &= \frac{1}{\kappa_n(1)^2 B^2}, \\
b_n(v, 1) &= \kappa_n(1)^{-1} v^{-1/2}, \\
W_{n-1}(u) &= u + [q_{n-1}(b_n(u, 1))]^{-2}, \\
e_n(v) &= W_{n-1}^{-1}(v), \\
u_n(v, b_n) &= e_n(v b_n^{1/\alpha}) b_n^{-1/\alpha}.
\end{aligned}
$$

It is important to notice that the definition of κ_n calls for values known from earlier in the induction namely the numbers $g_m(\phi_m)$ for $m > n.$ (See (43) above.)

However, before one can use these special functions, we need to know just when to apply them, i.e when and how much stock to resell. With this in mind, recall the definition of the functions h_m given by by the simultaneous equations

$$
\left.
\begin{aligned}
h_m(B, \phi) = b_m(v, \phi) = \phi^{-1} \kappa_m(\phi)^{-1} v^{-\alpha}, \\
B = b_m(v, 1) = \kappa_m(1)^{-1} v^{-\alpha}.
\end{aligned}
\right\}
$$

Solving, we obtain

$$
\begin{aligned}
h_m(B, \phi) &= \phi^{-1} \kappa_m(\phi)^{-1} \kappa_m(1) B = \psi_m B \\
&= h_m(1, \phi) B,
\end{aligned}
$$

so that, as asserted earlier in the finite-horizon section, the dependence on B is linear. Note that $\psi_N = \phi_N^{-1}$.

As for the carry-forward, we have the explicit forms

$$
u_n^*(v, \phi_{n+1}, b_n) = \begin{cases}
(\kappa_n(1)^{-1/\alpha} - 1) b_n^{-1/\alpha} & b_n v^\alpha < \kappa_n(1)^{-1}, \\
u_{n+1}(v, b_n) & \kappa_n(1)^{-1} < b_n v^\alpha < \phi_{n+1}{}^{-1} \kappa_n(\phi_{n+1})^{-1}, \\
(\kappa_n(\phi_{n+1})^{-1/\alpha} - 1) b_n^{-1/\alpha} & \phi_{n+1}^{-1} \kappa_n(\phi_{n+1})^{-1} < b_n v^\alpha,
\end{cases}
$$

and

$$
x_n^*(v, \phi_{n+1}, b_n) = \begin{cases}
b_n^{-1/\alpha} & b_n v^\alpha < \kappa_n(1)^{-1}, \\
x_n(v, b_1) & \kappa_n(1)^{-1} < b_n v^\alpha < \phi_{n+1}{}^{-1} \kappa_n(\phi_{n+1})^{-1}, \\
(\phi_{n+1} b_1)^{-1/\alpha} & \phi_{n+1}{}^{-1} \kappa_m(\phi_{n+1})^{-1} < b_n v^\alpha.
\end{cases}
$$

From here it is a small step to compute the indirect profit $Y_n(b_n)$ by applying $f^\#(x) = \frac{\alpha}{1-\alpha} x^{1-\alpha}$ to the formulas above. Thus we have

$$
Y_n(b_n) = \begin{cases}
\frac{\alpha}{1-\alpha} b_n^{(\alpha-1)/\alpha} & b_n v^\alpha < \kappa_n(1)^{-1}, \\
\frac{\alpha}{1-\alpha} x_n(v, b_1)^{1-\alpha} & \kappa_n(1)^{-1} < b_n v^\alpha < \phi_{n+1}{}^{-1} \kappa_n(\phi_{n+1})^{-1}, \\
\frac{\alpha}{1-\alpha} (\phi_{n+1} b_1)^{(\alpha-1)/\alpha} & \phi_{n+1}{}^{-1} \kappa_m(\phi_{n+1})^{-1} < b_n v^\alpha.
\end{cases}
$$

10 Appendix E: Linear dependence of profits on output

In this appendix we prove in the Cobb-Douglas case that

$$\overline{F}(v \cdot G(b_0), b_0) = \overline{F}(v,1) f^{\#}(G(b_0)),$$

so that in the square-root case we have[64]

$$\overline{F}(v b_0^{-2}, b_0) = \frac{1}{b_0}\overline{F}(v,1).$$

Our main conclusion is the result that

$$\overline{F}(\hat{b}(b_0), b_0) = \frac{1}{b_0}\overline{F}(\hat{b}(1),1),$$

which asserts that for an optimally carried forward stock, the future expected indirect profits are linearly dependent on current indirect profit b_0^{-1}.

As for the general Cobb-Douglas situation, if $f(x) = x^{1-\alpha}/(1-\alpha)$, so that $f^{\#}(G(b)) = \frac{\alpha}{1-\alpha} b^{-(1-\alpha)/\alpha}$, the formula at the head of this section in explicit terms is as follows:

$$\overline{F}(v b_0^{-1/\alpha}, b_0) = \frac{\alpha}{1-\alpha} b_0^{-(1-\alpha)/\alpha}\overline{F}(v,1).$$

For notation see section ?? above. Observe that in the under-invested regime, when $Y_n(b_n) = \frac{\alpha}{1-\alpha} b_n^{(\alpha-1)/\alpha}$, we have

$$\overline{F}(\hat{b}(b_n), b_n) = \overline{F}(\hat{b}(1) b_n^{-1/\alpha}, b_n) = Y_n(b_n)\overline{F}(\hat{b}(1),1),$$

so the linear dependence on Y_n continues to hold.

Proof. For transparency we write the proof in the square-root case.

