Dichotomy and infinite combinatorics: the theorems of Steinhaus and Ostrowski

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(*Received*)

Abstract

We define combinatorial principles which unify and extend the classical results of Steinhaus and Piccard on the existence of interior points in the distance set. Thus the measure and category versions are derived from one topological theorem on interior points applied to the usual topology and the density topology on the line. Likewise we unify the subgroup theorem by reference to a Ramsey property. A combinatorial form of Ostrowski's theorem (that a bounded additive function is linear) permits the deduction of both the measure and category automatic continuity theorems for additive functions.

1. *Introduction*

The motivating results of this paper are the following two classical results of real analysis.

Theorem S (Steinhaus' Theorem, [**91**]). *For S (Lebesgue) measurable and of positive measure, the difference set $S - S$ contains an interval around the origin.*

A function $f : \mathbb{R} \to \mathbb{R}$ is called *additive* if it satisfies the *Cauchy functional equation*

$$f(x + y) = f(x) + f(y).$$

Theorem O (Ostrowski's Theorem, [**79**], cf. [**49**]). *If $f : \mathbb{R} \to \mathbb{R}$ is additive and bounded above (or below) on a set of positive measure, then f is linear: $f(x) = cx$ for some c.*

The first thing to note is that measure theory is not the only context for such results (indeed, it is not even the most natural one, as we shall see). Both have topological (or category) versions, in which the set, or function, has the property of Baire (briefly, 'is Baire' – see [**29**], [**53**]). We use 'meagre' for 'of the first category' (in relation to a set that is a countable union of nowhere dense sets).

THEOREM P (Piccard's Theorem [**83**]). *For S Baire and non-meagre, the difference set $S-S$ contains an interval around the origin.*

THEOREM BM (Banach-Mehdi Theorem – [**3**, Th. 4 p. 35], [**57**]). *If $f:\mathbb{R}\to\mathbb{R}$ is additive and bounded above (or below) on a non-meagre Baire set S, then f is linear: $f(x)=cx$ for some c.*

One thus seeks the natural common generalization of measurability and the Baire property. We find this in the context of infinite combinatorics; see below.

Each of these pairs of theorems exemplifies a stark *dichotomy*, in which things are either very nice or very nasty. For Theorems O and BM on additive functions: it is known that non-linear additive functions exist in profusion, and can be constructed from *Hamel bases* (of the reals over the rationals – see e.g. [**51**], V.2, [**34**] – [**25**] §5.2). Since by Theorems O and BM non-linear additive functions are highly pathological, we may call the existence of such functions the *Hamel pathology*. But, in the presence of even the merest hint of good behaviour, an additive function is beautifully well-behaved – it is (continuous, and hence) linear. This we call the *Darboux dichotomy* (cf. Th. D §7). The general context for such results is that of *automatic continuity*, studied e.g. by us ([**12**], [**14**],[**13**]) for real analysis, Dales [**28**] for Banach algebras, and Hoffmann-Jørgensen in [**93**] §2 and also [**89**] and [**85**]. There is a general dichotomy in the background: normed groups ([**33**] I.7.2) are either topological or pathological ([**18**]; [**76**], [**77**] §4).

For Theorems S and P on interior points, the general context is that of topological groups (or normed groups: [**18**]). The difference set is either topologically small (has empty interior), or topologically large (contains an interval). This we call the *Steinhaus dichotomy.*

There is a third dichotomy (Theorems 6·1 and 6·2 below), concerning additive subgroups of the reals. Such a subgroup is either very small in some sense, or is the whole of the reals. This we call the *subgroup dichotomy.*

Our motivation for the thorough-going scrutiny of these classical results given below comes from Karamata's theory of *regular variation*. In the standard work on this, [**6**], Steinhaus' Theorem S is Theorem 1.1.1 and Ostrowski's Theorem O is Theorem 1.1.7. Firstly, our integrated approach to the measurable and Baire cases provides the answer to the long-standing foundational question raised there ([**6**], p.11). Secondly, it also lends itself to generalization far beyond the classical setting of real analysis (functions $f:\mathbb{R}\to\mathbb{R}$); see [**6**] Appendix 1 for background here. It is now known that the theory may be developed in the contexts of normed (cf. [**33**] Th. I.7.2) and topological groups ([**18**]; [**76**], [**77**]); we confine ourselves here to the classical setting of real analysis for concreteness and simplicity.

There is a third aspect. In addition to the foundational and contextual aspects above, we were motivated by a desire to reduce the number of hard proofs in the subject, and to simplify them as far as possible. Our approach was suggested by a detailed study of the hard proofs in [**6**] Ch. 1-3 (that is, on theory, rather than applications) detailed below, and in particular, of the thematic similarities between them. What emerges involves an interplay between sequential, non-sequential and group-theoretic aspects. This is particularly the case in regard to the Steinhaus theory and additive functions, with which we deal first (cf. [**6**] §1.1). For the Uniform Convergence Theorem (UCT), the main result of the subject, see [**8**]. For Delange's theorem ([**6**] Th. 2.0.1), see [**15**]; here the classical result disaggregates, according to which axioms of set theory one assumes. For [**6**]

Theorems 1.4.3 and 3.2.5, see [**16**]. For a monograph synthesis of our approach, that of *topological regular variation*, we refer to [**20**]. Our aim has been to reduce the number of hard proofs – not only here but elsewhere in the subject – to zero.

In §2 and 3 below we give the definitions needed to formulate our approach. In §4 we introduce our bitopological viewpoint (we reduce the measurable case to the Baire case by switching from the Euclidean topology to the density topology), and develop our main tool, the Category Embedding Theorem (Th. 4·1 and [**10**]), obtaining Theorems S and P as corollaries in §5. In §6 we give more on Steinhaus' Theorem. We turn to Theorems O and BM in §7. In §8 we give a converse to Ostrowski's theorem, Theorem O. We turn briefly to higher dimensions in §9. We close in §10 with a number of remarks.

2. *Baire category aspects*

In this paper the term 'Baire' will always be used in relation to Baire category ideas, which occur in several related aspects: the Baire categories (meagre, non-meagre sets); the Baire Category Theorem, that in a complete metric space the intersection of dense open sets is *dense* (in fact is an absolute $\mathcal{G}_\delta$ set, i.e. is $\mathcal{G}_\delta$ in any completion and is competely metrizable); Baire spaces (spaces where this theorem holds); sets with the Baire property (i.e. sets open modulo a meagre set); and Baire measurable functions (for which the inverse image of an open sets is a set with the Baire property). For general information on Baire spaces, we refer to [**29**] §3.9 (especially for topological completeness and the absolute $\mathcal{G}_\delta$ property), and also to [**46**] §8. Note that:
(i) a space that is the union of an absolute $\mathcal{G}_\delta$ set and a meagre set (call this 'an almost complete space') is a Baire space [**1**];
(ii) in a complete metric space, a set with the Baire property is the union of an absolute $\mathcal{G}_\delta$ set and a meagre set [**46**], 8.23;
(iii) a topologically well-defined subset, such as an analytic set (for which see [**42**], Cor. 2.9.4 – Nikodym's theorem), has the Baire property, so is almost complete by (i);
(iv) a Baire measurable function between separable metric spaces is continuous on a co-meagre absolute $\mathcal{G}_\delta$ set, (see [**18**], §11);
(v) a metric space X is Baire iff X has the Blumberg property: for any (arbitrary) function $f : X \to \mathbb{R}$ there is a dense subspace $D = D_f$ on which the restriction function $f|D$ is continuous. (See [**23**]; [**38**] §3.1 has a wide ranging discussion of Blumberg-type characterizations of Baire spaces. Compare [**58**] Prop. 7.8, for a more recent application.)
So, when working with 'decently defined' spaces and sets, all these notions merge into one notion, that of almost completeness.