We again refer to the formula (compare Appendix A):

$$\begin{aligned}
\gamma^{-1} F(v, b_0, \phi) \;=\; & \int_0^{b(v,1)} \left[f^{\#}(G(b_1)) + b_1(v - \hat{b}(1,b_1)) + \overline{F_+}(\hat{b}(1,b_1), b_1) \right] dQ_1 \\
& + \int^{b(v,\phi)} \left[f(v - u(v,b_1)) + \overline{F_+}(u(v,b_1), b_1) \right] dQ_1 \\
& + \int_{b(v,\phi)}^{b(v,1)} \left[\phi b_1 (v - \hat{b}(v,b_1)) + f^{\#}(G(\phi b_1)) + \overline{F_+}(\hat{b}(\phi,b_1), b_1) \right] dQ_1.
\end{aligned}$$

We begin by assuming inductively the property that for all $v > 0$

$$\overline{F_+}(v g^{-2} b_1^{-2}, g b_1) = \frac{1}{b_1}\overline{F_+}(v g^{-2}, g),$$

[64] Thus $H(w,b) = F(1/w^2, b)$ is homogeneous of degree -1.

and show that for all v we have

$$\overline{F}(vg^{-2}b_0^{-2}, gb_0) = \frac{1}{b_0}\overline{F}(vg^{-2}, g).$$

In the formula above replace b_0 by b_0g and v by $vg^{-2}b_0^{-2}$. We also make the substitution $h = b_1/(gb_0)$. We now factorize out b_0^{-1} using inductive assumptions and some simple manipulations. To see this done note the following calculations. First note that since $b(v,1) = K/\sqrt{v}$ (for some constant K) we have $b(v(gb_0)^{-2},1) = Kb_0/\sqrt{vg^{-2}} = b(vg^{-2},1)b_0$. Next we have

$$\begin{aligned}
& \overline{F_+}(u(vg^{-2}b_0^{-2}, b_1), b_1) \\
= \; & \overline{F_+}(u(vg^{-2}b_0^{-2}, hgb_0), hgb_0) \\
= \; & \overline{F_+}(\tilde{u}(vg^{-2}b_0^{-2}(hgb_0)^2)(hgb_0)^{-2}, hgb_0) \\
= \; & \overline{F_+}(\tilde{u}(vg^{-2}(hg)^2)(hgb_0)^{-2}, hgb_0) \\
= \; & \overline{F_+}(\tilde{u}(vg^{-2}(hg)^2)(hg)^{-2}, hg)/b_0 \\
= \; & \overline{F_+}(u(vg^{-2}, hg), hg)/b_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& f(vg^{-2}b_0^{-2} - u(vg^{-2}b_0^{-2}, hgb_0)) \\
= \; & f(vg^{-2}b_0^{-2} - \tilde{u}(vg^{-2}b_0^{-2}h^2g^2b_0^2)h^{-2}g^{-2}b_0^{-2}) \\
= \; & b_0^{-1}f(vg^{-2} - \tilde{u}(vg^{-2}h^2g^2)h^{-2}g^{-2}) \\
= \; & b_0^{-1}f(vg^{-2} - u(vg^{-2}, hg))
\end{aligned}$$

(since $u_n(v, b_n) = \tilde{u}(vb_n^{1/\alpha})b_n^{-1/\alpha}$). Finally,

$$\begin{aligned}
& \overline{F_+}(\hat{u}(1, gb_0h), gb_0h) \\
= \; & \overline{F_+}(\hat{u}(1,1)(gb_0h)^{-2}, gb_0h) \\
= \; & \overline{F_+}(\hat{u}(1,1)(gh)^{-2}, gh)b_0^{-1} \\
= \; & \overline{F_+}(\hat{u}(1, gh), gh)b_0^{-1}.
\end{aligned}$$

We thus obtain (dropping the display of the third term in view of its similarity to the first) that

$$\begin{aligned}
& \gamma^{-1}F(vg^{-2}b_0^{-2}, gb_0, \phi) \\
= \; & \int_0^{b(vg^{-2},1)} \left[\frac{1}{gb_0h} + gb_0h\left(\frac{1}{(gb_0h)^2} - \frac{1}{g^2b_0^2}\hat{u}(1, gh)\right) + \overline{F_+}(\hat{u}(1, gb_0h), gb_0h)\right] dQ_1(h) \\
+ \; & \int_{b(vg^{-2},1)}^{b(vg^{-2},\phi)} \left\langle f(vg^{-2}b_0^{-2} - u(vg^{-2}b_0^{-2}, b_1)) + \overline{F_+}(u(vg^{-2}b_0^{-2}, gb_0h), gb_0h)\right\rangle dQ_1(h) + \ldots \\
= \; & \int_0^{b(vg^{-2},1)} \left(\frac{1}{b_0}\left[\frac{1}{gh} + gh\left(\frac{1}{gh^2} - \hat{u}(1, gh)\right)\right] + \overline{F_+}(\hat{u}(1, gh), gh)b_0^{-1}\right) dQ_1(h)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{b(vg^{-2},1)}^{b(vg^{-2},\phi)} \left[b_0^{-1} f(vg^{-2} - u(vg^{-2},hg)) + \overline{F_+}(u(vg^{-2},hg),hg)/b_0 \right] dQ_1(h)\\
&+\ldots\\
=\ & b_0^{-1}\int_0^{b(vg^{-2},1)} \left(\left[\frac{1}{gh} + gh\left(\frac{1}{gh^2} - \hat{u}(1,gh)\right)\right] + \overline{F_+}(\hat{u}(1,gh),gh)\right) dQ_1(h)\\
&+ b_0^{-1}\int_{b(vg^{-2},1)}^{b(vg^{-2},\phi)} \left[(f(vg^{-2} - u(vg^{-2},gh)) + \overline{F_+}(u(vg^{-2},gh),gh)\right] dQ_1(h) + \ldots\\
=\ & \frac{1}{b_0}\gamma^{-1}F(vg^{-2},g,\phi).
\end{aligned}
$$

Taking averages, we obtain the required result.

11 Appendix F: computing residual income with unrealised holding gains added to book-value

In this section we confirm the formula for the residual income $\tilde{y}^{CV}_{i+1}$, i.e. including realised and adding unrealised holding gains, at the end of the period $[t_i, t_{i+1}]$ as a function of the input price b_i. The residual income is given by cases as follows:

$$
\tilde{y}^{CV}_{i+1} = \begin{cases}
\frac{1}{b_i} + b_i v_i - v_i(1+r)h_i, & \text{for } b_i < 1/\sqrt{v_i},\\
\frac{2}{b_i} - Rh_i\frac{1}{b_i^2}, & \text{for } 1/\sqrt{v_i} \le b_i \le b_i(v_i,1),\\
2\sqrt{x(v_i,b_i)} - Rh_i x(v_i,b_i), & \text{for } b_i(v_i,1) < b_i \le b_i(v_i,\phi),\\
\frac{1}{\phi b_i}(1 - u_i(1,\phi)) + \phi b_i v_i + \frac{u_i(1,\phi)}{\phi^2 b_i^2}(1+r)h_i - (1+r)v_i h_i, & \text{for } b_i > b_i(v_i,\phi),
\end{cases}
$$

where h_i is the unit cost of the asset holding v_i at the beginning of the period $[t_i, t_{i+1}]$ inclusive of all past holding gains and book-value includes all holding gains.

We study the residual income in the three investment regimes discussed in the last section. We assume that the time $t = t_i$ opening cash and asset position is respectively c_i and v_i and that the historic unit cost of the asset is h_i. Thus

$$B_i = c_i + v_i h_i$$

is the project's book-value for the last period. We use the notation $R = 1 + r$ (so that $\gamma = R^{-1}$).

11.0.1 Under-invested

In this regime we assume the price b_i is such the optimal asset holding $\hat{v}_i = x_i + u_i > v_i$. Thus $z_i = \hat{v}_i - v_i > 0$. Here $x_i = 1/b_i^2$.

First sub-range (a) We assume first that $x_i > v_i$ i.e. $b_i < 1/\sqrt{v_i}$. Then

$$\begin{aligned} B_{i+1} &= (1+r)c_i + \frac{2}{b_i} - b_i(x_i + u_i - v_i) + b_i u_i \\ &= (1+r)c_i + \frac{1}{b_i} + b_i v_i. \end{aligned}$$

Hence

$$y_{i+1} = \frac{1}{b_i} + b_i v_i - v_i R h_i.$$

Note that at the endpoint we have

$$y_{i+1} = \frac{2}{b_i} - v_i R h_i.$$

Also observe that

$$y'_{i+1} = -\frac{1}{b_i^2} + v_i < 0$$

in this range with zero slope at $b_i = 1/\sqrt{v_i}$.

In this cases the future profit measured in currency of time t_{i+1} is

$$\gamma V(u_i, \phi, b_i) - b_i u_i = [\gamma V(u_i(1), \phi, 1) - u_i(1)] b_i^{-1}.$$

Second sub-range (b) We assume next that $x_i = 1/b_i^2 < v_i$ i.e. $1/\sqrt{v_i} < b_i$.

We assume z_i is delivered at time t_i at price p_i The new cash asset position at time t_{i+1} is

$$c_{i+1} = (1+r)c_i + \frac{2}{b_i} - b_i(x_i + u_i - v_i), \quad v_{i+1} = u_i,$$

and so

$$\begin{aligned} B_{i+1} &= (1+r)c_i + \frac{2}{b_i} - b_i(x_i + u_i - v_i) + [(v_i - x_i)Rh_i + b_i(x_i + u_i - v_i)] \\ &= (1+r)c_i + \frac{2}{b_i} + (v_i - x_i)Rh_i. \end{aligned}$$

Note that $(v_i - x_i) + [x_i + u_i - v_i] = u_i$. The quantity B_{i+1} is the adjusted book-value because the term $(v_i - x_i)Rh_i$ contains the unrealised holding gain $(v_i - x_i)rh_i$.

Hence

$$\begin{aligned} y_{i+1} &= \frac{2}{b_i} + (v_i - x_i)Rh_i - Rh_i v_i \\ &= \frac{2}{b_i} - Rh_i \frac{1}{b_i^2}. \end{aligned}$$

This has a b_i plot peaking at $b_i = Rh_i$.

Note that at the left endpoint we have agreement with the formula of the earlier subrange:

$$y_{i+1} = \frac{1}{b_i} + b_i v_i - v_i R h_i = \frac{2}{b_i} - R h_i \frac{1}{b_i^2},$$

ensuring continuity across the two subcases.

Again notice that

$$y'_{i+1} = -\frac{2}{b_i^2} + 2h_i \frac{1}{b_i^3} = \frac{2}{b_i^3}(h_i - b_i) < 0,$$

provided $b_i > h_i$. This will be the case in this subrange provided $h_i \leq 1/\sqrt{v_i}$.

In this case the future profit measured in currency of time t_{i+1} is

$$\begin{aligned}
& \gamma V(u_i, \phi, b_i) - b_i(x_i + u_i - v_i) - (v_i - x_i) R h_i \\
= \; & [\gamma V(u_i, \phi, b_i) - b_i u_i] - (v_i - x_i)(R h_i - b_i) \\
= \; & [\gamma V(u_i(1), \phi, 1) - u_i(1)] b_i^{-1} - (v_i - b_i^{-2})(R h_i - b_i).
\end{aligned}$$

11.0.2 Overinvested

In this regime we assume over-stocking so $z_i = b_i - v_i < 0$. Here $x_i = 1/(\phi b_i)^2$. Now $-z_i$ is sold at time t_i at the current price ϕp_i. Computing at time t_{i+1}, we have

$$c_{i+1} = (1 + r)c_i + \frac{2}{\phi b_i} + \phi b_i(v_i - x_i - u_i), \quad v_{i+1} = u_i,$$

$$B_{i+1} = (1 + r)c_i + \frac{2}{\phi b_i} + \phi b_i(v_i - x_i - u_i) + u_i R h_i.$$

Note that B_{i+1} is the adjusted book-value at time t_{i+1} and contains as an addition to historic value the unrealized holding gain of the assets namely $r u_i$ giving us the term $u_i R h_i$.Here

$$\begin{aligned}
y_{i+1} &= \frac{2}{\phi b_i} + \phi b_i(v_i - x_i - u_i) + u_i R h_i - R v_i h_i \\
&= \frac{1}{\phi b_i} + \phi b_i(v_i - u_i) + u_i R h_i - R v_i h_i \\
&= \frac{1}{\phi b_i}(1 - u_i(1, \phi)) + \phi b_i v_i + \frac{u_i(1, \phi)}{\phi^2 b_i^2} R h_i - R v_i h_i.
\end{aligned}$$

Notice that

$$y'_{i+1} = \phi v_i + \frac{u_i(1, \phi))}{\phi^2 b_i^3}(\phi b_i - 2 R h_i) - \frac{1}{\phi b_i^2},$$

which is positive for large enough b_i so long as $v_i > 0$.