The term 'almost complete' (in the category sense above) is due to E. Michael (see [**58**]), but Frolík introduced the notion in terms of open 'almost covers' (i.e. open families that cover a dense subspace, see [**32**] §4) and demonstrated its relation to the existence of dense $\mathcal{G}_\delta$-subspaces. It was first named 'almost Čech-complete' by Aarts and Lutzer ([**1**] §4.1.2). For metric spaces our category definition in (i) above is equivalent (and more directly connects with completeness). Indeed, on the one hand a completely regular space is almost Čech-complete iff it contains a dense Čech-complete (or topologically complete) subspace, i.e. one that is absolutely $\mathcal{G}_\delta$ (is $\mathcal{G}_\delta$ in some/any compactification). On the other hand a metrizable Baire space X contains a dense completely metrizable $\mathcal{G}_\delta$-subset iff X is a completely metrizable $\mathcal{G}_\delta$-set up to a meagre set.

We summarize these observations in one formal assertion:

Proposition 2·1. *In an almost complete space, a subset B has the Baire property iff the subspace B is almost complete.*

Proof. If X is almost complete, then any subspace that is almost complete is a Baire set, because an absolute $\mathcal{G}_\delta$ has the Baire property in X. For the converse, for a Baire set $B \subseteq X$ with X almost complete, write $X = H_X \cup N_X$ with N_X meagre and H_X an absolute $\mathcal{G}_\delta$ and $B = (U \backslash M_B) \cup N_B$ with U open and M_B, N_B meagre. Without loss of generality M_B may be taken as a meagre $\mathcal{F}_\sigma$ subset of U (otherwise choose F_B a meagre $\mathcal{F}_\sigma$ containing M_B and let F_B and $N_B \cup (F_B \backslash M_B)$ replace M_B and N_B respectively). Intersecting the representations of X and B, one has $B = H_B \cup N'_B$ for $H_B := H_X \cap (U \backslash F_B)$, an absolute $\mathcal{G}_\delta$, and some meagre $N'_B \subseteq N_B \cup N_X$. So, B is almost complete. $\square$

Almost completely metrizable spaces may be usefully characterized by reference to a less demanding absoluteness condition than topological completeness (we recall the latter is equivalent to being an absolute $\mathcal{G}_\delta$). One may show that a non-meagre normed group is almost complete iff it is almost absolutely analytic (see [**78**], [**76**]). This generalizes to normed groups a result, observed by S. Levi in [**55**], which goes back to Kuratowski ([**53**] Cor. 1 p. 482 taken together with IV.2, p. 88). See [**75**] Cor. 2 and [**78**], for additional material.

Examples of Baire sets. By analogy with the projective hierarchy of sets (known also as the *Luzin hierarchy*, see e.g. [**46**], p. 313), which may be generated from the closed sets by iterating projection and complementation any finite number of times, we may form the closely associated hierarchy of sets starting with the closed sets and iterating any finite number of times the Souslin operation **S** (following the notation of [**42**]) and complementation, denoted analogously by **C** say. Thus in a complete metric space one obtains the family $\mathcal{A}$ of analytic sets, by complementation the family $\mathbf{C}\mathcal{A}$ of co-analytic sets, then $\mathbf{SC}\mathcal{A}$ which contains the previous two classes, and so on. By Nikodym's theorem ([**42**] Cor. 2.9.4) all these sets have the Baire property. One might call this the *Souslin hierarchy.* Its simplest members are open sets, or in the case of $\mathbb{R}$ the intervals.

One may go further and form the smallest σ-algebra (so with complementation allowed) closed under **S** and containing the closed sets; this contains the Souslin hierarchy (implicit through an iteration over the countable ordinals). Members of the latter family are referred to as the *C-sets* – see Nowik and Reardon [**71**].

3. *Infinite combinatorics and shift-compactness*

The field of infinite combinatorics has, largely under the influence of Erdős and his school, grown to have many applications, for example to Ramsey theory within additive combinatorics [**35**], and to number theory [**92**]. A theme of this paper is that an aspect of infinite combinatorics has powerful applications in analysis, in connection with measure-category duality (see e.g. [**80**]). Corresponding to 'almost everywhere' (a.e.) for 'off a null set', we write 'quasi everywhere' (q.e.) for 'off a meagre set'; we write 'for generically all points' for 'off a null/meagre set', according to context. The infinite combinatorics are generic in this sense; see Theorem 4·2 (Th. KBD) in §4 below. The key unifying concept is *shift-compactness,* taken from the probability literature (see [**81**]) and adapted from its hitherto special context of semi-groups of measures (under convolution) to the additive group of reals. We define it and two related concepts as follows.

Definitions. 1. Say that $S \subseteq \mathbb{R}$ is *shift-compact,* resp. *properly shift-compact* (shift-compact for bounded sequences in $\mathbb{R}$, resp. in S), and write $S \in \mathcal{S}h_{\mathbb{R}}$ resp. $S \in \mathcal{S}h$, if for any bounded/convergent sequence u_n there are $t \in \mathbb{R}$ and infinite $\mathbb{M} = \mathbb{M}_t$ such that
(i) $\{t + u_m : m \in \mathbb{M}\} \subseteq S$, and
(ii) $\lim_{\mathbb{M}}(t + u_m) \in S$.

2. Say that $S \subseteq \mathbb{R}$ is *null-shift-compact* (shift-compact for null sequences), $S \in \mathcal{S}h_*$, if for any null sequence $z_n \to 0$ there are $t \in \mathbb{R}$ and infinite $\mathbb{M} = \mathbb{M}_t$ such that
(i) $\{t + z_m : m \in \mathbb{M}\} \subseteq S$,
(ii) $t = \lim_{\mathbb{M}}(t + z_m) \in S$.

3. Say that $S \subseteq \mathbb{R}$ is *null-shift-precompact,* and write $S \in \mathcal{S}$, if for any null sequence $z_n \to 0$ there are $t \in \mathbb{R}$ and infinite $\mathbb{M} = \mathbb{M}_t$ such that
(i) $\{t + z_m : m \in \mathbb{M}\} \subseteq S$.

The asterisk notation in Definition 2 above suggests avoiding a suitably small exceptional set. Similar notation is used in [**6**] §2.9 (on differentiating asymptotic relations); compare also the notions of category quantifier and measure quantifier (see e.g. [**46**], 8.J and 17.26, and see Theorem KBD below). It is clear that $\mathcal{S}h_{\mathbb{R}} \subseteq \mathcal{S}_*$ and $\mathcal{S}h_{\mathbb{R}} \subseteq \mathcal{S}h$; writing $\overline{\mathcal{S}}$ for the family of closed sets in $\mathcal{S}$, we have:

Proposition 3·1. *If $A \subseteq \mathbb{R}$ is null–shift-compact, then A is shift-compact and so properly shift-compact:*

$$\overline{\mathcal{S}} \subseteq \mathcal{S}_* = \mathcal{S}h_{\mathbb{R}} \subseteq \mathcal{S}h \text{ and } \mathcal{S}_* \subseteq \mathcal{S}.$$

Proof. Let a_n be a convergent sequence (sequence in A) with limit a_0. Then $z_n := a_n - a_0$ is a null sequence, hence for some $t \in A$ and infinite $\mathbb{M}_t$ we have $t + z_n$ in A for $n \in \mathbb{M}_t$. Thus with $s := t - u_0$ we have $s + u_n = t + z_n \in A$ for $n \in \mathbb{M}_t$ and convergence through $\mathbb{M}_t$ to $s + u_0 = t \in A$. So A is shift-compact (properly shift-compact). □

These are forms of compactness (for a topological analysis of this insight involving open shifted-covers, and further applications, see [**18**]). They generalize their forerunner *universality in relation to null sequences,* introduced in a related context by Kestelman [**48**], where the more demanding requirement on the set $\mathbb{M}$ in Definition 3 above is that it be co-finite. The latter concept of universality is implicit in some of Banach's work (see e.g. [**3**]).