Here the future profit measured in currency of time t_{i+1} is

$$\gamma V(u_i(b_i, \phi), \phi, b_i) - u_i R h_i = V(u_i(1, \phi), \phi, 1) b_i^{-1} - R h_i u_i(1, \phi) b_i^{-2}.$$

11.0.3 Midrange

In this regime opening stock is partitioned between current and future needs. Here the cash/asset position at time t_{i+1} is

$$c_{i+1} = Rc_i + 2\sqrt{x(v_i, b_i)}, \quad v_{i+1} = u(v_i, b_i)$$

and

$$B_{i+1} = Rc_i + 2\sqrt{x(v_i, b_i)} + Rh_i u(v_i, b_i).$$

Again the term B_{i+1} is the adjusted book-value which contains the unrealised holding gain $rh_i u(v_i, b_i)$.So

$$\begin{aligned} y_{i+1} &= 2\sqrt{x(v_i, b_i)} + h_i R u(v_i, b_i) - Rh_i v_i \\ &= 2\sqrt{x(v_i, b_i)} + Rh_i[v_i - x(v_i, b_i)] - Rh_i v_i \\ &= 2\sqrt{x(v_i, b_i)} - Rh_i x(v_i, b_i). \end{aligned}$$

Here the future profit is

$$\gamma V(u_i(v_i, b_i), \phi, b_i) - u(v_i, b_i) R h_i.$$

12 Appendix G: Book-value in the Feltham Ohlson model

It bears remarking here that the framework of the Feltham-Ohlson model takes as its primitive a notion of accounting valuation, namely the historic book-value (from which `earnings' are defined once dividends are known). Formerly, implicit in their model is a **valuation function** $\varpi(.)$ defining the book value from the portfolio $H_t = (c, v_0, v_1, ..., v_t)$ of ex-dividend cash, c, and unused investment assets $v_0, ..., v_t$ where v_i was bought at times t_i and price p_i resulting in the historic cost book valuation of assets on hand being

$$B_t = c + v_0 p_0 + ... + v_t p_t.$$

That is, suppressing the information concerning the realized prices known at time t, the valuation takes the general form:

$$B_t = \varpi(t, H_t).$$

However, the realisation of the abnormal earnings stream $\tilde{N}_t = \{\tilde{x}_t\}$, as defined from $\varpi(.,.)$, is then predicted by a model M of its dynamics, which typically depends upon the current book value as initial condition[65], and so implies first of all a stochastic process $\tilde{M}_t = \{\tilde{x}_t^M\}$, i.e. stochastically generated prediction of the realised stream $\tilde{N}_t$, and then the price of equity via the identity (3). Thus predicted price of equity is affected by the accounting convention (which is the historic cost convention in the Feltham-Ohlson model). To see this more clearly, suppose ϖ' is an alternative accounting convention, yielding the alternative valuations

$$\begin{aligned} B'_t &= \varpi'(t, H_t), \\ \tilde{x}'_t &\equiv y_t - rB'_{t-1}, \end{aligned}$$

then, provided B'_t also satisfies the standard technical assumption (concerning the rate of convergence), we have, by the usual argument

$$B_t + \sum_{\tau=1} \gamma^\tau E_t(\tilde{x}_{t+\tau}) = W_t = B'_t + \sum_{\tau=1} \gamma^\tau E_t(\tilde{x}'_{t+\tau}).$$

So, abbreviating the summation of discounted expected values temporarily to V_t, we have

$$B_t + V_t[\tilde{N}_t] = B'_t + V_t[\tilde{N}'_t]. \tag{46}$$

It could therefore be shown that the same model of the earnings stream dynamics M gives a value

$$B'_t + V_t[\tilde{M}'_t],$$

[65] By formulae such as

$$W_t = B_t + \frac{\omega}{R - \omega}\tilde{x}_t + \frac{R}{(R - \omega)(R - \gamma)} x_t.$$

which is perhaps a better predictor of W_t than $B_t + V_t[\tilde{M}_t]$. Now observed actual discrepancies from the realization could either deny validity of the $AR(1)$ assumption, or require that any explanation absorb the discrepancy in a dividend policy consistent with the $AR(1)$ assumption via (2), i.e.

$$d_t^M = \tilde{\mathcal{J}}_t^M - B_t + (1 + r)B_{t-1}.$$

However, an alternative accounting convention could perhaps generate different model predictions closer to reality despite using the same underlying stochastic dynamics.

Evidently, the technical assumption proving (3), namely that $\gamma^\tau B_\tau \to 0$ as $\tau \to \infty$ (i.e. that book value does not grow faster than the bank yield $1 + r$), implicitly favours the **historic cost convention** (as perpetually unused stock is in the limit discounted to zero). However, the technical assumption may be satisfied by any other convention governing unused production input assets provided, for instance, that these assets are utilized almost surely within a uniformly bounded horizon. In reality there is an expiry date for most inputs and this guarantees that it is optimal to utilize them ahead of the best-before date.

Fortunately no such technicalities arise in a finite horizon; moreover, in that setting there is a identity corresponding to (3) that includes final book-value B_T (possibly as final dividend). There is thus an alternative convention directly justifiable by the definition of residual income itself. Inspection of an equivalent to the defining equation, namely

$$\tilde{\mathcal{J}}_t = B_t - (1 + r)B_{t-1} + d_t, \tag{47}$$

in which old book-value is interest-adjusted before being deducted from current book-value, suggests a common value rendering of the two book-values. We may therefore justifiably use as alternative accounting valuation the following function $\varpi'(.)$ in accord with the **common value accounting convention**, namely

$$B'_t = \varpi'(t, H) = c + v_0(1 + r)^t p_0 + v_1(1 + r)^{t-1} p_1 + \ldots + v_t p_t.$$

The valuation B'_t thus includes in c the interest on cash in the bank from recorded earlier revenues and also attracts cost-of-capital charges on top of historic costs.