In qualitative measure theory (that is, measure theory in which one is concerned only with whether the measure of a set is zero or positive, rather than with its numerical value), it is often the case that a measure-theoretic theorem has a category-theoretic analogue, in which we replace 'measurable function' by 'function having the Baire property' ([**53**], [**80**] – briefly, 'Baire function' below, and likewise 'Baire set'), and 'set of positive measure' by 'non-meagre set' (or set of second category). See [**80**] for a monograph treatment of such measure-category duality.

In previous work, on additive functions (Th. O) and related results (Th. S), it is the measure case that has been regarded as primary and the Baire (or topological, or category) case as secondary. As we shall see below, the reverse order is the more natural: it is the Baire case that is paramount. Indeed, we deduce the measure cases from the Baire cases, and do so by passing from the Euclidean topology to the density topology.

The tool whereby we interpret measurable functions as Baire functions is *refinement* of the usual metric (Euclidean) topology of the line $\mathbb{R}$ to a non-metric one: the *density topology* (see. e.g. [**46**], [**56**]). Recall that for T measurable, t is a (metric) density point of T if $\lim_{\delta\to 0}|T\cap I_\delta(t)|/\delta = 1$, where $I_\delta(t) = (t-\delta/2, t+\delta/2)$. By the Lebesgue Density Theorem almost all points of T are density points ([**36**] §61, [**80**] Th. 3.20). A set U is d-open (density-open = open in the density topology d) if (it is measurable and) each of its points is a density point of U. We mention five properties:

(i) The density topology is finer than (contains) the Euclidean topology ([**46**], 17.47(ii)). See [**56**] for a textbook treatment of other such fine topologies.

(ii) A set is Baire in the density topology iff it is (Lebesgue) measurable ([**46**], 17.47(iv)).

(iii) A Baire set is meagre in the density topology iff it is null ([**46**], 17.47(iii)). So (since a countable union of null sets is null) the conclusion of the Baire theorem holds for the line under d :

(iv) $(\mathbb{R}, d)$ is a Baire space, i.e., the conclusion of the Baire theorem holds ([**29**] §3.9).

(v) A function is d-continuous iff it is approximately continuous in Denjoy's sense ([**56**], p.1, 149).

The viewpoint of this paper and its relative [**19**] is *bitopological*: one moves between category and measure by moving between the Euclidean and the density topologies. Measure-category duality is the theme of the well-known book by Oxtoby, [**80**], at the end of which the density topology is briefly considered; one might almost regard the two papers as corresponding to 'the missing last chapter' of [**80**]. Related but rather different in emphasis is the book [**56**], on general fine topologies, with particular reference to the density topology and the classical fine topology of potential theory.

4. *Topology and Infinite Combinatorics*

Our principal tool is the Category Embedding Theorem below. At its heart is the condition below applied to a sequence of autohomeomorphisms which may be regarded as a category convergence to the identity; we call it weak to distinguish it from earlier usage as exemplified by Miller in [**62**]. We shall see that it is satisfied in the case of the Euclidean and density topologies by shifts induced by a null sequence $z_n \to 0$, namely the functions $h_n(x) := x + z_n$. In the condition below one should think of a sequence of homeomorphisms as being successively smaller shifts, just as in the preceding example; the condition says that locally the overlap between an open set and its successive images tends towards topological negligibility.

Definition (weak category convergence). A sequence of homeomorphisms h_n satisfies the *weak category convergence* condition if:

For any non-empty open set U, there is a non-empty open set $V \subseteq U$ such that, for each k,

$$\bigcap_{n\geqslant k} V\backslash h_n^{-1}(V) \text{ is meagre.} \tag{1}$$

Equivalently, for each k, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geqslant k)\ h_n(t) \in V.$$

This is a weak form of 'convergence to the identity'; for background and applications, see [**19**].

In Theorem 4·1 below, the topological space X may be assumed to be non-meagre (of second category) in itself, and the Baire set T to be non-meagre, as otherwise there is nothing to prove. To verify that X is non-meagre, one would typically assume that X is a Baire space (see §2,3). We recall that the real line is a Baire space under both the Euclidean and the density topologies. For the illustrations of the power and generality of Theorem 4·1 in contexts far beyond the real line see [**19**], [**18**], [**16**], [**74**]. Here we use it to unify category and measure versions of classical results in analysis such as those in the title.

In the theorem the term 'embedding' is motivated by the applications which follow. We write ω for $\{0, 1, 2, \ldots\}$.

THEOREM 4·1 (Category Embedding Theorem, CET). *Let X be a topological space and $h_n : X \to X$ be homeomorphisms satisfying* (1). *Then, for any Baire set T, for quasi-all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Take T Baire and non-meagre. We may assume that $T = U\backslash M$ with U non-empty and open and M meagre. Let $V \subseteq U$ satisfy (1). Since the functions h_n are homeomorphisms, the set

$$M' := M \cup \bigcup_n h_n^{-1}(M)$$

is meagre. Writing 'i.o.' for 'infinitely often', put

$$\begin{aligned} W = \mathbf{h}(V) &:= \bigcap_{k\in\omega} \bigcup_{n\geqslant k} V \cap h_n^{-1}(V) = \limsup\, [h_n^{-1}(V) \cap V] \\ &= \{x : x \in h_n^{-1}(V) \cap V \text{ i.o.}\} \subseteq V \subseteq U. \end{aligned}$$

So for $t \in W$ we have $t \in V$ and

$$v_m := h_m(t) \in V, \tag{2}$$

for infinitely many m – for $m \in \mathbb{M}_t$, say. Now W is co-meagre in V. Indeed

$$V\backslash W = \bigcup_{k\in\omega} \bigcap_{n\geqslant k} V\backslash h_n^{-1}(V),$$

which by (1) is meagre.

Take $t \in W\backslash M' \subseteq U\backslash M = T$, as $V \subseteq U$ and $M \subseteq M'$. Thus $t \in T$. For $m \in \mathbb{M}_t$, we have $t \notin h_m^{-1}(M)$, since $t \notin M'$ and $h_m^{-1}(M) \subseteq M'$; but $v_m = h_m(t)$ so $v_m \notin M$. By (2), $v_m \in V\backslash M \subseteq U\backslash M = T$. Thus $\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T$ for t in a co-meagre set.

To deduce that quasi-all $t \in T$ satisfy the conclusion of the theorem, put $S := T\backslash \mathbf{h}(T)$;

then S is Baire and $S \cap \mathbf{h}(T) = \varnothing$. If S is non-meagre, then by the preceding argument there are $s \in S$ and an infinite $\mathbb{M}_s$ such that $\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S$, i.e. $s \in \mathbf{h}(S) \subseteq \mathbf{h}(T)$, a contradiction. □

The last step in the proof above is an implicit appeal to yet another dichotomy a 'generic dichotomy' – see [**16**]. The conclusion of the theorem has a natural interpretation in the case of shifts. To justify it we shall need to prove that (1) holds for shifts. We begin with the easier of two verifications of (1).

THEOREM 4·1E (Verification Theorem for shifts in the Euclidean topology). *Let V be an open interval in $\mathbb{R}$. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,*

$$H_k := \bigcap\nolimits_{n \geqslant k} V \backslash (V + z_n) \text{ is empty.}$$

That is, the sequence $h_n(x) := x - z_n$ satisfies (1) under the Euclidean topology.

Proof. Let $V = (a, b)$ with $a < b$. The result is clear if the sequence is ultimately zero, so we may assume that the null sequence has an infinite subsequence $\mathbb{M}$ consisting either of positive terms only or of negative terms only. Consider first the case where the subsequence is positive. Then, for all n so large that $a + z_n < b$, we have $V \backslash h_n^{-1}(V) = (a, a + z_n)$, and so $\bigcap_{n \geqslant k} V \backslash h_n^{-1}(V)$ is empty for any $k \in \omega$. The same argument applies if the subsequence is negative, but with the end-points exchanged. □

THEOREM 4·1D (Verification Theorem for shifts in the density topology). *Let V be measurable and non-null. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,*

$$H_k := \bigcap\nolimits_{n \geqslant k} V \backslash (V + z_n) \text{ is null, so meagre in the } d\text{-topology.}$$

That is, the sequence $h_n(x) := x - z_n$ satisfies (1) under the d-topology.