Thus cost of unused stock recorded in both B'_t and B'_{t-1} on this convention cancel each other out in the (47) calculation of residual income, allowing treatment of unused 'investment stock in place' just like interest on any earlier cash deposits sitting in the bank. This has two important consequences:

(i) current value residual income attributable to immediate utilization of investment stock is **increased** by comparison to the historic cost convention, which would not include any holding gains on the investment stock;

(ii) residual income attributable to investment stock that has been in place for multiple periods is **decreased** relative to the historic cost convention.

Both these factors properly reflect return from investment in rewarding the record of profitable activity from investment and down-playing unprofitable activity. Note that any unused stock sold back will also increase the value of residual income as a cash addition.

We stress that both conventions must of necessity give rise to the same value of the firm by (46), and either earnings stream may be interpreted from the other, for instance

$$V_t[\tilde{N}_t]_{\text{hist}} = (V_t[\tilde{N}_t^0] + B_t^0)_{\text{current}} - (B_t)_{\text{hist}},$$

or as

$$(\tilde{e}_t)_{\text{hist}} = (\tilde{e}_t^0)_{\text{current}} + [B_t - (1+r)B_{t-1}]_{\text{hist}} - [B_t^0 - (1+r)B_{t-1}^0]_{\text{current}}.$$

However, as each gives a different interpretation to the term `residual income', each offers a different route to predicting managerial activity and predicted residual earnings stream. In each dividends are left outside the scope of equity-value computation.

We should point out an additional advantage of the modified convention that well serves our purposes. If we employ a model of economic activity with constant expected return then the common value convention automatically gives constant returns to unused stock.

12.0.4 Example: A stylised two period residual income model

Suppose we start with $x + u$ units of capital at $t = 0$ purchased for p_0 a unit[66] and we plan to use x of the units in the first period and u of the units in the second period[67] with a square root returns function operating in both periods, that is:

$$\text{opening net assets } B_0 = p_0(u + x).$$

We assume a square root returns function.

Version 1: stylised model under historic cost convention We compute the two periods' respective earnings and residual incomes under the historic cost convention

$$\begin{array}{ll}
B_1 = 2\sqrt{x} + p_0 u, & B_2 = (1+r)2\sqrt{x} + 2\sqrt{u}, \\
B_1 - B_0 = v_1, & B_2 - B_1 = v_2, \\
v_1 = 2\sqrt{x} - p_0 x, & v_2 = 2\sqrt{u} - p_0 u + 2\sqrt{x} r, \\
\tilde{e}_1 = v_1 - r p_0(u + x), & \tilde{e}_2 = v_2 - r(2\sqrt{x} + p_0 u), \\
= 2\sqrt{x} - (1+r)p_0 x - r p_0 u, & = 2\sqrt{u} - (1+r)p_0 u.
\end{array}$$

Note that the revenue $2\sqrt{x}$ included in B_1 arises at the end of the first period (i.e. time $t = 1$). As a check, note the value of the firm at time $t = 0$ is

$$B_0 + \frac{\tilde{e}_1}{1+r} + \frac{\tilde{e}_2}{(1+r)^2}$$

[66] Assume this is financed by the owners initial equity investment.

[67] In order to make the simplest representation we shall assume that u is the dynamically optimal second-period usage; that is, even though the firm could buy or sell more units after observing the second period input price of capital it is not optimal to buy or sell capital. Our immediate object here is to map the the two models into a common notation rather than to concentrate on optimization. Once the mapping is established we will return to optimization issues.

$$
\begin{aligned}
&= p_0(u+x) + \frac{2\sqrt{\bar{x}} - (1+r)p_0x - rp_0u}{1+r} + \frac{2\sqrt{\bar{u}} - (1+r)p_0u}{(1+r)^2} \\
&= p_0u + \frac{2\sqrt{\bar{x}} - rp_0u}{1+r} + \frac{2\sqrt{\bar{u}} - (1+r)p_0u}{(1+r)^2} \\
&= \frac{2\sqrt{\bar{x}}}{1+r} + \frac{2\sqrt{\bar{u}}}{(1+r)^2}.
\end{aligned}
$$

Version 2: stylised model under `common values' convention And now we compute using the common value accounting convention, as given below equation (47):

$$
\begin{aligned}
B'_1 &= 2\sqrt{\bar{x}} + p_0u(1+r), & B'_2 &= (1+r)2\sqrt{\bar{x}} + 2\sqrt{\bar{u}}, \\
B'_1 - B'_0 &= v'_1, & B'_2 - B'_1 &= v'_2, \\
v'_1 &= 2\sqrt{\bar{x}} - p_0x + p_0ur, & v'_2 &= 2\sqrt{\bar{u}} - p_0u(1+r) + 2\sqrt{\bar{x}}r, \\
\tilde{e}'_1 &= v'_1 - rp_0(u+x), & \tilde{e}'_2 &= v'_2 - r(2\sqrt{\bar{x}} + p_0u(1+r)), \\
&= 2\sqrt{\bar{x}} - (1+r)p_0x, & &= 2\sqrt{\bar{u}} - (1+r)^2p_0u.
\end{aligned}
$$

Here

$$
\begin{aligned}
& p_0(u+x) + \frac{2\sqrt{\bar{x}} - (1+r)p_0x}{1+r} + \frac{2\sqrt{\bar{u}} - (1+r)^2p_0u}{(1+r)^2} \\
&= \frac{2\sqrt{\bar{x}}}{1+r} + \frac{2\sqrt{\bar{u}}}{(1+r)^2}.
\end{aligned}
$$

Observe that B'_1 includes the current income and the interest-adjusted historic valuation of unused stock left languishing; hence the residual income $\tilde{e}'_1$ comprises the profit on current production using stock valued at the interest-adjusted historic valuation (as it was bought one period ago). Similarly, B'_2 includes the current cash revenue and deposited cash revenues from the previous period (compounded up); consequent on the treatement in B'_1 of unused stock, the residual income $\tilde{e}'_2$ here equals the profit from final production using long unused stock valued at the interest-adjusted historic valuation (bought two periods ago). Recalling

$$
(\tilde{e}_t)_{\text{hist}} = (\tilde{e}_t)_{\text{current}} + [B_t - (1+r)B_{t-1}]_{\text{hist}} - [B'_t - (1+r)B'_{t-1}]_{\text{current}},
$$

we have

$$
(B_1)_{\text{hist}} - (B'_1)_{\text{current}} = (2\sqrt{\bar{x}} + p_0u) - (2\sqrt{\bar{x}} + p_0u(1+r)) = -p_0ur
$$

$$
(B_2)_{\text{hist}} - (B'_2)_{\text{current}} = 0.
$$

13 Appendix H: Monotonicity[68] of $V^{\#}$

Recall that $u(v,b)$ is the optimal carry-forward when the current resource price is b and the stock v held is such that no units of resource are acquired nor resold. We need to consider the marginal valuation

$$\begin{aligned} P(b) &= F(u(v,b),b) - u(v,b)F'(u(v,b),b) \\ &= \frac{1}{b}[F(\bar{u},1) - \bar{u}F'(\bar{u},1)] \end{aligned}$$

(note that $\bar{u}(vb^2) = u(v,b)/b^2$),or dropping the second variable

$$P(b) = \frac{1}{b}[F(U(b^2v)) - U(b^2v)F'(U(b^2v))].$$

which is of central importance to us. It represents the benefit of the future value of $w = b^2v$ relative to the current price level b. Here $U(w)$ denotes the solution to the equation

$$f'(w - U(w)) = F'(U(w)).$$

We need to know that $P(b)$ is decreasing with b.

13.1 An equivalent formulation

Put $b = \sqrt{w/v}$ (i.e. $w = b^2v$), and since w increases with b,write

$$P(\sqrt{w/v}) = \frac{\sqrt{v}}{\sqrt{w}}[F(U(w)) - U(w)F'(U(w))]$$

or since v is constant we ask to show that the following is decreasing with w :

$$\frac{1}{\sqrt{w}}[F(U(w)) - U(w)F'(U(w))].$$

This is the ratio of future profit to current 'profit' $2\sqrt{w} - w\frac{1}{\sqrt{w}} = \sqrt{w}$ in which the current cost is measured at the marginal value of $1/\sqrt{w}$.

This leads to a further simplification. Put $u = U(w)$, so that $w = V(u)$, where $w = V(u)$ is the inverse function to $u = U(w)$. Thus $V(u)$ solves the equation

$$f'(V(u) - u) = F'(u).$$

(Compare Appendix D.) We therefore consider the ratio

$$\frac{F(u) - uF'(u)}{\sqrt{V(u)}}.$$

[68] We gratefully acknowledge the contribution of Graham Brightwell to this appendix.

13.2 An integral inequality

Let

$$
\begin{aligned}
\Psi(u) &= F(u) - uF'(u) \\
&= \int_0^{1/\sqrt{u}} (\frac{1}{b_1} dQ_1 + \int_{1/\sqrt{u}}^{1/(\phi_1\sqrt{u})} \sqrt{u}\, dQ_1 + \int_{1/(\phi_1\sqrt{u})}^{1} \frac{1}{\phi_1 b_1} dQ_1
\end{aligned}
$$

we consider

$$
\frac{\Psi(u)}{\sqrt{V(u)}} = \frac{\Psi(u)}{\sqrt{u}} \cdot \sqrt{\frac{u}{V(u)}}.
$$

The left-hand side is an intertemporal comparison of the future use of the apportioned resource $\sqrt{u}$, against the immediate use of the entire resource $V(u)$. On the right-hand side $\Psi(u)/\sqrt{u}$ compares the future profit from use of u to the immediate profit from the use of u on its own (which would have led to a benefit $f^{\#}(u) = \sqrt{u}$).

We now let

$$
\begin{aligned}
K(u) &=_{def} \Psi(u)/\sqrt{u} \\
&= \frac{1}{\sqrt{u}} \int_0^{1/\sqrt{u}} \frac{1}{b_1} dQ_1 + \int_{1/\sqrt{u}}^{1/(\phi_1\sqrt{u})} dQ_1 + \frac{1}{\sqrt{u}} \int_{1/(\phi_1\sqrt{u})}^{1} \frac{1}{\phi_1 b_1} dQ_1 \\
&= \frac{1}{\sqrt{u}} X + Y
\end{aligned}
$$

where $X = X_1 + X_2$ and

$$
X_1 = \int_0^{1/\sqrt{u}} \frac{1}{b_1} dQ_1, \qquad X_2 = \int_{1/(\phi_1\sqrt{u})}^{1} \frac{1}{\phi_1 b_1} dQ_1, \qquad Y = \int_{1/\sqrt{u}}^{1/(\phi_1\sqrt{u})} dQ_1.
$$

Similarly, we may consider the ratio of future marginal benefit $F'(u)$ to the immediate marginal benefit of using u namely $1/\sqrt{u}$. We thus put

$$
\begin{aligned}
L(u) &=_{def} \sqrt{u} F'(u) \\
&= \sqrt{u} \int_0^{1/\sqrt{u}} b_1 dQ_1 + \int_{1/\sqrt{u}}^{1/(\phi\sqrt{u})} dQ_1 + \phi\sqrt{u} \int_{1/(\phi\sqrt{u})}^{1} b_1 dQ_1 \\
&= \sqrt{u} Z + Y,
\end{aligned}
$$

where $Z = (Z_1 + Z_2)$ and

$$
Z_1 = \int_0^{1/\sqrt{u}} b_1 dQ_1, Z_2 = \phi \int_{1/(\phi\sqrt{u})}^{1} b_1 dQ_1
$$

Note that the `apportionment ratio' is

$$
\frac{V(u)}{u} = \frac{1}{u}\left(u + F'(u)^{-2} \right) = 1 + L(u)^{-2}.
$$

The ratio of interest is thus

$$A(u) =_{def} \frac{K(u)}{\sqrt{1 + L(u)^{-2}}} = \frac{-(u)}{\sqrt{u}} \cdot \sqrt{\frac{u}{V(u)}},$$

and we wish to show $A'(u) < 0$.