Proof. Suppose otherwise. Then for some k, $|H_k| > 0$. Write H for H_k. Since $H \subseteq V$, we have, for $n \geqslant k$, that $\emptyset = H \cap h_n^{-1}(V) = H \cap (V + z_n)$ and so a fortiori $\emptyset = H \cap (H + z_n)$. Let u be a metric density point of H. Thus for some interval $I_\delta(u) = (u - \delta/2, u + \delta/2)$ we have

$$|H \cap I_\delta(u)| > \frac{3}{4}\delta.$$

Let $E = H \cap I_\delta(u)$. For any z_n, we have $|(E + z_n) \cap (I_\delta(u) + z_n)| = |E| > \frac{3}{4}\delta$. For $0 < z_n < \delta/4$, we have $|(E + z_n) \backslash I_\delta(u)| \leqslant |(u + \delta/2, u + 3\delta/4)| \leqslant \delta/4$. Put $F = (E + z_n) \cap I_\delta(u)$; then $|F| > \delta/2$. But $\delta \geqslant |E \cup F| = |E| + |F| - |E \cap F| \geqslant \frac{3}{4}\delta + \frac{1}{2}\delta - |E \cap F|$. So

$$|H \cap (H + z_n)| \geqslant |E \cap F| \geqslant \frac{1}{4}\delta,$$

contradicting $\emptyset = H \cap (H + z_n)$. This completes the proof. □

Now for the promised interpretation of Th. 4·1 to the case of shifts: this is the Kestelman-Borwein-Ditor Theorem (Th. 4·2 (Th. KBD) below), due in this form in the measure case to Borwein and Ditor [**21**], answering a question of Erdős [**30**] (see [**60**] for more on this). The result was already known much earlier albeit in somewhat weaker form by Kestelman ([**48**] Th. 3), and rediscovered by Trautner [**94**] (see [**6**] p. xix and footnote p. 10). See also [**17**] for a homotopic generalization. As with the CET, the set T here may be assumed to be non-meagre/non-null, since otherwise there is nothing to prove.

THEOREM 4·2 (Kestelman-Borwein-Ditor Theorem: Theorem KBD). *Let* $\{z_n\} \to 0$ *be a null sequence of reals. If* T *is Baire or Lebesgue measurable, then for generically all* $t \in T$ *there is an infinite set* $\mathbb{M}_t$ *such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Th. 4·1 may be applied to $h_n(x)$ as above in view of Th. 4·1E or 4·1D respectively in the category/measure cases. $\square$

For a weakening of the assumptions here see Theorem 7·2, the 'No Trumps Theorem' of §7. The theorem justifies the definitions in §3.

THEOREM 4·3. *A non-meagre Baire (non-null measurable) subset* S *of the real line is shift-compact, hence null-shift-compact.*

Proof. Without loss of generality suppose $u_n \to u$. So we may put $z_n := u_n - u \to 0$ and apply Th. 4·2 (Th. KBD) to obtain for quasi all $s \in S$:
(i) $\{s + z_m : m \in \mathbb{M}\} = \{s - u + u_m : m \in \mathbb{M}\} \subseteq S$,
(ii) $\lim_{\mathbb{M}}((s - u) + u_m) = \lim_{\mathbb{M}}(s + z_m) = s \in S$.
Thus for any such s, putting $t := s - u$ yields $\{t + u_m : m \in \mathbb{M}\} \subseteq S$ and $\lim_{\mathbb{M}}(t + u_m) \in S$. $\square$

We now change from a topological to a bitopological setting ([**47**], cf. [**56**]) in which we have two distinct but related topologies in play. This bitopological viewpoint enables us to unify Theorems S and P. The common generalization is of course a category theorem. This refers to shift-invariant topologies on $\mathbb{R}$, i.e. topologies under which translation is continuous. The density topology is an example; for other 'fine' topologies with this property, see [**75**].

THEOREM 4·4 (Topological, or Category, Interior Point Theorem). *Let* $\mathbb{R}$ *be given a shift-invariant topology* τ *under which it is a Baire space and suppose the homeomorphisms* $h_n(x) = x + z_n$ *satisfy* (1), *whenever* $\{z_n\} \to 0$ *is a null sequence (in the Euclidean topology). For* S *Baire and non-meagre in* τ, *the difference set* $S - S$ *contains an interval around the origin.*

Proof. Suppose otherwise. Then for each positive integer n we may select

$$z_n \in (-1/n, +1/n)\backslash(S-S).$$

Since $\{z_n\} \to 0$ (in the Euclidean topology), the Category Embedding Theorem (Th. 4·1) applies, and gives an $s \in S$ and an infinite $\mathbb{M}_s$ such that $\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S$. Then for any $m \in \mathbb{M}_s$,

$$s + z_m \in S \text{ , i.e. } z_m \in S - S,$$

a contradiction. □

5. *Steinhaus Dichotomy*

We deduce the two classical motivating theorems, Th. P and Th. S – which constitute the Steinhaus dichotomy. We give two proofs, both brief, one from Th. **??**, the other from Th. 4·2 (Th. KBD) (though these are essentially the same).

Proofs of Theorem P

First Proof. Apply Theorem 4·4, since by Th. 4·1E (1) holds. □

Second Proof. Suppose otherwise. Then, as before, for each positive integer n we may select $z_n \in (-1/n, +1/n)\backslash(S-S)$. Since $z_n \to 0$, by Theorem KBD, for quasi-all $s \in S$ there is an infinite $\mathbb{M}_s$ such that $\{s + z_m : m \in \mathbb{M}_s\} \subseteq S$. Then for any $m \in \mathbb{M}_s$, $s + z_m \in S$, i.e. $z_m \in S - S$, a contradiction. □

Remark. See [**18**] for a derivation from here of the more general result (due to Pettis, [**82**]) that for S, T Baire and non-meagre in the Euclidean topology, the difference set $S - T$ contains an interval.

Proofs of Theorem S

First Proof. Arguing as in the first proof above, by Th. 4·1D (1) holds and S, in the density topology, is Baire (being measurable) and non-meagre ([**46**] 17.47(iii)). □

Second Proof. Arguing as in the second proof above, Th. 4·2 (Th. KBD) applies. □

Just as with the Pettis extension of Piccard's result, so also here, Steinhaus proved that for S, T non-null measurable $S - T$ contains an interval. This too may be derived from the CET (Th. 4·1); see [**18**].

Unlike some of the results above, these results extend to topological groups. See e.g. [**26**] Th. 4.6 p.1175 for the positive statement, and the closing remarks for a negative one.

6. *Subgroup Dichotomy*

The next result gives (with its refinement, Theorem 6·2 below) the third of the sharp dichotomies of this paper. It concerns additive subgroups of the reals. These may be small or large in a number of senses, to be made precise below. For instance, a subgroup may be discrete (the integers, for example) or dense (the rationals); we use Kronecker's theorem (see [**37**] XXIII, Th. 438) to split theses two cases. In analysis one needs a stronger dichotomy in which 'large' means total – the entire real line. Theorem 6·1 below takes small in the the dual senses of category and measure, and is a direct consequence of Th. S (see Remark 1 after Theorem 6·2).

THEOREM 6·1 (Subgroup Theorem, cf. [**6**, Cor. 1.1.4], [**54**]). *For an additive Baire (resp. measurable) subgroup S of $\mathbb{R}$, the following are equivalent:*

(i) $S = \mathbb{R}$,
(ii) *S is non-meagre (resp. non-null).*

First Proof. By Th. 4·4 , for some interval I containing 0, we have $I \subseteq S - S \subseteq S$, and hence $\mathbb{R} = \bigcup_n nI = S$. □

Interest in the theorem comes typically from examples such as the one below.