As a preliminary we compute that

$$K'(u) = -\frac{1}{2}u^{-3/2}\int_0^{1/\sqrt{u}} \frac{1}{b_1} dQ_1 - \frac{1}{2}u^{-3/2}\int_{1/(\phi_1\sqrt{u})}^1 \frac{1}{\phi_1 b_1} dQ_1 < 0,$$

so that we have

$$K(u) = \frac{X}{\sqrt{u}} + Y, \qquad -K'(u) = \frac{X}{2u^{3/2}}.$$

Likewise

$$L'(u) = \frac{1}{2}u^{-1/2}\int_0^{1/\sqrt{u}} b_1 dQ_1 + \frac{1}{2}u^{-1/2}\phi\int_{1/(\phi\sqrt{u})}^1 b_1\sqrt{u}\, dQ_1$$

and again

$$L(u) = \sqrt{u}Z + Y, \qquad L'(u) = \frac{Z}{2\sqrt{u}}.$$

Now

$$\begin{aligned} A'(u) &= \frac{K'(u)}{\sqrt{1 + L(u)^{-2}}} + \frac{K(u)}{(1 + L(u)^{-2})^{3/2}}\frac{L'(u)}{L(u)^3} \\ &= \frac{1}{L(u)^3\,(1 + L(u)^{-2})^{3/2}}\left[K'(u)(L(u)^3 + L(u)) + K(u)L'(u)\right] \end{aligned}$$

We thus need to show that[69]

$$-K'(u)(L(u)^3 + L(u)) > K(u)L'(u)$$

or

$$\frac{1}{2}u^{-3/2}X\left[\sqrt{u}Z + Y + L^3\right] > \frac{Z}{2\sqrt{u}}\left[\frac{X}{\sqrt{u}} + Y\right],$$

i.e.

$$\frac{X}{u}\left[\sqrt{u}Z + Y + L^3\right] > Z\left[\frac{X}{\sqrt{u}} + Y\right].$$

[69] Alternatively, we require that

$$-K'(u)L(u)^3 > K(u)L'(u) + K'(u)L(u)$$

and the left-hand side is positive. Thus the condition is satisfied in any u interval where KL is decreasing in u.

or, on subtracting $XZ/\sqrt{u}$ from each side

$$\frac{X}{u}\left[Y + L^3\right] > YZ$$

or with $X^* = X/\sqrt{u}$ and $Z^* = \sqrt{u}Z$ we require for monotonicity that

$$X^*\left[Y + L^3\right] > YZ^*. \tag{48}$$

13.3 Verification

In this section we show that

$$X^*\left[Y + L^3\right] > YZ^*,$$

provided ϕ_1 is not too small, namely provided

$$\phi_1 > \exp(-1.65396\sigma), \tag{49}$$

(so that for a typical annual standard deviation σ of 30% we require $\phi_1 > 60\%$). Alternatively for a given ϕ_1 this requires that

$$\sigma > \frac{\ln 1/\phi_1}{1.65396}.$$

Recall that

$$Y = \int_{1/\sqrt{u}}^{1/(\phi_1\sqrt{u})} dQ_1 = Q_1(1/(\phi_1\sqrt{u})) - Q_1(1/(\sqrt{u}))$$

and

$$X^* = \frac{1}{\sqrt{u}}\left(\int_0^{1/\sqrt{u}} \frac{1}{b_1}dQ_1 + \int_{1/(\phi_1\sqrt{u})}^{1} \frac{1}{\phi_1 b_1}dQ_1\right)$$

$$Z^* = \sqrt{u}\left(\int_0^{1/\sqrt{u}} b_1 dQ_1 + \int_{1/(\phi_1\sqrt{u})}^{1} \phi_1 b_1 dQ_1\right)$$

and $L = \sqrt{u}F'(u)$ so that

$$L = \sqrt{u}F'(u) = \sqrt{u}Z + Y = Z^* + Y.$$

The argument divides as $L \le 1$ or $L > 1$ i.e. $Z^* \le 1 - Y$ or $Z^* > 1 - Y$.

Remark. The optimal hedge $u = \hat{u}$ is such that

$$\int_0^{1/\sqrt{u}} b_1 dQ_1 + \int_{1/(\phi_1\sqrt{u})}^{1} \phi_1 b_1 dQ_1 + \frac{1}{\sqrt{u}}\int_{1/\sqrt{u}}^{1/(\phi_1\sqrt{u})} dQ_1 = 1,$$

i.e.

$$L = \sqrt{\hat{u}},$$

so the two cases we consider when $u = \hat{u}$ are accordingly $\sqrt{\hat{u}} < 1$ or $\sqrt{\hat{u}} \ge 1$ respectively.

13.4 Case $Z^* \le 1 - Y$.i.e. $L \le 1$

Now we claim that when $L \le 1$ we have the stronger strict inequality, which evidently implies (48), that

$$X^* > Z^*.$$

Indeed by Jensen's Inequality $KL > 1$ so if $L \le 1$ then $K > 1 \ge L$ so in particular

$$X^* + Y > Z^* + Y.$$

Remark. We have just verified that in this case

$$0 > K(u)L'(u) + K'(u)L(u)$$

that is KL is decreasing in u.

13.5 Case $Z^* \ge 1 - Y$.i.e. $L \ge 1$, or $\hat{u} > 1$

Here we aim to show that provided

$$\phi_1 > \exp(-1.65396\sigma).$$

the condition (48) holds.

13.5.1 We prove a tighter condition ...

By Jensen

$$\begin{aligned} XZ &= \left(\int_0^{1/\sqrt{\bar{u}}} \frac{1}{b_1} dQ_1 + \int_{1/(\phi_1 \sqrt{\bar{u}})}^{1} \frac{1}{\phi_1 b_1} dQ_1 \right) \left(\int_0^{1/\sqrt{\bar{u}}} b_1 dQ_1 + \int_{1/(\phi \sqrt{\bar{u}})}^{1} \phi b_1 dQ_1 \right) \\ &> (1 - Y)(1 - Y) \end{aligned}$$

i.e.

$$X^* Z^* = XZ > (1 - Y)^2$$

To satisfy (48), i.e.

$$X^*(Y + L^3) \ge YZ^*,$$

it is equivalent to have

$$X^* Z^*(Y + L^3) \ge Y(Z^*)^2$$

and thus enough to have

$$(1 - Y)^2(Y + L^3) > Y(Z^*)^2.$$

13.5.2 A monotonicity argument

We consider

$$\Delta = (1 - Y)^2(Y + [Y + Z^*]^3) - Y(Z^*)^2$$

Treating Z^* as a free variable, so that $\Delta = \Delta(Z^*)$, and Y fixed at it true value (when u is given) observe that this difference Δ is strictly positive (and the strict inequality is true) when $Z^* = 1 - Y$. Indeed

$$\Delta(1 - Y) = (1 - Y)^2(Y + 1) - Y(1 - Y)^2 = (1 - Y)^2.$$

Recall that $L = Z^* + Y > 1$ i.e. $Z^* > 1 - Y$. So we check that $\Delta(Z^*)$ is increasing in Z^*.