EXAMPLE. For an additive extended real-valued function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, $D_f := \{x : f(x) < \infty\}$ is a subgroup of $\mathbb{R}$ and is the domain of definition of the corresponding real-valued 'partial' function f.

For a more intricate example, where the domain of definition of a 'partial function' is a subgroup of $\mathbb{R}$, see [**6**] Lemma 3.2.1. The task there is to give additional conditions under which the subgroup is all of $\mathbb{R}$, so that the partial function is in fact total. (Compare [**6**] Th. 3.2.5.) Sometimes a sufficient additional condition is density of the subgroup (in $\mathbb{R}$), which in turn may be reduced to the existence of two rationally incommensurable elements in the subgroup (by Kronecker's Theorem). Theorem 6·2 below uses related but stronger conditions than density guaranteeing 'totality'. See [**6**] Th. 1.10.2, where one uses density to show that in fact the domain of definition contains a co-countable set, so that the Subgroup Theorem applies.

Here we develop a combinatorial version, in the language of Ramsey theory [**35**].

DEFINITIONS. 1. Say that a set $S \subseteq \mathbb{R}$ has the *strong (weak) Ramsey distance property* if for any convergent sequence $\{u_n\}$ there is an infinite set (a set with two members) $\mathbb{M}$ such that

$$\{u_n - u_m : m, n \in \mathbb{M} \text{ with } m \neq n\} \subseteq S.$$

Thinking of the points of S as those having a particular colour, S has the strong Ramsey distance property if any convergent sequence has a subsequence all of whose pairwise distances have this colour.

2. Motivated by Lemma 6·4 below, say that $S \subseteq \mathbb{R}$ has the *finite covering property*, if there is an interval I and finite number of points $\{x_i : i = 1, ..., m\}$, such that the shifts $\{S + x_i : i = 1, ..., m\}$ cover I. When S is a subgroup, these shifted copies of S are just S-cosets.

THEOREM 6·2 (Combinatorial Steinhaus Theorem). *For an additive subgroup S of $\mathbb{R}$, the following are equivalent:*

(i) $S = \mathbb{R}$,
(ii) *S contains a non-meagre Baire, or a non-null measurable set,*
(iii) *S is shift-compact,*
(iv) *S is null-shift-compact,*
(v) *S has the strong Ramsey distance property,*
(vi) *S has the weak Ramsey distance property,*
(vii) *S has the finite covering property.*
(viii) *S has finite index in* $\mathbb{R}$.

The proof follows Lemmas 6·3 and 6·4 below; for an application see [**10**]. Theorem 6·2 effects a transition from topological through combinatorial to algebraic notions bringing out different aspects of the Steinhaus Theorem. See Laczkovich [**54**] for a topological study of proper subgroups of $\mathbb{R}$ (cf. §10 Remark 4). Taking a topological view, [**18**] shows that any open covering of a shift-compact subset of $\mathbb{R}$ yields a finite 'shifted-subcovering', i.e. consisting of shifted copies of a finite number of members of the open covering.

Lemma 6·3 (Finite Index Property). *An additive subgroup of* $\mathbb{R}$ *has finite index iff it coincides with* $\mathbb{R}$.

Proof. Suppose an additive subgroup S has finite index, n say, so that the quotient $\mathbb{R}/S$ is a finite group of order n. Then for each $x \in \mathbb{R}$, denoting S-cosets by $[x]$, one has $n[x/n] = [0]$, by Lagrange's theorem. That is, $x = n(x/n) \in S$, i.e. S is $\mathbb{R}$ itself. □

We use the above result in combination with the next observation, which is actually an instance (by specialization to $\mathbb{R}$) of the Finite Index Lemma due to B. H. Neumann (see [**69**], [**70**], or [**33**], Lemma 7.3): if an abelian group can be covered by a finite number of cosets of subgroups, then one of the subgroups has finite index. The dual reformulation, that a proper additive subgroup of $\mathbb{R}$ does not have the finite covering property, is actually what we need to prove that (vi) implies (i).

Lemma 6·4 (Finite-covering characterization). *For S an additive subgroup of* $\mathbb{R}$*, some open interval is covered by a finite union of cosets S of* $\mathbb{R}$ *iff S has finite index in* $\mathbb{R}$ *iff* $S = \mathbb{R}$.

Proof. For S a subgroup of $\mathbb{R}$ note first that S is either countable or dense (or both). Indeed, if S is uncountable, then it contains two elements which are rationally incommensurable, so there are two elements s, s' of S such that $ps + qs'$ is non-zero for all non-zero integers p, q. (Otherwise there are non-zero integers p, q such that $ps + qs' = 0$ in which case $s/s' = -q/p \in \mathbb{Q}$.) But then the subgroup of S comprising the points $ps+qs'$ for p, q integers is dense in $\mathbb{R}$, since s'/s is irrational (again by Kronecker's Theorem).

Suppose that a finite number of cosets of S, say $\{[x_i]_S : i = 1, 2, ..., m\}$, cover an interval (a, b) with $a < b$, in which case S is uncountable and so dense. We will show that these cosets in fact cover all of $\mathbb{R}$. Indeed, by density of S, for any $x \in \mathbb{R}$ we may choose $s \in S \cap (x - b, x - a)$ and so $x \in s + (a, b)$. But $[s + x_i] = [x_i]$, so one of these covers

x. That is, $\{[x_i] : i = 1, 2, ..m\}$ cover $\mathbb{R}$, i.e. S has finite index in $\mathbb{R}$ (as in Neumann's lemma). By the preceeding lemma, $S = \mathbb{R}$. The converse is clear. $\square$

Proof of Th. 6·2. It is clear that (i) implies (ii) and from Th. 4·2 (Th. KBD) that (ii) implies (iii), and by Proposition 3·1 that (iii) is equivalent to (iv) (since S is a subgroup). To see that (iii) implies (v) observe that, as S is shift-compact there are t and an infinite $\mathbb{M}$ such that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq S.$$

As S is a subgroup, for distinct m and n in $\mathbb{M}$

$$u_n - u_m = (t + u_n) - (t + u_m) \in S,$$

giving (v). Clearly (v) implies (vi). To prove (vi) implies (vii) we may assume that $S \neq \mathbb{R}$ (otherwise there is nothing to prove) and, by aiming for a contradiction, that S does not have the finite covering property. Suppose that $v_0, ..., v_{n-1}$ have been selected with $v_k < 1/(k+1)^2$ and $v_m + ... + v_{n-1} \notin S$ for each $m < n-1$. We want to select $v_n < 1/(n+1)^2$ such that for each $m < n$

$$v_m + ... + v_n \notin S, \text{ or equivalently } v_n \notin S - (v_m + ... + v_{n-1}). \tag{4}$$

Thus we require that

$$\begin{aligned} v_n &\in \bigcap\nolimits_{m<n} \left(0, 1/\left(n+1\right)^2\right) \setminus \left(S - (v_m + ... + v_{n-1})\right) \\ &= \left(0, 1/\left(n+1\right)^2\right) \setminus \bigcup\nolimits_{m<n} \left(S - (v_m + ... + v_{n-1})\right). \end{aligned}$$

If we cannot select such a v_n, then

$$\bigcup\nolimits_{m<n} S - (v_m + ... + v_{n-1}) \supseteq \left(0, 1/\left(n+1\right)^2\right),$$

and so S does have the finite covering property, contradicting our assumptions. (The induction can be started, as S is a subgroup, so cannot contain any interval, in particular $(0, 1)$.) Thus after all, the induction can proceed. Put $u_n := v_1 + ... + v_n$; then $\{u_n\}$ is convergent. By (v), there is a set $\mathbb{M}$ such that for m and n in $\mathbb{M}$ with $m < n$,

$$v_m + ... + v_{n-1} = u_n - u_m \in S.$$

This contradicts (4), so after all S does have the finite covering property, i.e. (vii) holds. Lemma 6·4 shows that (vii) implies (viii) and Lemma 6·3 that (viii) implies (i). $\square$

Second Proof of Th. 4·2 *(Subgroup Theorem).* Immediate from Th. 6·2. $\square$

Remarks. 1. As a consequence of Th. 4·2 (Th. KBD) on shift compactness, Th. S stands between Th. 6·2 (which employs shift-compactness) and Th. 6·1 (which Th. S implies).