We differentiate with respect to Z^* to obtain

$$\Delta' = (1 - Y)^2(3[Y + Z^*]^2) - 2YZ^*.$$

If

$$(1 - Y)^2 > \frac{1}{6}$$

(i.e. $Y < 1 - 1/\sqrt{6}$,which we term `the $\sqrt{6}$ condition') we obtain

$$\Delta' > \frac{1}{2}[Y + Z^*]^2 - 2YZ^* \geq 0,$$

since

$$[Y + Z^*]^2 \geq 4YZ^*.$$

We are now done, i.e. the inequality holds in the case that $Z^* > 1 - Y$ subject to the $\sqrt{6}$ condition on Y holding and in fact

$$\Delta > (1 - Y)^2 > \frac{1}{6}. \tag{50}$$

We now study the $\sqrt{6}$-condition and show that this is implied by (49).

13.6 The $\sqrt{6}$condition

By this we mean

$$Y = P[\phi_1 t \leq b_1 \leq t] < 1 - \frac{1}{\sqrt{6}},$$

with $\phi_1 < 1$ where $t = 1/\phi_1\sqrt{u}$. Now let $P_1(t) = P[b_1 \leq t]$ then

$$P[\phi_1 t \leq b_1 \leq t] = P_1(t) - P_1(\phi_1 t)$$

is easy to study. We have

$$P_1(t) = \Phi((\log t - m)/\sigma).$$

For $t > 0$ the function $P_1(t) - P_1(\phi_1 t)$ has a maximum when

$$t = e^m / \sqrt{\phi_1}$$

equal to

$$1 - 2\Phi(\frac{1}{2\sigma}\log\phi_1),$$

so that for $t > 0$ the $\sqrt{6}$-condition holds uniformly in t provided ϕ_1 is large enough, namely

$$1 - 2\Phi(\frac{1}{2\sigma}\log\phi_1) < 1 - \frac{1}{\sqrt{6}}$$

or

$$\log\phi_1 > 2\Phi^{-1}(\frac{1}{2\sqrt{6}})\sigma = -1.65396\sigma$$

i.e.

$$\phi_1 > \exp(-1.65396\sigma).$$

13.7 N-fold version

In this section of the appendix we indicate why in the mutiple-period setting it is still true that provided ϕ_1 is large enough ψ is decreasing in price b_1 and so $V^{\#}$ is increasing in q-income. One identifies a condition for monotonicity analogous to that of the two period model and verifies that it holds provided that the forthcoming discount factor ϕ_1 approaches unity. This kind of argument does not give an explicit bound for ϕ_1 although the condition (49) still needs to hold. The idea of the proof is to demonstrate that the new condition analogous to (48) reduces back to the old condition (48) when we ignore certain terms. As the old condition (48) is a strict inequality and in fact (50) holds, we deduce our result by showing that the additional terms tend to zero as ϕ_1 tends to unity. The additional terms contain as factors the two integrals

$$\int_{b(v)}^{b(v,\phi)} \frac{x(v,b_1)/v - x'(v,b_1)}{\sqrt{x(v,b_1)}} dQ_1, \qquad \int_{b(v)}^{b(v,\phi)} \frac{[x(v,b_1)/v - x'(v,b_1)]}{x(v,b_1)^{3/2}} dQ_1.$$

13.8 Postscript: Monotonicity of y(b)

For the record we prove thaty that $y(b)$ is decreasing

We have since $y = \sqrt{x(v,b)}$ that

$$\frac{1}{y} = F_u(u(v,b),b) = bF_u(\tilde{u},1) = bF_u(b^2(v - y^2),1)$$

so

$$-\frac{1}{y^2}\frac{dy}{db} = F_u(\tilde{u},1) + bF_{uu}(\tilde{u},1)(2bu - 2yb^2\frac{dy}{db})$$

so

$$\left(2yb^3F_{uu}(\bar{u},1) - \frac{1}{y^2}\right)\frac{dy}{db} = F_u(\bar{u},1) + F_{uu}(\bar{u},1)2\bar{u}.$$

The lhs bracket is negative (since F is concave). So we must now prove the rhs is positive. Here $x(u, b_1) = u,$ (in this period no resources can be carried forward), so

$$F_u(\bar{u},1) = \int_0^{b_1(\bar{u})} b_1 dQ(b_1) + f'(\bar{u})\int_{b_1(\bar{u})}^{b_1(\bar{u},\phi)} dQ(b_1) + \phi\int_{b_1(\bar{u},\phi)}^{1} b_1 dQ(b_1)$$

and

$$F_{uu}(\bar{u},1) = f''(\bar{u})\int_{b_1(\bar{u})}^{b_1(\bar{u},\phi)} dQ(b_1).$$

But $f'(\bar{u}) = \bar{u}^{1/2}$, and $2\bar{u}f''(\bar{u}) = -\bar{u}^{1-3/2} = -\bar{u}^{1/2}$ thus

$$\begin{aligned}
& F_u(\bar{u},1) + F_{uu}(\bar{u},1)2\bar{u} \\
= & \int_0^{b_1(\bar{u})} b_1 dQ(b_1) + \phi\int_{b_1(\bar{u},\phi)}^{1} b_1 dQ(b_1) \\
& +[f'(\bar{u}) + 2\bar{u}f''(\bar{u})]\int_{b_1(\bar{u})}^{b_1(\bar{u},\phi)} dQ(b_1) \\
= & \int_0^{b_1(\bar{u})} b_1 dQ(b_1) + \phi\int_{b_1(\bar{u},\phi)}^{1} b_1 dQ(b_1) > 0.
\end{aligned}$$