2. The role of the finite covering property may be clarified by noting that, in the context of the theorem above, one may validly add the property 'S is closed' yielding amended equivalent conditions (ii)$'$-(vii)$'$, and then obtain a further equivalent condition:

(ix)$'$ *S is closed and contains an interval around* 0.

Indeed, if S is null-shift-compact, then it is closed. For suppose that $s_n \to s_0$ with $s_n \in S$ for all n. Then $z_n := s_0 - s_n$ is a null sequence and, as S is null-shift-compact, there is $s \in S$ such that $s + (s_0 - s_n) \in S$ infinitely often. But $s_n \in S$ and $s \in S$, so $s_0 \in S$. So S is closed. In particular, if S is closed and has the finite covering property, then by Baire's Theorem some coset of S contains an interval and so S itself contains an interval, (a, b) say. So its midpoint, s say, is in S. Shifting by $-s$, the set S contains an interval about 0, $(-\delta, \delta)$ say, yielding (vii)$'$. So, S also contains $(-n\delta, n\delta)$ for each $n \in \mathbb{N}$, so S is $\mathbb{R}$ (as in the first proof of Th. 6·1).

7. *Darboux Dichotomy*

We recall the following result (due to Darboux in 1875 – see [**51**]) and give its proof, as it is short.

THEOREM D (Darboux's Theorem). *If $f : \mathbb{R} \to \mathbb{R}$ is additive and locally bounded at some point, then f is linear.*

Proof. By additivity we may assume that f is locally bounded at the origin. So we may choose $\delta > 0$ and M such that, for all t with $|t| < \delta$, we have $|f(t)| < M$. For $\varepsilon > 0$ arbitrary, choose any integer N with $N > M/\varepsilon$. Now provided $|t| < \delta/N$, we have

$$N|f(t)| = |f(Nt)| < M, \text{ or } |f(t)| < M/N < \varepsilon,$$

giving continuity at 0. Linearity easily follows (see e.g. [**6**] Th.1.1.7). □

We now formulate the classical Ostrowski Theorem in what we term its strong form, as it includes both its classical measure-theoretic version and the Baire analogue due to Banach (see the Corollaries below, and also §8). Below we say that f is locally bounded (locally bounded above, or below) on a set S if any point t has a neighbourhood I such that f is bounded (resp. above, or below) on $S \cap I$.

THEOREM 7·1 (Strong Ostrowski Theorem). *For a null-shift-precompact set S (i.e. $S \in \mathcal{S}$), if $f : \mathbb{R} \to \mathbb{R}$ is additive and bounded (locally, above or below) on S, then f is locally bounded and hence linear.*

Proof. Suppose that f is not locally bounded in any neighbourhood of some point x. Then we may choose $z_n \to 0$ such that $f(x + z_n) \geqslant n$, without loss of generality (otherwise replace f by $-f$). So $f(z_n) \geqslant n - f(x)$. Since S is null-shift-precompact, there are $t \in \mathbb{R}$ and an infinite $\mathbb{M}_t$ such that

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq S,$$

implying that f is unbounded on S locally at t (since $f(t + z_n) = f(t) + f(z_n)$), a contradiction. So f is locally bounded, and by Darboux's Theorem (Th. D) f is continuous and so linear. □

Since any non-empty interval is null-shift-precompact, Theorem 7·1 embraces Darboux's Theorem. A weaker result, with the condition S null-shift-precompact strengthened to S universal, was given by Kestelman in [**49**]. So Theorems O and BM are now immediate corollaries of the above and of Theorem KBD. Theorem 7·1 is the basis of the Darboux dichotomy; that in turn is connected with the Steinhaus dichotomy, because an additive function bounded on a Baire non-meagre (measurable non-null) set A is bounded on the difference set $A-A$ and so on an interval contained in $A-A$. That is, the Darboux dichotomy based upon a 'thick set', an interval, may have its basis refined to a thinner set, just so long as the difference set is 'thick'. F. B Jones [**45**] refined this basis further by observing that it is enough for A to be analytic, so long as the subgroup generated by A is the reals (see [**14**]).

The boundedness conditions above lead naturally to a consideration of the level sets of a function and their combinatorial properties. The classical measure and category contexts appeal to various forms of localization. The nub is that, when a non-negligible set is decomposed into a countable union of nice sets, one of these is non-negligible. This is captured in the combinatorics below. Here we go beyond the null sequences.

Definitions. 1. For the function $h : \mathbb{R} \to \mathbb{R}$, the (symmetric) *level sets* of h are defined by

$$H^r := \{t : |h(t)| < r\}.$$

2. For $\{T_k : k \in \omega\}$ a countable family of sets of reals, we write $\mathbf{NT}(\{T_k : k \in \omega\})$ to mean that, for every bounded/convergent sequence $\{u_n\}$ in $\mathbb{R}$, some T_k contains a translate of a subsequence of $\{u_n\}$, i.e. there are $k \in \omega$, $t \in \mathbb{R}$ and infinite $\mathbb{M}_t \subseteq \omega$, such that

$$\{t + u_n : n \in \mathbb{M}_t\} \subseteq T_k$$

(see §10 Remark 7 for background on the origins of this notation).

We shall see that the **NT** property is a common generalization of both measurability and the Baire property. (Specializing to the case when $T_k = S$ for all k, we see that S is null-shift-precompact iff $\mathbf{NT}(S)$ holds.) This allows a formulation of when a function may be regarded as having 'nice' level sets. Our next result shows that **NT** captures classical notions of localization. Since $\mathbb{R}$ is the union of the level sets of a function, we have as an immediate corollary of Theorem KBD:

Theorem 7·2 (No Trumps Theorem, cf. [**10**]). *For $h : \mathbb{R} \to \mathbb{R}$ measurable or Baire $\mathbf{NT}(\{H^k : k \in \omega\})$ holds.*

As an illustration of its usefulness, we derive a common combinatorial generalization, with a weaker hypothesis, of Theorems O and BM. See also [**7**] for a common analysis of measurable and Baire subadditivity via **NT**.

Theorem 7·3 (Generalized Fréchet-Banach Theorem). *If $h : \mathbb{R} \to \mathbb{R}$ is additive and its level sets H^k satisfy $\mathbf{NT}(\{H^k : k \in \omega\})$, then h is locally bounded and so continuous and linear.*

Proof. Suppose that h is not locally bounded at the origin. Then we may choose $z_n \to 0$ such that $h(z_n) \geqslant n$, without loss of generality (if not replace h by $-h$). But there are $s \in \mathbb{R}$, $k \in \omega$ and an infinite $\mathbb{M}_s$ such that

$$\{s + z_m : m \in \mathbb{M}_s\} \subseteq H^k,$$

so

$$h(s + z_m) = h(s) + h(z_m) > h(s) + m,$$

so that h is unbounded on H^k, a contradiction as $|h| < k$ on H^k. Thus h is locally bounded and additive; hence by Darboux's Theorem (cf. the proof of Th. 7·1) we conclude that h is continuous and so linear. □

By Th. 7·2 this result embraces its classical counterpart for h measurable or Baire (due to Fréchet in 1914 and Banach in 1920; see [**51**]).

COROLLARY 7·4 (Fréchet-Banach Theorem). *If $f : \mathbb{R} \to \mathbb{R}$ is additive and measurable or Baire, then f is continuous and so linear.*

There is also a further combinatorial generalization of the classical Ostrowski Theorem, by reference to functions with 'nice' level sets.

THEOREM 7·1′ (Combinatorial Ostrowski Theorem, cf. [**10**]). *For $h(x)$ an additive function, $h(x)$ is continuous and $h(x) = cx$ for some constant c iff $\mathbf{NT}(\{H^k : k \in \omega\})$ holds.*

Proof. If $\mathbf{NT}(\{H^k : k \in \omega\})$ holds then by Th. 7·3 above h is linear. Conversely, if $h(x) = cx$, then $H^k = \{t : |ct| < k\}$ is for each $k = 1, 2, ...$ an interval and hence null-shift-precompact. □

The Subgroup Theorem may also be similarly restated. For this, we need a variant on $\mathbf{NT}(S)$ in which null-shift-precompact is strengthened to universal for null sequences; the corresponding notation is $\mathbf{NT}_A(S)$, where the suffix A denotes 'almost all', i.e. 'for all but a finite number of'.

THEOREM 6·2′ (Combinatorial Steinhaus Theorem Restated). *For an additive subgroup S of $\mathbb{R}$, the following are equivalent, to each other than to (ii)-(viii) of Th. P:*

(i) $S = \mathbb{R}$,
(iii)′ $\mathbf{NT}_A(S)$,
(iv)′ $\mathbf{NT}(S)$.

8. *Converse Ostrowski Theorem: pseudocompactness*

The Strong Ostrowski Theorem (Th. 7·1 above) draws its strength from the weaker hypothesis that $f \in \mathcal{A}dd$ be bounded above on a null-shift-precompact set T (rather than a Baire meagre/measurable non-null set). As noted, by Th. 4·2 (Th. KBD) this embraces both the measure and the category analogue (cf. Mehdi's Theorem [**57**] on the continuity of a mid-point convex function bounded on a non-meagre Baire set); for an alternative bitopological approach to the two cases see [**19**] (again using the density and Euclidean topologies).

Here we take up further the topological perspective linking shift-compactness with compactness; for any space T (not necessarily a subset of reals as before), let $\mathcal{B}(T)$, $\mathcal{B}^+(T)$, $\mathcal{C}(T)$ denote respectively the sets of real-valued functions that are bounded, bounded above, or continuous on T. Recall (cf. [**29**] §3.10, or for a group perspective [**26**] §6) that a space T is *pseudocompact* if $\mathcal{C}(T) \subseteq \mathcal{B}(T)$, equivalently $\mathcal{C}(T) \subseteq \mathcal{B}^+(T)$, i.e. every continuous function on T is bounded/bounded above. (In a separable metric space, in particular when T is a subset of $\mathbb{R}$ equipped with the usual topology, this property is of course equivalent to any of countable compactness, compactness, sequential compactness – again, see [**29**] §3.10.)

For $T \subseteq \mathbb{R}$ with $T \in \mathcal{S}$, the Strong Ostrowski Theorem (Th. 7·1) implies a 'reverse' inclusion to that involved in pseudocompactness, namely $\mathcal{A}dd(T)\cap\mathcal{B}^+(T) \subseteq \mathcal{C}(T)$, since for $f \in \mathcal{A}dd$, $f|T$ bounded implies in particular $f|T$ continuous. In the definition below we narrow the class of functions under the scope of pseudocompactness. This allows null-shift-compactness (i.e. membership in $\mathcal{S}_*$; recall that $\mathcal{S}_* \subset \mathcal{S}$) to be viewed as an *additive (sequential) compactness* property and yields a natural converse to Ostrowski's theorem.

Definition. Say that $T \subseteq \mathbb{R}$ is *$\mathcal{A}dd$-pseudocompact* (additively pseudocompact) if every function of $\mathcal{A}dd(T)$ in $\mathcal{C}(T)$ is bounded, i.e. $\mathcal{A}dd(T)\cap\mathcal{C}(T) \subseteq \mathcal{B}(T)$, equivalently $\mathcal{A}dd(T)\cap\mathcal{C}(T) \subseteq \mathcal{B}^+(T)$.

Theorem 8·1 (Additive-pseudocompactness, or Converse Ostrowski, Theorem). *Let $f \in \mathcal{A}dd(\mathbb{R})$ and let $T \in \mathcal{S}_*$. If f is continuous on T, then f is bounded above on bounded intervals of $\mathbb{R}$. In particular, T is additively pseudocompact.*

The conclusions remain valid for the more general class of functions f satisfying

$$\bar{k}_v(t) := \inf{}_{\delta>0} \sup{}_{|z|<\delta} |f(t+v+z) - f(v+z)| < \infty, \text{ for all } v, t. \tag{5}$$

Proof. If not, we may take f unbounded above; suppose that $f(u_n) \to \infty$ for $f \in \mathcal{A}dd(\mathbb{R})$ and $\{u_n\}$ bounded. Suppose without loss of generality that $u_n \to u$. Then $z_n := u_n - u \to 0$. As $T \in \mathcal{S}_*$, for some $t \in T$ and some infinite $\mathbb{M}_t$ we have $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$. Hence $f(t) + f(z_m) = f(t + z_m) \to f(t)$, for $m \in \mathbb{M}_t$, because f is continuous on T. Thus, taking limits with $m \in \mathbb{M}_t$, we have

$$f(u) - f(t) = \lim f(u + z_m) - f(t + z_m) = \infty,$$

a contradiction, as $f(u) - f(t)$ is finite. With the more general assumption on $f(.)$, the fact that $z_m \to 0$ yields the conclusion

$$\inf{}_{\delta>0} \sup{}_{|z|<\delta} |f(u+z) - f(t+z)| = \infty,$$

contradicting (5) with $v = u - t$. □

We include for completeness the Strong Ostrowski Theorem (see the opening paragraph) in this language:

THEOREM 7·1″ (Converse Additive-pseudocompactness Theorem). *For $f \in \mathcal{A}dd(\mathbb{R})$ and $T \in \mathcal{S}$ bounded, if f is bounded above on T, then f is continuous on $\mathbb{R}$.*

REMARKS. 1. The above analysis, mutatis mutandis, may be repeated in a Euclidean space $\mathbb{R}^d$, or for homomorphisms between metric groups.

2. There is a link here to the Ger-Kuczma investigation of 'sets of automatic continuity' (see [**34**] or [**51**], Ch. IX, X): the class of null-shift-precompact sets $\mathcal{S}$ is included in one of their classes. Specifically: for f additive, a null-shift-precompact T is a 'set of automatic continuity of f' given boundedness from above. That is, f bounded above on T implies f is continuous. See [**12**] for more on this.

3. The condition (5) is motivated by regular variation (for which see [**6**], or [**8**]). Indeed, we may say (following [**6**], Ch. 2, OR, Ch. 3, $O\Pi$) that $f(.)$ is O-regularly varying at u, if $\bar{k}_u(s) < \infty$, for all s. Thus (5) asserts that $f(.)$ is O-regularly varying at all points u.

9. *Higher Dimensions*

The Subgroup Theorem holds in $\mathbb{R}^N$ and more generally in any topological vector space $\mathbb{V}$ – this is a matter only of a change in vocabulary: one need only replace open intervals by open balls throughout. However, more interestingly, the Combinatorial Steinhaus Theorem (Th. 6·2) actually implies its own higher-dimensional analogue in relation to properties (i) to (vi). The other two require special treatment.

THEOREM 9·1 (Multi-dimensional Combinatorial Steinhaus Theorem). *For an additive subgroup T of a topological vector space $\mathbb{V}$(in particular $\mathbb{R}^N$), the following are equivalent:*

(i) $T = \mathbb{V}$,

(ii) *T contains a non-meagre Baire set (or a non-null measurable set if $\mathbb{V}$ is a locally compact group),*

(iii) $\mathbf{NT}_A(T)$,

(iv) $\mathbf{NT}(T)$,

(v) *T has the strong Ramsey distance property,*

(vi) *T has the weak Ramsey distance property.*

Proof. We need only prove (vi) implies (i). So suppose that (vi) holds for a subgroup $T \subseteq \mathbb{V}$. For any non-zero vector v in $\mathbb{V}$, let $S = T \cap Lin\{v\}$. We claim that Theorem 6·2 implies $S = Lin\{v\}$, so that $v \in S$. Thereupon $T = \mathbb{V}$ is immediate. Now, up to homomorphism, S is a subgroup of $\mathbb{R}$, so to establish the claim it suffices to observe that property (vi) for T implies the corresponding property (vi) of Theorem 6·2 for S, now regarded as a subgroup of $Lin\{v\}$. Indeed, given a convergent $\{u_n\} \subseteq Lin\{v\}$, there is a non-empty set $\mathbb{M}$ such that

$$\{u_n - u_m : m, n \in \mathbb{M} \text{ and } m \neq n\} \subseteq T.$$

But for $m, n \in \mathbb{M}$ we trivially have

$$u_n - u_m \in Lin\{v\} \cap T = S.$$

So the claim is established, and hence too our theorem. $\square$

Remarks. 1. The finite covering property, and hence also the finite index property (via density, as in Lemma 6·4), may be added to the list of equivalents. This involves not only a similar one-dimensional argument but also a reference to the proof method of Theorem 6·2, as follows.

If $T \subseteq \mathbb{V}$ has the weak Ramsey distance property, then as in the proof of Theorem 6·2 T has the finite covering property (involving the covering of some ball in $\mathbb{V}$).

Next, suppose that T has the finite covering property and that a finite number of cosets of T, say $\{[x_i]_T : i = 1, 2, ..., m\}$ covers a ball B. W.l.o.g. the ball is centered at 0. (If B is centered at z say, put $z_i = x_i - z$, then $\{[z_i]_T : i = 1, 2, ..., m\}$ covers the ball $B - z$ centred at 0.) Then $T \cap Lin\{v\}$ has the covering property for any non-zero vector v in $\mathbb{V}$ (by reference to the ball centered at o), so by Lemma 6·3 and 6·4 is the whole of $Lin\{v\}$. That is, $T = \mathbb{V}$.

2. See [**50**] and [**72**] for the topological-vector-space version of Steinhaus's Theorem.

10. *Concluding remarks*

1. Topological groups and shifts.

Just as we generalized the Combinatorial Steinhaus Theorem (Th. 6·2) from one to higher dimensions above, some of the results here can be generalized to topological groups; see [**13**] and [**18**] for details. We point out, however, that some of the work above does not extend in this way. For, we have made use of the density topology to unify the measure and category cases. But it is known that the real line cannot be made into a topological group under the density topology, see [**86**] (Proof of Prop. 1.9) and more recent work of Heath and Poerio ([**39**]).

In the above we work with shifts, so fixing one variable in the Cauchy functional equation and reducing the effective dimension from two to one.

We note there is a connection between Neumann's lemma (see §6) and the 'closed $\mathfrak{S}$-topology' generated, for $\mathfrak{S}$ a family of subgroups, by declaring as closed sets all cosets $S + a$ for $S \in \mathfrak{S}$. Here all the shift $x \to x + a$ are continuous and so one obtains a semi-topological group (cf. [**33**] §I.7), just as in the case of normed groups mentioned in §1 and 2.

2. Namioka's theorem.

The dimension reduction just mentioned is relevant to the relationship between separate and joint continuity for functions of two variables. The prototypical result here is Namioka's theorem [**68**], that separate continuity implies joint continuity, not everywhere but generically – off a small set. See [**77**] for further literature and recent work on joint versus one-sided continuity.

3. Analyticity.

In a compact metrizable topological group, a subgroup of interest may well not be a $\mathcal{G}_\delta$ and so not be completely metrizable. It has been noticed, in particular by van Mill in his elementary proof [**59**] of the very important Effros Open Mapping Theorem, that analyticity suffices. The theme that analytic topological groups, or even normed groups, are adequate for most purposes is explored in recent work by one of us: [**75**] lays out the basis of this vewpoint, and applications are given in separable groups in [**76**] and [**77**]. The non-separable context is considered in [**78**].

4. Neumann's Lemma.

Extensions of Neumann's Lemma for abelian groups are considered in [**22**] and also by Muthuvel in [**65**], [**66**], [**67**]. The fact that a finite number of proper cosets cannot cover the reals follows from a number of theorems including those of Laczkovich [**54**] and Miller and Muthuvel [**63**]. Our notion of finite covering property is equivalent to the property of being 'syndetic near 0' studied in [**40**], indeed the more general notion of syndetic subset of a semigroup provides a natural context for Neumann's Lemma. For results analogous to Theorem S but in the semigroup or amenable group context see Beiglböck [**4**] in connection with Jin's [**44**].

5. Negligibles.

The meagre and null sets in the work above may be thought of as negligible. One generalization is in the theory of sigma-ideals ([**46**] §15.C); another is in the work of Fremlin [**31**] on measure spaces with negligibles.

6. Quantitative versus qualitative measure theory.

As mentioned in the Introduction, we work largely with qualitative rather than quantitative measure theory here. The only place where we use quantitative measure theory is in the proof of Th. 4·1D. The distinction between the two is related to the limits of measure-category duality. For background on this, see e.g. [**80**].

7. No Trumps.

The term No Trumps in Theorem 7·2, a combinatorial principle, is used in close analogy with earlier combinatorial principles, in particular Jensen's Diamond $\diamondsuit$ [**43**] and Ostaszewski's Club $\clubsuit$ [**73**]. Our proof of Th. 7·2 makes explicit an argument implicit in [**5**], p. 482 (and repeated in [**6**], p. 9), itself inspired by [**27**] (see also [**8**], [**19**]). The intuition behind our formulation may be gleaned from forcing arguments in [**60**], [**61**]. Applied to the level sets, it is equivalent to the UCT, as is shown in [**8**]; it also plays a key role in the theory of subadditive functions, for which see [**7**].

8. Countability conditions.

Topological (or Čech) completeness (see [**29**] §3.9) and with it category methods are within general topology the notions that, roughly speaking, require some countability to be present; this is also true of metrizability and of p-spaces (see [**2**] §7, and also [**75**] §5), a generalization of both, whence the link between category aspects as here and metric aspects as in normed groups [**18**]. Countability is also present in measure theory via σ-additivity, and this is the basis of the measure-category duality in [**80**] and here. The use of a countable family of sets in the capture of translated subsequences in **NT** thus appears just as natural as the use in metrization theory of countable families with characteristically metric properties such as point-finiteness or local finiteness or star-refinement (see e.g. [**29**] §4.4).

9. *Finite versus infinite dimensions.*

The topological condition of Theorem 9·1 (Baire-non-meagre) naturally translates into a measure-theoretic condition in locally compact groups by appeal to Haar measure. We note that in infinite-dimensional vector spaces, despite the absence of local compactness and Haar measure, it is nevertheless possible to use the *Haar-null* sets of Christensen [**24**] (cf. [**93**] §2.4, p. 374), and the *prevalent* sets (valid analogues of the notion of a translation-invariant set containing "almost every" point) of [**41**]. For recent work here, see for example [**88**].

10. Models of set theory.

The programme begun here has implications for models of set theory, such as that of Solovay [**90**] in which all sets of reals are measurable (and have the Baire property), or Shelah's model [**87**] in which all sets of reals have the Baire property. In shift-compactness (Th. 4·2) one has a unifying concept for measurability and the Baire property.

11. *Genericity.*

Our use of genericity is related to genericity in other contexts. In mathematical logic, genericity in the sense of forcing is linked in the case of Cohen's original forcing argument to genericity in the sense of Baire category. See Mostowski's book [**64**], Ch. IX (especially p. 132, and p.127, where Mostowski attributes this link to Ryll-Nardzewski) for the earliest exposition here, and also [**46**], I.8B, II.16D for a modern textbook treatment. There is a similar link between Solovay's 'random forcing' argument in the measure context. See [**75**] for further examples – the general framework is provided in [**95**].

Acknowledgements. We are most grateful to the referee for his thorough and constructive report, which led to many improvements, particularly on the algebraic side. We thank Imre Leader for his comments.

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